In conclusion I would like to express my gratitude to the Academician V. A. Fock and Iu. V. Novozhilov for their attention and discussion of this work.

[1] F. Low, Phys. Rev. **97**, 1392 (1955).
[2] Lehmann, Symanzik, and Zimmerman, Nuovo cimento **1**, 1 (1955).
[3] G. Chew and F. Low, Phys. Rev. **101**, 1570 (1956).
[4] A. Klein, Phys. Rev. **102**, 913 (1956).
[5] M. Gell-Mann and M. L. Goldberger, Phys. Rev. **91**, 398 (1953).

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SPACE-TIME CORRELATION FUNCTIONS FOR A SYSTEM OF PARTICLES WITH ELECTROMAGNETIC INTERACTION

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A closed set of equations is obtained for the random functions $N_{\mathbf{qp}}(t)$. This set determines the number of particles at a given point **p**, **q** in phase space at time t, and the vector and scalar potentials **A**, φ. Chains of coupled equations for the moments of the random functions have been obtained by averaging from this set of equations. The equations are solved under the assumption that the random process in the system is stationary and uniform. Expressions are obtained which permit determination of space-time correlation functions of currents, densities, and vector potentials from a knowledge of simultaneous (equilibrium) correlation functions. Expressions are obtained for correlation functions of "extraneous" random electromagnetic fields and currents. In the absence of space dispersion these expressions become the familiar formulae derived by Leontovich and Rytov phenomenologically. An explicit expression is obtained for the complex dielectric constant of the medium

In determining temporal correlation functions of random processes or space-time correlation functions for random fields we can introduce the concept of "extraneous" random forces or fields for which the correlation functions are known. Thus Rytov[1] has investigated fluctuations of the electromagnetic field by introducing extraneous emf's or extraneous fields, for which the correlation functions are assumed to be known. Similarly, the theory of hydrodynamical fluctuations is constructed by introducing "extraneous terms" with known correlation functions into the equation of motion of a fluid.[2]

It is the object of the present paper to obtain a closed set of approximate equations for the space-time correlation functions of a system of particles with electromagnetic interaction.

A similar problem for a classical system of particles with Coulomb interaction was considered by Tolmachev,[3] who used Bogoliubov's method to obtain a chain of equations for nonsimultaneous correlation functions. Through an approximate solution of this chain of equations Tolmachev[4] obtained an expression which relates the space-time correlation function of a system of charged particles with Coulomb interaction to the correlation function for $\tau = 0$.

In his investigation[5] of the spectra of elementary excitations in a system of centrally-interacting particles, the present author used the following equation for the random function:

$$N_{\mathbf{qp}}(t) = \sum_{i=1}^{N} \delta(\mathbf{q} - \mathbf{q}_i)\, \delta(\mathbf{p} - \mathbf{p}_i),$$

which defines the number of particles in the phase space region $d\mathbf{q}d\mathbf{p}$ around the point $\mathbf{q}, \mathbf{p}$ at time t. For the random function $N_{\mathbf{qp}}(t)$ there exists a multidimensional distribution function $F(\ldots N_{\mathbf{qp}} \ldots, t)$ for the probabilities of different values of $N_{\mathbf{qp}}$. The quantum analog of $N_{\mathbf{qp}}(t)$ is

$$N^{(Q)}_{\mathbf{qp}}(t) = (2\pi)^{-3}\int \Psi^{+}(\mathbf{q} - {}^1/_2\,\hbar\tau)\,\Psi(\mathbf{q} + {}^1/_2\,\hbar\tau)\,e^{-i\tau\mathbf{p}}d\tau,$$

in which Ψ^{+} and Ψ are quantized wave functions which satisfy the familiar commutation relations.

In Ref. 6 the equations for $N_{\mathbf{qp}}(t)$ and $N^{Q}_{\mathbf{qp}}(t)$ were used to obtain chains of equations for nonsimultaneous distribution functions for systems of centrally-interacting particles. In the present paper the same method is used to investigate space-time correlation functions for a classical system of particles with electromagnetic interaction.

1. The Hamiltonian function of a classical system of N charged particles can be represented as follows:

$$\mathscr{H} = \sum_{i=1}^{N}\frac{1}{2m}\left(\mathbf{P}_i - \frac{e}{c}\,\mathbf{A}(\mathbf{q}_i)\right)^2 + e\sum_{i=1}^{N}\varphi(\mathbf{q}_i) + \frac{1}{4\pi}\int \mathbf{E}\,\mathrm{grad}\,\varphi\, d\mathbf{q} + \frac{1}{8\pi}\int(\mathbf{E}^2 + \mathbf{H}^2)d\mathbf{q}, \qquad (1)$$

$$\mathbf{H} = \mathrm{curl}\,\mathbf{A}, \quad \mathbf{E} = -\,\mathrm{grad}\,\varphi - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}, \quad \mathrm{div}\,\mathbf{A} = 0.$$

Here a Coulomb calibration has been used, by which the Coulomb interaction between particles can be distinguished immediately. It is assumed that electron charges are compensated by uniformly distributed positive ion charges.

