APPLICATION OF THE LOW INTEGRAL EQUATION METHOD TO THE PROBLEM OF PROTON-PROTON SCATTERING

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An integral equation of the Low type is derived and investigated in relation to the proton-proton scattering problem. The connection with the phenomenological quantum-theoretical treatment of proton-proton scattering is established. It is shown that the total scattering cross section observed in the medium-energy region can be obtained by solving the derived equation, taking into account only the most intense scattered wave $^1S_0$, provided one neglects in the equation the contribution from states that involve a number of real mesons and nucleon-antinucleon pairs in addition to two nucleons. Moreover, it must be assumed that the inhomogeneous term of the equation is computed in the nonrelativistic single-meson approximation.

CONTEMPORARY quantum field theory meets with serious difficulties in trying to explain even the most fundamental phenomena of nuclear physics. These difficulties are related to a great extent to the inapplicability of perturbation theory methods to the investigation of properties of interacting meson* and nucleon fields and to the fact that it thus becomes necessary to develop methods of calculation which are not derived from perturbation theory. Further development of such methods will enable us to find out to what extent does the present quantum field theory correspond to reality, and which of its features will remain in a future theory.

One of the methods not based on perturbation theory is the method of Low equations.†

Applying this method to a single nucleon problem, Chew and Low[3] found out that a whole class of phenomena taking place at low energies in the presence of one nucleon can be explained sufficiently well even with the help of the present theory.

It thus becomes interesting to try to apply the method of Low equations to a two-nucleon problem.

This work considers the problem of elastic nuclear proton-proton scattering from the point of view of the Low-type integral equation.

Section 1 of this work contains the derivation of the integral equation for proton-proton scattering in a form convenient for the subsequent calculations. States that contain in addition to two nucleons a certain number of real mesons and nucleon-antinucleon pairs are neglected. This is permissible in the region of nonrelativistic energies corresponding to the elastic proton-proton scattering. This equation contains, as an unknown function, a matrix element which is related on the energy shell to the proton-proton scattering amplitude. The inhomogeneous term of the equation is computed in the nonrelativistic single-meson approximation.

No perturbation theory, in the form of expansion in terms of the coupling constant of meson and nucleon fields, is used anywhere in the derivation of the integral equation, nor in its solution.

A very important circumstance is the fact that the obtained equation contains only renormalized quantities.

In Sec. 2 the integral equation is used to find the scattering cross-section assuming that the latter is determined by the $^1S_0$ phase.

In the integration with respect to the energy of the intermediate state, a cutoff limit is introduced equal to 5μ (μ is the rest mass of the meson).* It is important that the dependence of the scattering cross-section on the cutoff limit is very weak.

The expression obtained for the scattering cross-section agrees very well with the experimental data in the intermediate energy region from about 0.1 to 11 Mev.

1. INTEGRAL EQUATION FOR THE PROTON-PROTON SCATTERING

Consider the matrix element of the operator

*Here and in the following the word "meson" means "π meson."

†Equations similar to those of Low also considered in Ref. 2.

*We let $c = \hbar = 1$.

$$S(\mathbf{k}'\nu') = a(\mathbf{k}'\nu', 0)\, H_I(0) - H_I(0)\, a(\mathbf{k}'\nu', 0)$$
$$= -\bar{u}^{\nu'}(\mathbf{k}')\left(\sum_{\lambda=1}^{4}\gamma_\lambda \frac{\partial}{\partial x_\lambda} + m\right)\psi(x)\Big|_{x=0},$$

taken between the functionals of the states of the system consisting of one and two protons:

$$(\Omega(\mathbf{q}'\mu'),\ S(\mathbf{k}'\nu')\,\Omega_{in}(\mathbf{k}\nu\mathbf{q}\mu)). \qquad (1)$$

Here $\psi(x)$ is the nucleon field operator; $H_I(0)$ is the energy density of the interaction of meson and nucleon fields $H_I(x)$ (with the renormalization counterterms for $x = 0$ introduced); $\mathbf{k}$, $\mathbf{q}$ and $\mathbf{k}'$ $\mathbf{q}'$ are the space components of the proton four-momenta k, q and k', q', for which $\mathbf{k} + \mathbf{q} = \mathbf{k}' + \mathbf{q}'$ in the center of mass coordinate system, but in general $k_0 = q_0 \neq k_0' = q_0'$; ν, μ and ν', μ' are the spin variables; $u^{-\nu}(k) = u^{\nu *}(k)\gamma_4$; $u^{\nu}(k)$ and γ_λ are the bispinors and Dirac matrices; m is the proton mass; and

$$a(\mathbf{k}\nu, 0) = a(\mathbf{k}\nu, x_0)\,|_{x_0=0} = \int d_3\mathbf{x} \exp(-i\mathbf{k}\mathbf{x})\, u^{\nu *}(\mathbf{k})\,\psi(x)\,|_{x_0=0}.$$