If the vector potential is regarded as the field coordinate and the quantity

$$\Pi = -\,\mathbf{E}/4\pi c = \dot{\mathbf{A}}/4\pi c^2 + \mathrm{grad}\,\varphi/4\pi c$$

as the field momentum, one can introduce a distribution function

$$f(\mathbf{q}_1 \ldots \mathbf{q}_N,\ \mathbf{P}_1 \ldots \mathbf{P}_N, \ldots \mathbf{A}(\mathbf{q}) \ldots, \ldots \Pi(\mathbf{q}) \ldots, t),$$

for the probabilities of different values of the coordinates and momenta of both the particles and the field. The equation for f can be obtained by means of a Hamiltonian equation corresponding to the Hamiltonian (1). The equilibrium solution of this equation is the Gibbs distribution.

The equation for f could be used to determine the space-time correlation functions. In the present work, however, we shall use a different method which is essentially as follows. We introduce the random function

$$N_{\mathbf{qP}}(t) = \sum_{i=1}^{N}\delta(\mathbf{q} - \mathbf{q}_i)\,\delta(\mathbf{P} - \mathbf{P}_i),$$

which defines the number of particles in the phase-space region $d\mathbf{q}d\mathbf{P}$ around the point $\mathbf{q}, \mathbf{P}$ at time t. By means of this definition we represent the Hamiltonian (1) in the equivalent form

$$\mathscr{H} = \int\frac{1}{2m}\left(\mathbf{P} - \frac{e}{c}\,\mathbf{A}(\mathbf{q})\right)^2 N_{\mathbf{qP}}d\mathbf{q}d\mathbf{P} + e\int\varphi(\mathbf{q})N_{\mathbf{qP}}d\mathbf{q}d\mathbf{P} + \frac{1}{4\pi}\int\mathbf{E}\,\mathrm{grad}\,\varphi\,d\mathbf{q} + \frac{1}{8\pi}\int(\mathbf{E}^2 + \mathbf{H}^2)\,d\mathbf{q}. \qquad (2)$$

The positively charged ionic background can be taken into account explicitly if $\int N_{\mathbf{qP}}d\mathbf{P}$ is replaced by $\int N_{\mathbf{qP}}d\mathbf{P} - n_+$, where n_+ is the number of ions per unit volume.

Using the Hamiltonian (2), we obtain Hamilton's equations for the particles and variables which characterize the field:

$$\dot{\mathbf{q}} = \frac{\partial}{\partial \mathbf{P}}\frac{\delta\mathscr{H}}{\delta N_{\mathbf{qP}}} = \left(\mathbf{P} - \frac{e}{c}\,\mathbf{A}\right)/m, \qquad (3)$$

$$\dot{\mathbf{P}} = -\frac{\partial}{\partial \mathbf{q}}\frac{\delta H}{\delta N_{\mathbf{qP}}} = -\frac{\partial}{\partial \mathbf{q}}\int\frac{e^2}{|\mathbf{q}-\mathbf{q}'|}N_{\mathbf{q}'\mathbf{P}'}\,d\mathbf{q}'d\mathbf{P}' - \frac{1}{2m}\frac{\partial}{\partial \mathbf{q}}\left(\mathbf{P} - \frac{e}{c}\mathbf{A}\right)^2, \qquad (4)$$

$$\dot{\mathbf{A}} = \delta\mathscr{H}/\delta\Pi = 4\pi c^2\Pi - c\,\mathrm{grad}\,\varphi, \qquad (5)$$

$$\dot{\Pi} = -\frac{\delta H}{\delta \mathbf{A}} = \frac{1}{4\pi}\Delta\mathbf{A} + \frac{e}{cm}\int\left(\mathbf{P} - \frac{e}{c}\,\mathbf{A}\right)N_{\mathbf{qP}}\,d\mathbf{P}, \qquad (6)$$

$$\Delta\varphi = -\,4\pi e\int N_{\mathbf{qP}}\,d\mathbf{P}, \quad \mathrm{div}\mathbf{A} = 0. \qquad (7)$$

From Eqs. (4) and (5) and the continuity condition we derive an equation for the random function $N_{\mathbf{qp}}(t)$:

$$\frac{\partial N_{\mathbf{qP}}}{\partial t} + \frac{1}{m}\left(\mathbf{P} - \frac{e}{c}\,\mathbf{A}\right)\frac{\partial N_{\mathbf{qP}}}{\partial \mathbf{q}} - \frac{\partial}{\partial \mathbf{q}}\int\frac{e^2}{|\mathbf{q}-\mathbf{q}'|}N_{\mathbf{q}'\mathbf{P}'}d\mathbf{q}'d\mathbf{P}'\cdot\frac{\partial N_{\mathbf{qP}}}{\partial \mathbf{P}} - \frac{1}{2m}\frac{\partial}{\partial \mathbf{q}}\left(\mathbf{P} - \frac{e}{c}\,\mathbf{A}\right)^2\frac{\partial N_{\mathbf{qP}}}{\partial \mathbf{P}} = 0. \qquad (8)$$

In (5) → (8) we have a closed set of equations for the random functions $N_{\mathbf{qP}}$ and $\mathbf{A}$, Π, φ.

It will be convenient hereinafter to use, instead of the canonical variables $\mathbf{q}$, $\mathbf{P}$, the variables $\mathbf{q}$, $\mathbf{p} = \mathbf{P} - e\mathbf{A}/c$ and the field variables $\mathbf{A}$, $\dot{\mathbf{A}} = 4\pi c^2\Pi - c\,\mathrm{grad}\,\varphi$. Then $N_{\mathbf{qP}}d\mathbf{q}d\mathbf{P}$ becomes $N_{\mathbf{qp}}d\mathbf{q}d\mathbf{p}$, where $N_{\mathbf{qp}}(t)$ is the number of particles in a region $d\mathbf{q}d\mathbf{p}$ around a point $\mathbf{q}, \mathbf{p}$ of phase space at time t. After the transformation, Eqs. (5)–(8) become

$$\frac{\partial N_{\mathbf{qp}}}{\partial t} + \frac{\mathbf{p}}{m}\frac{\partial N_{\mathbf{qp}}}{\partial \mathbf{q}} - \left\{e\frac{\partial\varphi}{\partial \mathbf{q}} + \frac{e}{c}\dot{\mathbf{A}} - \frac{e}{mc}\,\mathbf{p}\times\mathbf{H}\right\}\frac{\partial N_{\mathbf{qp}}}{\partial \mathbf{p}} = 0; \qquad (9)$$

$$\Delta\mathbf{A} - \frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2} = \frac{1}{c}\frac{\partial}{\partial t}\,\mathrm{grad}\,\varphi - \frac{4\pi e}{cm}\int\mathbf{p}'N_{\mathbf{qp}'}d\mathbf{p}'; \qquad (10)$$

$$\Delta\varphi = -4\pi e \int N_{\mathbf{qp}}\, d\mathbf{p}, \quad \operatorname{div} \mathbf{A} = 0. \tag{11}$$

The Hamiltonian function (2) in these variables has the simple form

$$\mathscr{H} = \int \frac{p^2}{2m} N_{\mathbf{qp}}\, d\mathbf{p}\, d\mathbf{q} + e \int \varphi(\mathbf{q}) N_{\mathbf{qp}}\, d\mathbf{q}\, d\mathbf{p} - \frac{1}{8\pi} \int (\operatorname{grad} \varphi)^2 d\mathbf{q} + \frac{1}{8\pi} \int \left\{ \left(\frac{1}{c} \frac{\partial \mathbf{A}}{\partial t} \right)^2 + (\operatorname{curl} \mathbf{A})^2 \right\} d\mathbf{q}. \tag{12}$$

Equations (9) — (11) correspond in external form to the set of self-consistent equations for the particle distribution function and electric and magnetic field strengths which was considered by Vlasov[7] and a number of other authors. Unlike the set of self-consistent equations, (9) — (11) are given for the random functions $N_{\mathbf{qp}}(t)$, $\mathbf{A}$, $\dot{\mathbf{A}}$ and in this case for a total description it is also necessary to know the distribution function

$$F(\ldots N_{\mathbf{qp}}(t) \ldots, \mathbf{A} \ldots, \ldots \dot{\mathbf{A}} \ldots).$$

This procedure permits us to obtain from (9) — (11) a system of chains of equations for functions which determine average values of different combinations of $N_{\mathbf{qp}}(t)$ and $\mathbf{A}(\mathbf{q}, t)$ at different points of space and at different times. For this purpose, (9) — (11) must be averaged by means of F after they are multiplied by $N_{\mathbf{q'p'}}(t')$, $\mathbf{A}(\mathbf{q'}, t')$ or their combination. The first two equations of the chains are obtained by direct averaging of (9) — (11). Using a bar to denote averaging by means of F, we obtain

$$\frac{\partial \overline{N}_{\mathbf{qp}}}{\partial t} + \frac{\mathbf{p}}{m} \frac{\partial \overline{N}_{\mathbf{qp}}}{\partial \mathbf{q}} - \frac{\partial}{\partial \mathbf{q}} \int \frac{e^2}{|\mathbf{q} - \mathbf{q}'|} \overline{N_{\mathbf{q'p'}} d\mathbf{q}' d\mathbf{p}' \frac{\partial N_{\mathbf{qp}}}{\partial \mathbf{p}}} + \overline{\left(-\frac{e}{c} \frac{\partial \mathbf{A}}{\partial t} + \frac{e}{mc} \mathbf{p} \times \mathbf{H} \right) \frac{\partial N_{\mathbf{qp}}}{\partial \mathbf{p}}} = 0, \tag{13}$$

$$\Delta \overline{\mathbf{A}} - \frac{1}{c^2} \frac{\partial^2 \overline{\mathbf{A}}}{\partial t^2} = \frac{1}{c} \frac{\partial}{\partial t} \operatorname{grad} \int \frac{e^2}{|\mathbf{q} - \mathbf{q}'|} \overline{N}_{\mathbf{q'p'}}\, d\mathbf{q}'\, d\mathbf{p}' - \frac{4\pi e}{cm} \int \mathbf{p}' \overline{N}_{\mathbf{q'p'}}\, d\mathbf{p}'. \tag{14}$$