The functional $\Omega_{in}(\mathbf{kq})$ (we shall leave out the spin variables for brevity) can be represented in the form

$$\Omega_{in}(\mathbf{kq}) = a_{in}^{+}(\mathbf{k})\,\Omega(\mathbf{q}). \qquad (2)$$

In the above equation the operator $a_{in}^{+}(\mathbf{k})$ is the asymptotic form of the operator $\exp(-ik_0x_0) \times a^{+}(\mathbf{k}, x_0)$ [the operator $a^{+}(\mathbf{k}, x_0)$ is the Hermitian conjugate of the operator $a(\mathbf{k}, x_0)$]:

$$a_{in}^{+}(\mathbf{k}) = \exp(-ik_0x_0)\, a^{+}(\mathbf{k}, x_0) \text{ when } x_0 \to -\infty$$

and, by virtue of the equations of motion, it satisfies the relationship

$$a^{+}(\mathbf{k}, 0) = a_{in}^{+}(\mathbf{k}) + i\int_{-\infty}^{0} dx_0 \int d_3\mathbf{x} \exp(ikx - iPx)\, S^{+}(\mathbf{k}) \exp(iPx), \qquad (3)$$

where x_0 is the time; $kx = \mathbf{kx} - k_0x_0$; $Px = \mathbf{Px} - P_0x_0$; $(\mathbf{P}, P_0)$ – is the operator of the four-dimensional energy-momentum vector of the interacting meson and nucleon fields; the operator $S^{+}(\mathbf{k})$ is the Hermitian conjugate of $S(\mathbf{k})$.

For $k_0 = q_0 = k_0' = q_0'$, the matrix element (1) is related by a simple relationship to the proton-proton scattering amplitude $T(\mathbf{k}'\mathbf{q}', \mathbf{kq})$ with initial proton quantum numbers $\mathbf{k}, \mathbf{q}$ and final quantum numbers $\mathbf{k}', \mathbf{q}'$:

$$T(\mathbf{k}'\mathbf{q}', \mathbf{kq}) = -\frac{k_0}{4\pi}\,(\Omega(\mathbf{q}'),\ S(\mathbf{k}')\,\Omega_{in}(\mathbf{kq})). \qquad (4)$$

Our problem is to derive an integral equation for the matrix element (1).

We shall start from the anti-commutator relationship for the nucleon field operators $\psi_\alpha(x)$ and $\psi_\beta^{+}(0)$, taken at the same instant of time $x_0 = 0$:

$$\{\psi_\alpha(x),\ \psi_\beta^{+}(0)\} = \delta_{\alpha\beta}\delta(x).$$

Multiplying both sides of the above equation by $u_\alpha^{*}(\mathbf{k}')\,u_\beta(\mathbf{k})\exp(-i\mathbf{k}'\mathbf{x})$, integrating with respect to $\mathbf{x}$, and summing over α and β we obtain

$$\left\{\int d_3\mathbf{x} \exp(-i\mathbf{k}'\mathbf{x})\, u^{*}(\mathbf{k}')\,\psi(x),\ \psi^{+}(0)\,u(\mathbf{k})\right\} = u^{*}(\mathbf{k}')\,u(\mathbf{k}). \qquad (5)$$

It follows from this that the operator expression on the left hand side of (5) is a number, and consequently its matrix element taken between the functionals $\Omega(\mathbf{q}')$ and $\Omega(\mathbf{q})$, which are mutually orthogonal for $\mathbf{q}' \neq \mathbf{q}$, is equal to zero:

$$(\Omega(\mathbf{q}'), \left\{\int d_3\mathbf{x} \exp(-i\mathbf{k}'\mathbf{x})\, u^{*}(\mathbf{k}')\,\psi(x),\ \psi^{+}(0)\,u(\mathbf{k})\right\}\Omega(\mathbf{q})) = 0. \qquad (6)$$

Introducing a complete orthonormal set of eigenvectors of the states Ω_{in}^{n} for the operator $(\mathbf{P}, P_0)$, and integrating with respect to the variable $\mathbf{x}$, we can rewrite Eq. (6) in the form:

$$\sum_{n(p^n=0)} (\Omega_{in}^{n}, \psi^{+}(0)\,u(\mathbf{k})\,\Omega(\mathbf{q}))\,(\Omega_{in}^{n}, \psi^{+}(0)\,u(\mathbf{k}')\,\Omega(\mathbf{q}'))^{*} + \sum_{n(p^n=\mathbf{q}-\mathbf{k}')} (\Omega(\mathbf{q}'), \psi^{+}(0)\,u(\mathbf{k})\,\Omega_{in}^{n})\,(\Omega(\mathbf{q}), \psi^{+}(0)\,u(\mathbf{k}')\,\Omega_{in}^{n})^{*} = 0, \qquad (7)$$

where $\mathbf{p}^n$ is the momentum of the intermediate state described by the functional Ω_{in}^{n}.