In Eq. (13) the scalar potential φ was eliminated by means of (11).

Averages of the products of random functions are called moments; therefore (13) and (14) are the first and second moments of $N_{\mathbf{qp}}(t)$ and $\mathbf{A}(\mathbf{q}, t)$.

Keeping in mind the relations

$$\overline{N_{\mathbf{qp}}(t)} = N f_1(\mathbf{q}, \mathbf{p}, t),$$

$$\overline{N_{\mathbf{qp}}(t) N_{\mathbf{q'p'}}(t)} = N(N-1) f_2(\mathbf{q}, \mathbf{p}, \mathbf{q}', \mathbf{p}', t) + N f_1(\mathbf{q}, \mathbf{p}, t)\, \delta(\mathbf{q} - \mathbf{q}')\, \delta(\mathbf{p} - \mathbf{p}'),$$

where f_1 and f_2 are the first and second distribution functions, it follows from (13) and (14) that self-consistent equations for the distribution function f_1 and the electric and magnetic field strengths are obtained only when correlation effects can be neglected.

All second moments can be expressed in terms of the following three second moments:

$$M^{(2)} = \overline{N_{\mathbf{qp}}(t) N_{\mathbf{q'p'}}(t')}; \quad S_\alpha^{(2)} = \overline{N_{\mathbf{qp}}(t) A_\alpha(\mathbf{q}', t')};$$
$$A_{\alpha\beta}^{(2)} = \overline{A_\alpha(\mathbf{q}, t) A_\beta(\mathbf{q}', t')}, \quad \alpha, \beta = 1, 2, 3. \tag{15}$$

The equation for $M^{(2)}$ which is obtained by using Eq. (9) for $N_{\mathbf{qp}}(t)$ is the following:

$$\frac{\partial M^{(2)}}{\partial t} + \frac{\mathbf{p}}{m} \frac{\partial M^{(2)}}{\partial \mathbf{q}} - \frac{\partial}{\partial \mathbf{q}} \int \frac{e^2}{|\mathbf{q} - \mathbf{q}'|} \frac{\partial}{\partial \mathbf{p}} M^{(3)}(\mathbf{q}, \mathbf{p}, t, \mathbf{q}', \mathbf{p}', t', \mathbf{q}'', \mathbf{p}'', t)\, d\mathbf{q}''\, d\mathbf{p}'' + \overline{\left(\frac{e}{m} \mathbf{E} + \frac{e}{cm} \mathbf{p} \times \mathbf{H} \right) N_{\mathbf{q'p'}}(t') \frac{\partial N_{\mathbf{qp}}(t)}{\partial \mathbf{p}}} = 0. \tag{16}$$

Equations for $S_\alpha^{(2)}$ and $A_{\alpha\beta}^{(2)}$ are easily obtained in a similar manner. Equation (16) and the corresponding equations for $S_\alpha^{(2)}$ and $A_{\alpha\beta}^{(2)}$ relate the second and third moments. This chain can be extended. In order to obtain an approximately closed set of equations, say for the second moments, the third moments in (16) and in the corresponding equations for $S_\alpha^{(2)}$ and $A_{\alpha\beta}^{(2)}$ must be expressed in terms of second and first moments.

2. In the present paper we shall consider the case of a system of particles with electromagnetic interaction in which a stationary, uniform, random process occurs (see the review articles by Iaglom[8] and Obukhov[9]). Here the second moments $A_{\alpha\beta}^{(2)}$, and the moments $S_\alpha^{(2)}$ and $M^{(2)}$ integrated over the momenta, depend only on the absolute values of the time difference $\tau = |t - t'|$ and the coordinate difference $r = |\mathbf{q} - \mathbf{q}'|$. In this case we also have $\overline{N_{\mathbf{qp}}(t)} = \overline{N}_{\mathbf{p}}$, i.e., the average particle distribution is unchanged in time and homogeneous in space, and $\overline{\mathbf{E}} = \overline{\mathbf{H}} = \overline{\mathbf{A}} = 0$. Deviation of the number of particles from the average value is denoted by $\delta N_{\mathbf{qp}}(t)$. Since $\overline{\mathbf{E}} = \overline{\mathbf{H}} = \overline{\mathbf{A}} = 0$ the deviations from the average values of these functions coincide with the functions themselves.