Matrix elements of the operator $\psi^{+}(0)\,u(\mathbf{k})$, appearing in Eq. (7) (the matrix elements of the operator $\psi^{+}(0)\,u(\mathbf{k}')$ are obtained from the matrix elements of the operator $\psi^{+}(0)\,u(\mathbf{k})$ by replacing the quantum numbers $\mathbf{k}, \mathbf{q}$ by $\mathbf{k}', \mathbf{q}'$) can be transformed into the corresponding matrix elements of the operator $S^{+}(\mathbf{k})$ with the aid of the equations:

$$(\Omega_{in}^{n}, \psi^{+}(0)\,u(\mathbf{k})\,\Omega(\mathbf{q})) = \delta_{n,\mathbf{kq}} - \frac{(\Omega_{in}^{n}, S^{+}(\mathbf{k})\,\Omega(\mathbf{q}))}{2k_0 - E_n + i\varepsilon} \quad (\mathbf{p}^n = 0),$$

$$(\Omega(\mathbf{q}'), \psi^{+}(0)\,u(\mathbf{k})\,\Omega_{in}^{n}) = -\frac{(\Omega(\mathbf{q}'), S^{+}(\mathbf{k})\,\Omega_{in}^{n})}{k_0 - q_0' + E_n} \quad (\mathbf{p}^n = \mathbf{q} - \mathbf{k}'), \qquad (8)$$

where E_n is the energy of the intermediate state with momentum $\mathbf{p}^n$; $\epsilon > 0$ and $\epsilon \to 0$; $\delta_{n,\mathbf{kq}} = 0$ if $\Omega^n_{in} \neq \Omega_{in}(\mathbf{kq})$, and $\delta_{n,\mathbf{kq}} = 1$ if $\Omega^n_{in} = \Omega_{in}(\mathbf{kq})$.

Equations (8) are easy to prove if we write the operator $\psi^+(0)$ on the left hand side of (8) in the form

$$\psi^+(0) = \sum_{\mathbf{p}} [u^*(\mathbf{p})\, a^+(\mathbf{p}, 0) + v^*(\mathbf{p})\, b(\mathbf{p}, 0)]$$

(the bispinor $v(\mathbf{p})$ and the operator $b(\mathbf{p}, 0)$ pertain to antinucleons) and then, following the calculational methods of Klein,[4] we substitute for $a^+(\mathbf{p}, 0)$ and $b(\mathbf{p}, 0)$ their expressions from (3) and from an analogous formula for the antinucleon operator $b(\mathbf{p}, 0)$. Remembering relations (2) and taking into account the orthogonality and normalization properties of bispinors

$$u^{\nu^*}(\mathbf{k})\, u^{\mu}(\mathbf{k}) = \delta_{\nu\mu}, \quad v^{\nu^*}(-\mathbf{k})\, u^{\mu}(\mathbf{k}) = 0,$$

we arrive at Eq. (8).

Thus, substituting (8) into (7), we get

$$(k_0 - k_0')^{-1} [(\Omega(\mathbf{q}), S(\mathbf{k})\, \Omega_{in}(\mathbf{k}'\mathbf{q}'))^* - (\Omega(\mathbf{q}'), S(\mathbf{k}')\, \Omega_{in}(\mathbf{kq}))]$$

$$= 2 \sum_{n(\mathbf{p}^n = 0)} \frac{(\Omega(\mathbf{q}'), S(\mathbf{k}')\, \Omega^n_{in})\, (\Omega(\mathbf{q}), S(\mathbf{k})\, \Omega^n_{in})^*}{(2k_0' - E_n - i\varepsilon)\, (2k_0 - E_n + i\varepsilon)} - V, \qquad (9)$$

where

$$V = - \sum_{n(\mathbf{p}^n = \mathbf{q} - \mathbf{k}')} \frac{(\Omega(\mathbf{q}'), S^+(\mathbf{k})\, \Omega^n_{in})\, (\Omega(\mathbf{q}), S^+(\mathbf{k}')\, \Omega^n_{in})^*}{E_n^2 - (q_0 - k_0')^2}.$$

In the first term of the right-hand side of Eq. (9) the summation is over all those intermediate states of the system consisting of two nucleons and of a certain arbitrary number $l \geq 0$ of real mesons and nucleon-antinucleon pairs, described by the functionals Ω^n_{in}, which are consistent with the laws of conservation of charge and momentum. In the expression for V the summation is over all those intermediate states of the system consisting of a certain arbitrary number $l \geq 0$ of real mesons and nucleon-antinucleon pairs, described by functionals Ω^n_{in}, which have a momentum $\mathbf{p}^n = \mathbf{q} - \mathbf{k}'$ and are consistent with the law of conservation of charge.