The averages of the products of deviations from average values will, as usual, be called central moments. The subsequent solution will be obtained in the approximation where it is possible to neglect third central moments in an equation for second central moments.

For the purpose of deriving equations for second central moments in the present approximation we first obtain from (9) — (11) the equations for the deviations of $\delta N_{qp}(t)$ and $\mathbf{A}$ from their mean values:

$$\frac{\partial \delta N_{\mathbf{qp}}}{\partial t}+\frac{\mathbf{p}}{m}\frac{\partial \delta N_{\mathbf{qp}}}{\partial \mathbf{q}}-\frac{\partial}{\partial \mathbf{q}}\int\frac{e^2}{|\mathbf{q}-\mathbf{q}'|}\delta N_{\mathbf{q}'\mathbf{p}'}\,d\mathbf{q}'\,d\mathbf{p}'\frac{\partial \overline{N}_{\mathbf{p}}}{\partial \mathbf{p}}-\frac{e}{mc}\frac{\partial \mathbf{A}}{\partial t}\frac{\partial \overline{N}_{\mathbf{p}}}{\partial \mathbf{p}}=0, \quad (17)$$

$$\Delta \mathbf{A}-\frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2}=\frac{1}{c}\frac{\partial}{\partial t}\operatorname{grad}\int\frac{e^2}{|\mathbf{q}-\mathbf{q}'|}\delta N_{\mathbf{q}'\mathbf{p}'}\,d\mathbf{q}'\,d\mathbf{p}'-\frac{4\pi e}{mc}\int \mathbf{p}'\,\delta N_{\mathbf{qp}'}\,d\mathbf{p}'. \quad (18)$$

In (17) and (18) terms containing products of the deviations δN_{qp} and $\mathbf{A}$ were omitted because these terms in the equations for second central moments give third central moments, which we are neglecting in the present approximation. The term $(e/mc)[\mathbf{p}\times\mathbf{H}]\partial\overline{N}_p/\partial\mathbf{p}$ vanishes because $\overline{N}_p = f(p^2)$.

Three random functions will be of the greatest importance for what follows. These are $n_q(t) = \int N_{qp}\,dp$ — the number of particles at a given point of three-dimensional space at a given time, $\mathbf{j}(\mathbf{q},t) \times (e/m)\int \mathbf{p}N_{qp}\,d\mathbf{p}$ — the current, and $\mathbf{A}(\mathbf{q},t)$ — the vector potential. These random functions form one scalar and two vector fields, all of which are stationary, uniform, and isotropic.

In virtue of the condition $\operatorname{div}\mathbf{A} = 0$ the vector field $\mathbf{A}$ is purely rotational. The vector field of the currents can also be divided into potential and rotational parts: $\mathbf{j} = \mathbf{j}^{(p)} + \mathbf{j}^{(r)}$.

According to a theorem proved by Obukhov[9] the correlation functions of the vector field can be represented as the sum of potential and rotational components. Moreover,

$$\overline{n_{\mathbf{q}}(t)\,\mathbf{j}^{(r)}(\mathbf{q}',t')}=0,$$

i.e., uniform and isotropic scalar and rotational vector fields are not correlated. Finally,

$$\overline{\mathbf{A}\mathbf{j}^{(p)}}=\overline{\mathbf{j}^{(p)}\mathbf{j}^{(r)}}=\overline{\operatorname{grad} n_{\mathbf{q}}\cdot\mathbf{j}^{(r)}}=\overline{\operatorname{grad} n_{\mathbf{q}}\cdot\mathbf{A}}=0,$$

i.e., homogeneous and isotropic rotational and potential vector fields are not correlated.

It follows that in (18) there can remain only the rotational component of the vector on the right-hand side, i.e.,

$$\Delta \mathbf{A}-\frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2}=-\frac{4\pi}{c}\mathbf{j}^{(r)}. \quad (19)$$

Equations (17) and (19) will be the basis of our further study.

Before proceding from (17) and (19) to the equations for second moments, we shall divide the former into two parts such that one part defines the potential part of the vector moments and the other defines the rotational part. We assume that the random function $\delta N_{qp}(t)$ can be divided into two parts: $\delta N_{qp} = \delta N_{qp}^{(r)} + \delta N_{qp}^{(p)}$ which satisfy the equations

$$\frac{\partial \delta N^{(p)}_{\mathbf{qp}}}{\partial t}+\frac{\mathbf{p}}{m}\frac{\partial \delta N^{(p)}_{\mathbf{qp}}}{\partial \mathbf{q}}-\frac{\partial}{\partial \mathbf{q}}\int\frac{e^2}{|\mathbf{q}-\mathbf{q}'|}\delta N^{(p)}_{\mathbf{q}'\mathbf{p}'}\,d\mathbf{q}'\,d\mathbf{p}'\frac{\partial \overline{N}_{\mathbf{p}}}{\partial \mathbf{p}}=0, \quad (20)$$