Identity (9) can be looked upon as the first of the infinite set of equations for the infinite number of matrix elements entering the first term of the right-hand side of identity (9). The quantity V determined by Eq. (10) plays the part of the inhomogeneous term in this equation. On the energy surface $E_n = 2k_0$ or $E_n = 2k_0'$, the desired matrix elements are related to the transition amplitudes for different scattering processes, leading to the formation of two protons, by simple relationships similar to (4).

If we use the nonrelativistic approximation, we can neglect in the first term of the right-hand side of (9) the states which, in addition to two nucleons, contain a certain number $l > 0$ of real mesons and nucleon-antinucleon pairs. In that case identity (9) can be considered as an equation for the matrix element (1), in which the inhomogeneous term V can be computed in the nonrelativistic single-meson approximation. Going over to the continuous spectrum and substituting integration for summation, we get

$$(k_0 - k_0')^{-1} [(\Omega(\mathbf{q}), S(\mathbf{k})\, \Omega_{in}(\mathbf{k}'\mathbf{q}'))^*$$

$$- (\Omega(\mathbf{q}'), S(\mathbf{k}')\, \Omega_{in}(\mathbf{kq}))] = \frac{1}{2(2\pi)^3} \times \qquad (11)$$

$$\sum \int \frac{(\Omega(\mathbf{q}'), S(\mathbf{k}')\, \Omega_{in}(\mathbf{ts}))\, (\Omega(\mathbf{q}), S(\mathbf{k})\, \Omega_{in}(\mathbf{ts}))^*\, \delta(t + s)}{(k_0' - t_0 - i\varepsilon)\, (k_0 - t_0 + i\varepsilon)} d_3\mathbf{t} d_3\mathbf{s} - V,$$

where

$$V \sim g^2 \frac{(\bar{u}(\mathbf{q}')\, \gamma_5 u(\mathbf{k}))\, (\bar{u}(\mathbf{k}')\, \gamma_5 u(\mathbf{q}))}{[(\mathbf{q} - \mathbf{k}')^2 + \mu^2]^{1/2} + (q_0 - k_0')^2\, [(\mathbf{q} - \mathbf{k}')^2 + \mu^2]^{1/2}}. \qquad (12)$$

Here $\Omega_{in}(\mathbf{ts})$ is the functional of the intermediate state of the system consisting of two protons with momenta $\mathbf{t}$ and $\mathbf{s}$; the symbol Σ indicates summation over spins of protons in the intermediate state; the inhomogeneous term V is computed in the nonrelativistic single-meson approximation for the charge-symmetric meson theory with pseudo-scalar coupling; g^2 is the coupling constant of the meson and nucleon fields; $\gamma_5 = \gamma_1\gamma_2\gamma_3\gamma_4$; μ is the meson rest mass.

Equation (11), where the inhomogeneous term V is determined by formula (12), is the desired integral equation for the matrix element (1).

In order to establish a connection with the phenomenological theory of scattering from the potential U, it is convenient to use the following symbolic expression for the wave function $\psi_{\mathbf{kq}}(0)$ (at the time $t = 0$) of the system of two colliding protons with initial momenta $\mathbf{k}$ and $\mathbf{q}$:[5]

$$\psi_{\mathbf{kq}}(0) = \varphi_{\mathbf{kq}} + \frac{1}{k_0 + q_0 - K - U + i\varepsilon} U \varphi_{\mathbf{kq}},$$

where K is the kinetic-energy operator for the system, and $\varphi_{\mathbf{kq}}$ is the eigenfunction of the operator K corresponding to the free motion of protons with momenta $\mathbf{k}$ and $\mathbf{q}$. We take the scalar product of this equation with $U\varphi_{\mathbf{k'q'}}$. From the resultant equation we subtract its complex conjugate, in which the primed and unprimed quantum numbers have been transposed. If we now introduce a complete orthonormal set of functions $\psi_{\mathbf{ts}}(0)$, we ob-

tain an equation analogous to (11) for the quantity

$$R(\mathbf{k'q'}, \mathbf{kq}) = (\varphi_{\mathbf{k'q'}}, U\psi_{\mathbf{kq}}(0)),$$

which plays the part of the matrix element (1) (Ref. 5).

2. INVESTIGATION OF THE INTEGRAL EQUATION FOR THE $^1S_0$ WAVE COMPUTATION OF THE SCATTERING CROSS-SECTION

In this work we confine ourselves to dealing with the most intensive scattered wave $^1S_0$ for which the orbital angular momentum, the spin, and the total angular momentum are all equal to zero. We assume accordingly that the matrix element

$$(\Omega(\mathbf{q'}\mu'), S(\mathbf{k'}\nu')\Omega_{in}(\mathbf{k}\nu\mathbf{q}\mu)),$$

related to the proton-proton scattering amplitude by Eq. (4), is independent of the directions of momenta $\mathbf{k} = -\mathbf{q}$, $\mathbf{k'} = -\mathbf{q'}$ and is antisymmetric with respect to each pair of spin variables ν, μ and ν', μ'.

Introducing dimensionless energies

$$x_1 = m^{-1}(k_0 - m), \quad x_2 = m^{-1}(k_0' - m), \tag{13}$$

we let

$$(\Omega(\mathbf{q'}\mu'), S(\mathbf{k'}\nu')\Omega_{in}(\mathbf{k}\nu\mathbf{q}\mu)) = -m^{-2}c(x_1)F(x_2x_1)\chi(\nu'\mu')\chi(\nu\mu), \tag{14}$$

where $c(x_1) = m(kk_0)^{-1/2} \sim (2x_1)^{-1/4}$; the spin function $\chi(\nu\mu)$ is determined with the help of two relations $\chi(\nu\mu) = -\chi(\mu\nu)$ and $\chi(1-1) = 1$ (each of the spin variables ν, μ can take on the two values 1 and -1).

Substituting (14) into (11), summing over spin variables, and integrating over the directions of the momentum $\mathbf{t} = -\mathbf{s}$, we obtain an equation for the dimensionless function $F(x_2x_1)$:

$$(x_1 - x_2)^{-1}[c(x_1)F(x_2x_1) - c(x_2)F^*(x_1x_2)] = \frac{1}{4\pi^2}\int_0^r \frac{F(x_2x')F^*(x_1x')}{(x' - x_1 - i\varepsilon)(x' - x_2 + i\varepsilon)}dx' - W(x_1x_2). \tag{15}$$

Here $W(x_1x_2)$ is the isotropic part of the inhomogenous term V of (11), determined by (12), in units m^{-3}; $x' = m^{-1}(t_0 - m)$.

The range of integration over the dimensionless energy x' in the right hand side of (15) is from zero to the cutoff limit r, which is set equal to 0.19 (the kinetic energy in the laboratory system is then equal to 5μ, where μ is the meson rest mass). This value of the cutoff limit is the value chosen in meson-nucleon scattering problems.

Using a well-known relation

$$\frac{1}{x' - x \pm i\varepsilon} = \mathrm{P}\frac{1}{x' - x} \mp i\pi\delta(x' - x),$$

we rewrite equation (15) in the form

$$(x_1 - x_2)^{-1}\left\{c(x_1)F(x_2x_1) - c(x_2)F^*(x_1x_2) - \frac{i}{4\pi}[F(x_2x_2)F^*(x_1x_2) + F(x_2x_1)F^*(x_1x_1)]\right\} = \frac{1}{4\pi^2}\mathrm{P}\int_0^r \frac{F(x_2x')F^*(x_1x')}{(x' - x_1)(x' - x_2)}dx' - W(x_1x_2). \tag{16}$$

At first we assume that $W(x_1x_2) = W(x_2x_1)$ is a product of functions depending only on one variable:

$$W(x_1x_2) = w(x_1)w(x_2).$$

We seek a solution of (16) in the form:

$$F(x_2x_1) = a_0(x_2) + a_1(x_2)(x_1 - x_2). \tag{17}$$

(Expression (17) for the function $F(x_2x_1)$ does not apply for $x_1 \to 0$, $x_2 \to 0$, since $W(x_1x_2)$ becomes infinite in the "non-physical region" of energies $x_1 < 0$, $x_2 < 0$, near $x_1 = x_2 = 0$.)

Substituting (17) into (16), carrying out the integration over the variable x', expanding both sides in powers of difference $x_1 - x_2$ for fixed x_2, and equating terms of the same powers of $x_1 - x_2$, we obtain the following equations for the desired functions $a(x)$ and $a_1(x)$:

$$c(a_0 - a_0^*) = \frac{i}{2\pi}a_0a_0^*; \tag{18}$$

$$2c'a_0 - 2ca_0^{*\prime} + 2ca_1 + 2ca_1^* - \frac{i}{\pi}a_0a_0^{*\prime} + \frac{i}{2\pi}a_0a_1^* - \frac{i}{2\pi}a_0^*a_1 = \frac{1}{2\pi^2}\left[(a_0a_1^* + a_0^*a_1)\ln\frac{r}{x} - \frac{1}{x}a_0a_0^* + ra_1a_1^*\right] - 2W(xx); \tag{19}$$

$$c''a_0 - ca_0^{*\prime\prime} - \frac{i}{2\pi}a_0a_0^{*\prime\prime} + \frac{1}{2\pi^2x}a_0a_0^{*\prime} - \frac{1}{4\pi^2x^2}a_0a_0^* + W'(xx) + a_1\left(2c' - \frac{i}{2\pi}a_0^{*\prime} + \frac{1}{2\pi^2x}a_0^* - \frac{1}{2\pi^2}a_0^{*\prime}\ln\frac{r}{x}\right) + a_1^{*\prime}\left(2c + \frac{i}{2\pi}a_0 - \frac{1}{2\pi^2}a_0\ln\frac{r}{x}\right) - \frac{r}{2\pi^2}a_1a_1^{*\prime} = 0. \tag{20}$$

Here $c = c(x)$, $a_0 = a_0(x)$, $a_1 = a_1(x)$; primes denote derivatives with respect to x of corresponding functions.