$$\frac{\partial \delta N^{(r)}_{\mathbf{qp}}}{\partial t}+\frac{\mathbf{p}}{m}\frac{\partial \delta N^{(r)}_{\mathbf{qp}}}{\partial \mathbf{q}}-\frac{e}{mc}\frac{\partial \mathbf{A}}{\partial t}\frac{\partial \overline{N}_{\mathbf{p}}}{\partial \mathbf{p}}=0,$$
$$\Delta \mathbf{A}-\frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2}=-\frac{4\pi}{c}\int \mathbf{p}\,\delta N^{(r)}_{\mathbf{qp}}\,d\mathbf{p}=-\frac{4\pi}{c}\mathbf{j}^{(r)}. \quad (21)$$

The Hamiltonian function is changed accordingly. Thus in the variables $\mathbf{q}$, $\mathbf{p}$, $\mathbf{A}$, $\dot{\mathbf{A}}$ the expression for the departure of the Hamiltonian function from its average value is

$$\delta\mathscr{H}=\int\frac{p^2}{2m}\left(\delta N^{(p)}_{\mathbf{qp}}+\delta N^{(r)}_{\mathbf{qp}}\right)d\mathbf{q}\,d\mathbf{p}$$
$$+\frac{1}{2}\int\frac{e^2}{|\mathbf{q}-\mathbf{q}'|}\delta N^{(p)}_{\mathbf{qp}}\,\delta N^{(p)}_{\mathbf{q}'\mathbf{p}'}\,d\mathbf{q}'\,d\mathbf{p}'\,d\mathbf{q}\,d\mathbf{p}$$
$$+\frac{1}{8\pi}\int\left\{\left(\frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}\right)^2+(\operatorname{curl}\mathbf{A})^2\right\}d\mathbf{q}. \quad (22)$$

The foregoing partition of the initial equations is somewhat analogous to the partition of the set of self-consistent equations for the distribution function f_1 and the electric and magnetic fields into rotational and longitudinal parts, as was done by Vlasov in Ref. 7 and elsewhere.

In the present case this partition is justified by the fact that the second moments calculated by means of the random function $\delta N_{qp}^{(p)} + \delta N_{qp}^{(r)}$ separate into the sum of uncorrelated potential and rotational components.

We have, for example,

$$\overline{\delta n_{\mathbf{q}}(t)\,\delta n_{\mathbf{q}'}(t')}=\overline{\delta n^{(p)}_{\mathbf{q}}(t)\,\delta n^{(p)}_{\mathbf{q}'}(t')}$$
$$+\overline{\delta n^{(p)}_{\mathbf{q}}(t)\,\delta n^{(p)}_{\mathbf{q}'}(t')}+\overline{2\delta n^{(p)}_{\mathbf{q}}(t)\,\delta n^{(r)}_{\mathbf{q}'}(t')}.$$

Let us consider the last term. In virtue of the stationary, uniform, and isotropic conditions we have

$$\overline{\delta n^{(p)}_{\mathbf{q}}(t)\,\delta n^{(r)}_{\mathbf{q}'}(t')}=\Pi(\tau,r).$$

Using the continuity equations

$$\frac{\partial}{\partial t}\delta n^{(p)}_{\mathbf{q}}+\operatorname{div}\mathbf{j}^{(p)}=0,\quad \operatorname{div}\mathbf{j}^{(r)}=0,\quad \frac{\partial}{\partial t}\delta n^{(r)}_{\mathbf{q}}=0,$$

we obtain $\partial\Pi(\tau,r)/\partial\tau = 0$ and thus $\Pi(\tau,r) =$

$\Pi(0, r)$. But from (22) it follows that $\Pi(0, r) = 0$, so that the term in question really vanishes.

The separation of the correlation function for currents into rotational and potential components follows directly from Obukhov's theorem. The same holds true for $\overline{j(q, t) A(q', t')}$.

Let us now consider the correlation of the scalar and vector fields:

$$\overline{\delta n_q(t) j(q', t')} = \overline{\delta n_q^{(p)} j^{(p)}} + \overline{\delta n_q^{(r)} j^{(r)}} + \overline{\delta n_q^{(p)} j^{(r)}} + \overline{\delta n_q^{(r)} j^{(p)}}.$$

The third term vanishes because the uniform and isotropic scalar and rotational vector fields are not correlated. The last term in this case can be represented by

$$\overline{\delta n_q^{(r)} j_\alpha^{(p)}(q', t')} = B(\tau, r) x_\alpha / r.$$

Using the continuity equations, we obtain

$$\operatorname{div}_{q'} \overline{\delta n_q^{(r)}(t) j_{q'}^{(p)}(t')} = 0.$$

Consequently,

$$\frac{\partial}{\partial x_\alpha}(B x_\alpha / r) = \frac{2B}{r} + B' = 0,$$

whence it follows that $B = C/r^2$, where C is a constant of integration. From the boundary condition of the correlation function at zero $C = 0$ and $B = 0$.