Equation (18) is obtained by letting the coefficients of $(x_1 - x_2)^{-1}$ in the left hand side of (16) be equal to zero. (Because of the relation

$$\lim_{x_1 \to x_2}\mathrm{P}\int_0^r \frac{dx'}{(x' - x_1)(x' - x_2)} = -\frac{r}{x_2(r - x_2)} \sim -\frac{1}{x_2}$$

the right-hand side of (16) does not contain such terms and has a finite limit for $x_1 \to x_2$.) Equation (18) thus shows that we can go to the limit $x_1 \to x_2$

in the left-hand side of (16), and is therefore the necessary consequence of the existence of a limit in the right hand side of (16) as $x_1 \to x_2$.

Equations (19) and (20) are obtained by equating the coefficients of the zeroth and first power of the difference $x_1 - x_2$ correspondingly in the left and right-hand sides of (16). (So far we do not consider coefficients of higher powers of the difference $x_1 - x_2$.)

Equation (18) and (19) have the property that each gives only one relation each for the real and imaginary parts of the functions contained in them. Thus, Eqs. (18), (19), and (20) give jointly four relations for the two complex functions $a_0(x)$ and $a_1(x)$.

Let us note here that Eq. (18) is the unitarity condition for the scattering matrix. This condition is therefore contained in the initial integral equation (16).

To solve the set of equations (18), (19), and (20), we start with Eq. (18) which leads to

$$a_0(x) = 4\pi c(x)[f(x) - i]^{-1}, \tag{21}$$

where $f(x)$ is a real function, so far arbitrary. We shall solve Eqs. (19) and (20) by successive approximations: first, we determine $f(x)$ from (19) assuming that $a_1(x) = 0$, then we put the obtained zeroth approximation for $f(x)$ into (20) to find $a_1(x)$ in the first approximation.

Direct substitution of (21) into (19), with the condition that $a_1(x) = 0$, leads to

$$f'(x) = -\frac{1}{\pi x} - (4\pi)^{-1}(2x)^{1/2}W(xx)[f^2(x) + 1]. \tag{22}$$

The quantity $W(xx)$ is the isotropic part of the inhomogeneous term V determined from Eq. (12) on the energy surface $k_0 = k_0'$ in units m^{-3}. Noting that in the non-relativistic approximation $(\bar{u}(p')\gamma_5 u(p)) \sim (2m)^{-1}\chi(\sigma, p' - p)$, where the vector σ is composed of the Pauli spin matrices $\sigma_1, \sigma_2, \sigma_3$, separating out the isotropic part of the term V, and introducing the dimensionless energy $x = m^{-1}(k_0 - m)$, we obtain approximately the following expression for the quantity $W(xx)$ for the value of the coupling constant $g^2 = 14.7$:

$$W(xx) \sim 0.8x(x + 0.006)^{-3/2}.$$

In our method the choice of the solution of Eq. (16) depends on the determination of the constant of integration in the solution of Eq. (22), which is an unknown function of the coupling constant. In principle this choice can be made by comparison with theory for low energies. However, we shall determine the constant of integration by requiring that the scattering cross-section for a certain energy coincide with the experimental value. For example, comparison with the proton-proton scattering data for the energy $x = 0.01$ leads to $f(0.01) \sim 0.9$. In this case the relative weight of the second term of the right hand side of equation (22) is ~ 0.003 compared to the first term; thus in the neighborhood of $x = 0.01$ the function $f(x)$ satisfies the following equation to a high degree of accuracy.

$$f'(x) = -\pi^{-1}x^{-1}. \tag{23}$$

Equation (23) will hold even better for $x < 0.01$, for as x gets smaller the relative weight of the second term of the right-hand side of eq. (22) will diminish only because $\pi^{-1}x^{-1}$ increases and $(2x)^{1/4}W(xx)$ decreases.

The solution of Eq. (23) has the form $f(x) = \pi^{-1}\ln(C/x)$ where the constant of integration C is determined from the condition $f(0.01) \sim 0.9$, and is approximately equal to the cutoff limit r. We thus obtain the following expression for the function $f(x)$:

$$f(x) = \frac{1}{\pi}\ln\frac{r}{x}. \tag{24}$$

Let us now consider the determination of $a_1(x)$ in the first approximation. Introducing

$$A \equiv 2c + \frac{i}{2\pi}a_0 - \frac{1}{2\pi^2}a_0 \ln\frac{r}{x},$$

let us rewrite Eq. (20) in the form

$$\frac{1}{2}a_0A^{**} - \frac{1}{2}a_0^{**}A + W'(xx) + a_1A^{**} + a_1^{**}A - \frac{r}{2\pi^2}a_1a_1^{**} = 0.$$

Inserting here the zeroth approximation for $a_0(x)$ [Eqs. (21) and (24)] and noting that in this case $A = 0$, we get

$$W'(xx) - \frac{r}{2\pi^2}a_1a_1^{**} = 0.$$

Since the function $W'(xx)$ is real, the solution of this equation has the form:

$$a_1(x) = e^{i\alpha}b(x), \text{ when } b(x) = b^*(x),$$
$$b^2(x) = 4\pi^2 W(xx)/r + \beta. \tag{25}$$

Here α and β are arbitrary real constants of integration. We shall see later that the constant β must be equal to zero and that the magnitude of the constant α is unimportant.

We obtained expressions (21), (22), and (25) for functions $a_0(x)$ and $a_1(x)$ by an approximate solution of the set of equations (18), (19), and (20). Actually, however, these expressions are the exact solutions of the set (18), (19), and (20) under the condition that $\beta = 0$. Moreover, the function $F(x_2x_1)$ determined from them with Eq. (17) is an

exact solution of the integral equation (16) under the assumption that $W(x_1x_2) = w(x_1)w(x_2)$. (Of course, the equality of the coefficients of the first, second, third... powers of the difference $x_1 - x_2$ on the right and left hand side of Eq. (16) follow from this automatically. We did not consider these equalities before.)

Both the first and the second statement are easily proved by direct verification. Let us prove the first statement, for example. To do this it is sufficient to verify the following equalities, obtained from (19) by striking out terms containing only the unknown function $a(x)$, and also $a_0^*(x)$ and $a_0^{*\prime}(x)$:

$$2ca_1 + 2ca_1^* + \frac{i}{2\pi}a_0a_1^* - \frac{i}{2\pi}a_0^*a_1 = \frac{1}{2\pi^2}(a_0a_1^* + a_0^*a_1)\ln\frac{r}{x} + \frac{r}{2\pi^2}a_1a_1^* - 2W(xx).$$

The above becomes obvious if we rewrite it in the form

$$a_1A^* + a_1^*A = \frac{r}{2\pi^2}a_1a_1^* - 2W(xx)$$

and if we recall the relation (25) and that $A = 0$.

Thus, the function $F(x_2x_1)$ determined from Eqs. (17), (21), (24), and (25) is the exact solution of the integral equation (16), where $W(x_1x_2)$ has the form $W(x_1x_2) = w(x_1)w(x_2)$. If $W(x_1x_2) \neq w(x_1)w(x_2)$ then the function $F(x_1x_2)$ obviously does not satisfy Eq. (16) itself, but it satisfies an equation obtained from (16) by replacing $W(x_1x_2)$ by

$$[W(x_1x_1)W(x_2x_2)]^{1/2}.$$

Furthermore, insofar as for this value of $F(x_2x_1)$ [Eqs. (17), (21), (24), and (25)] the relative weight of the term $W(x_1x_2)$ in the right hand side of (16) is negligibly small compared to the first term on the right hand side of (16) (it is no bigger than 0.01 for $x_1 \leq 0.03$ and $x_2 \leq 0.03$ if the coupling constant is $g^2 = 14.7$), the function $F(x_2x_1)$ found by us is an approximate solution of the original integral equation (16) itself. In fact, when $x_1 \leq 0.03$ and $x_2 \leq 0.03$, this function $F(x_2x_1)$ coincides with $a_0(x_2)$ [when $a_1(x_2) = 0$, $F(x_2x_1) = a_0(x_2)$ is an exact solution of (16) for $W(x_1x_2) = 0$], since for these values of x_1 and x_2 the absolute magnitude of the term $a_1(x_2)(x_1 - x_2)$ is not more than 0.05 of the absolute magnitude of the quantity $a_0(x_2)$. Thus the approximate solution $F(x_2x_1)$ we obtained for (16) depends weakly on the difference $x_1 - x_2$, which characterizes the degree of removal from the energy surface $x_1 = x_2$.