3. We now obtain the equations for second moments and investigate the solutions of these equations.

Equation (20) agrees with the corresponding equation in Ref. 6, where equations for the correlation functions of a centrally-interacting particle system were considered. Multiplying (20) by $\delta N_{q'p'}(t')$ and averaging, we obtain an equation for $M(r, \tau, p, p')$. The superscript "p" can be omitted here.

$$\frac{\partial M}{\partial t} + \frac{p}{m}\frac{\partial M}{\partial q} - \frac{\partial}{\partial q}\int \frac{e^2}{|q - q'|} M(q' - q'', \tau, p'', p')\, dq''\, dp' \frac{\partial \overline{N}_p}{\partial p} = 0. \quad (23)$$

Expanding M in a Fourier integral with respect to r, and using a Laplace transform with respect to τ, we have

$$M = \frac{1}{(2\pi)^4 i}\iint M_{ks}(p, p') e^{s\tau - ikr}\, ds\, dk \quad (24)$$

after which, when the solution is found by Landau's method in Ref. 10, we arrive at the following expression:

$$\int M(r, \tau, q, p')\, dp\, dp' \equiv M(\tau, r) = \frac{1}{(2\pi)^4 i}\iint \frac{\int \frac{M_k(0, p)}{s - ikp/m} dp}{1 + \frac{4\pi e^2}{k^2}\int \frac{ik\, \partial \overline{N}_p / \partial p}{s - ikp/m} dp} e^{s\tau - ikr} ds dk. \quad (25)$$

Equation (25) agrees with Tolmachev's solution.[4] The solution of (25) expresses $M(\tau, r)$ in terms of the correlation function for $\tau = 0$:

$$g(r) = M(0, r)/\bar{n}^2 - \delta(r)/\bar{n}; \quad \bar{n} = \int \overline{N}_p\, dp.$$

An expression for this function is known in some cases.

We now consider Eqs. (21) for $\delta N_{qp}^{(r)}(t)$ and **A**. Multiplying both equations by $\delta N_{q'p'}^{(r)}(t)$ and averaging, we obtain for the functions

$$M^{(r)} = \overline{\delta N_{qp}^{(r)}(t)\, \delta N_{q'p'}^{(r)}(t')}, \quad S'^{(r)} = \overline{\delta N_{q'p'}(t') A(q, t)}$$

the closed set of equations

$$\frac{\partial M^{(r)}}{\partial t} + \frac{p}{m}\frac{\partial M^{(r)}}{\partial q} - \frac{e}{mc}\frac{\partial S^{(r)}}{\partial t}\frac{\partial \overline{N}_p}{\partial p} = 0, \quad (26)$$

$$\Delta S^{(r)} - \frac{1}{c^2}\frac{\partial^2 S^{(r)}}{\partial t^2} = -\frac{4\pi}{c}\frac{e}{m}\int p' M^{(r)}\, dp'. \quad (27)$$

When in these equations we represent $M^{(r)}$ and $S^{(r)}$ by Fourier integrals with respect to the coordinates and perform a Laplace transformation with respect to time [see Eq. (24)], we obtain the following equations for the transformation amplitudes:

$$\left(s - i\frac{kp}{m}\right) M_{ks}^{(r)}(p, p') - \frac{es}{mc} S_{ks}^{(r)}(p')\frac{\partial \overline{N}_p}{\partial p} = M_k^{(r)}(0, p, p') - \frac{e}{mc} S_k^{(r)}(0, p')\frac{\partial \overline{N}_p}{\partial p}, \quad (28)$$

$$\left(\frac{s^2}{c^2} + k^2\right) S_{ks}^{(r)}(p') = \frac{4\pi e}{cm}\int p'' M_{ks}^{(r)}(p', p'')\, dp'' + \frac{s}{c^2} S_k^{(r)}(0, p') + \frac{1}{c^2}\left(\frac{\partial}{\partial t} S_k^{(r)}(\tau, p')\right)_{\tau=0}. \quad (29)$$

Eliminating $S_{ks}^{(r)}(p')$ in these equations, we obtain

$$\left(s - \frac{ikp}{m}\right) M_{ks}^{(r)}(p, p') - \frac{4\pi e^2 s}{m(s^2 + c^2k^2)}\int p'' M_{ks}^{(r)}(p', p'')\, dp'' \frac{\partial \overline{N}_p}{\partial p} = M_k(0, p, p') + \frac{e}{mc(s^2 + c^2k^2)}\left\{s\frac{\partial}{\partial t} S^{(r)}(\tau, p') - c^2k^2 S^{(r)}(\tau, p')\right\}_{\tau=0}\frac{\partial \overline{N}_p}{\partial p}. \quad (30)$$