Substituting the value obtained for $F(x_2x_1)$ into (14) and discarding the unimportant term $a_1(x_2)(x_1 - x_2)$, we arrive at the following expression for the sought matrix element (1):

$$(\Omega(q'),\ S(k')\Omega_{in}(k, q)) = \frac{4\pi}{(kk'k_0k_0')^{1/2}}\left(\frac{1}{\pi}\ln\frac{mr}{k_0' - m} - i\right)^{-1}.$$

Recalling Eq. (4), which relates the matrix element (1) to the proton-proton scattering amplitude on the energy surface $k_0 = k_0'$, we readily obtain an expression for the scattering cross-section of the $^1S_0$ wave:

$$d\sigma_{c.m.} = 8 \cdot 10^{-25}E^{-1}[1 + \pi^{-2}\ln^2(E_mE^{-1})]\ \text{cm}^2\,\text{sterad}^{-1}.$$

Here E is the energy of the incident proton in the laboratory system, in Mev, and E_m is the cutoff energy, equal to 5μ, in Mev.

The expression obtained for the scattering cross-section depends weakly on the cutoff energy E_m and is in good agreement with experimental data in the intermediate energy region from about 0.1 to 100 Mev.

The method considered here for solving Eq. (16) does not enable us to determine the dependence of the matrix element (1), and of the scattering cross-section, on the coupling constant g^2. This is due to the fact that in the solution of Eq. (22)

$$f = f[x, C(g^2)] \equiv \varphi^{-1}(x, g^2),$$

we determined the integration constant $C = C(g^2)$, which is an unknown function of g^2, by comparison with experiment for only one value of g^2, namely $g_0^2 = 14.7$. The dependence of the constant of integration, and also of the scattering cross-section, on the coupling constant g^2 could in principle be determined by comparison with the theory for low energies. We shall make here only a few general remarks about this dependence.

Taking it into account that $\varphi(x, g^2) \to 0$ for $g^2 \to 0$, we expand $\varphi(x, g^2)$ in powers of $(g/g_0)^2$ and single out the first term of the expansion

$$\varphi(x, g^2) = (g/g_0)^2\varphi_0(x) + (g/g_0)^4\varphi_1(x, g^2),$$

where $\varphi_1(x, g^2)$ has a finite limit different from zero for $g^2 \to 0$, and where obviously

$$\varphi_0(x) + \varphi_1(x, g_0^2) = \pi\ln^{-1}(rx^{-1}).$$

When $g \sim g_0$, the most important term in this expansion is the second term, since, because of smallness of the quantity $W(xx)$ we have $\varphi_0(x) \ll \varphi_1(x, g_0^2)$. [This last inequality is directly related to those estimates which have enabled us to substitute Eq. (22) for (23).]

For small g^2, to the contrary, the principal term is the first term. For such values of g^2 the scattering cross-section determined by it behaves as the fourth power of the coupling constant, g^4.

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SPACE-TIME CORRELATION FUNCTIONS FOR A SYSTEM OF PARTICLES WITH ELECTROMAGNETIC INTERACTION

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A closed set of equations is obtained for the random functions $N_{qp}(t)$. This set determines the number of particles at a given point **p**, **q** in phase space at time t, and the vector and scalar potentials **A**, φ. Chains of coupled equations for the moments of the random functions have been obtained by averaging from this set of equations. The equations are solved under the assumption that the random process in the system is stationary and uniform. Expressions are obtained which permit determination of space-time correlation functions of currents, densities, and vector potentials from a knowledge of simultaneous (equilibrium) correlation functions. Expressions are obtained for correlation functions of "extraneous" random electromagnetic fields and currents. In the absence of space dispersion these expressions become the familiar formulae derived by Leontovich and Rytov phenomenologically. An explicit expression is obtained for the complex dielectric constant of the medium

IN determining temporal correlation functions of random processes or space-time correlation functions for random fields we can introduce the concept of "extraneous" random forces or fields for which the correlation functions are known. Thus Rytov[1] has investigated fluctuations of the electromagnetic field by introducing extraneous emf's or extraneous fields, for which the correlation functions are assumed to be known. Similarly, the theory of hydrodynamical fluctuations is constructed by introducing "extraneous terms" with known correlation functions into the equation of motion of a fluid.[2]

It is the object of the present paper to obtain a closed set of approximate equations for the space-time correlation functions of a system of particles with electromagnetic interaction.

A similar problem for a classical system of particles with Coulomb interaction was considered by Tolmachev,[3] who used Bogoliubov's method to obtain a chain of equations for nonsimultaneous correlation functions. Through an approximate solution of this chain of equations Tolmachev[4] obtained an expression which relates the space-time correlation function of a system of charged particles with Coulomb interaction to the correlation function for $\tau = 0$.

In his investigation[5] of the spectra of elementary excitations in a system of centrally-interacting particles, the present author used the following equation for the random function:

$$N_{qp}(t) = \sum_{i=1}^{N} \delta(\mathbf{q} - \mathbf{q}_i)\,\delta(\mathbf{p} - \mathbf{p}_i),$$