We now divide (30) by $s - ikp/m$, multiply by pp' and integrate over p, p'. We direct the vector $\int p' M_{ks}^{(r)}(p'', p')\, dp'$ along the y axis. For the divergence of the vector we have

$$\operatorname{div}_{q'}\int p' M^{(r)}(q'', q', p'', p')\, dp' = 0,$$

since $\operatorname{div}_q j^{(r)}(q, t) = 0$. Therefore $k\int p' M_{ks}^{(r)}(p'', p')\, dp' = 0$ and, consequently, the vector k can be directed along the x axis. Then the integral of the second term in the left-hand side of (30) can be represented as

$$\frac{4\pi e^2 s}{m^2 (s^2 + c^2 k^2)\varkappa T} \int \frac{p_y \mathbf{p}' (\mathbf{p}''\mathbf{p})\, M_{\mathbf{k}s}^{(\mathbf{r})}(\mathbf{p}', \mathbf{p}'')}{s - ikp_x/m} \overline{N}_{\mathbf{p}}\, d\mathbf{p}\, d\mathbf{p}'\, d\mathbf{p}''. \quad (31)$$

In (31) the integrals which contain the products $p_x p_x''$ and $p_z p_z''$ vanish since $\overline{N}_{\mathbf{p}} = f(p^2)$. Using the notation

$$\int \mathbf{p}\mathbf{p}' M_{\mathbf{k}s}^{(\mathbf{r})}(\mathbf{p}, \mathbf{p}')\, d\mathbf{p}\, d\mathbf{p}' = \overline{(\mathbf{j}(\mathbf{q}, t)\, \mathbf{j}(\mathbf{q}', t'))}_{\mathbf{k}s},$$

we obtain

$$(\mathbf{j}^{(\mathbf{r})} \mathbf{j}^{(\mathbf{r})})_{\mathbf{k}s} = \frac{\displaystyle\int \frac{\mathbf{p}\mathbf{p}' M_{\mathbf{k}}^{(\mathbf{r})}(0, \mathbf{p}, \mathbf{p}')}{s - i\mathbf{k}\mathbf{p}/m}\, d\mathbf{p}\, d\mathbf{p}' + \frac{e/mc}{s^2 + k^2 c^2} \int (\mathbf{p}\mathbf{p}') \frac{\partial \overline{N}_{\mathbf{p}}}{\partial \mathbf{p}} \left\{ s \frac{\partial}{\partial t} \mathbf{S}_{\mathbf{k}}^{(\mathbf{r})}(\tau, \mathbf{p}') - c^2 k^2 \mathbf{S}_{\mathbf{k}}^{(\mathbf{r})}(\tau, \mathbf{p}') \right\}_{\tau=0} d\mathbf{p}\, d\mathbf{p}'}{\displaystyle 1 + \frac{4\pi e^2 s}{m^2 (s^2 + c^2 k^2)\varkappa T} \int \frac{p_y^2 \overline{N}_{\mathbf{p}}}{s - ikp_x/m}\, d\mathbf{p}}. \quad (32)$$

From (32) we obtain an expression for the current correlation function:

$$\overline{\mathbf{j}^{(\mathbf{r})}(\mathbf{q}, t)\, \mathbf{j}^{(\mathbf{r})}(\mathbf{q}', t')} = \frac{1}{(2\pi)^4 i} \iint (\mathbf{j}^{(\mathbf{r})} \mathbf{j}^{(\mathbf{r})})_{\mathbf{k}s}\, e^{s\tau - i\mathbf{k}\mathbf{r}}\, ds\, d\mathbf{k}. \quad (33)$$

This relation expresses the space-time correlation function for currents in terms of simultaneous correlation functions.

From a comparison of (25) and (33) it follows that the space-time density correlation function is largely determined by the form of the expression

$$1 + \frac{4\pi e^2}{k^2} \int \frac{i\mathbf{k}\, \partial \overline{N}_{\mathbf{p}} / \partial \mathbf{p}}{s - i\mathbf{k}\mathbf{p}/m}\, d\mathbf{p}, \quad (34)$$

which is the dielectric constant of a medium in which random longitudinal oscillations are occurring. When (34) is equated to zero we obtain the dispersion equation which was investigated by Vlasov[7] and Landau.[10]

From (33) it follows that the space-time correlation function for eddy currents is determined by

$$1 + \frac{4\pi e^2 s}{(s^2 + c^2 k^2) m^2 \varkappa T} \int \frac{p_y^2 N_{\mathbf{p}}}{s - ikp_x/m}\, d\mathbf{p}. \quad (35)$$

It will follow subsequently that (35) is associated with the dielectric constant of a medium in which random transverse oscillations are taking place.* When this expression is equated to zero we obtain the dispersion equation for transverse oscillations which was considered by Vlasov.[7]

By setting the real part of (34) equal to zero we obtain the frequency of longitudinal random oscillations. For example, for long waves the frequency and decrement σ_k of random longitudinal oscillations are given by the formulas[10]

$$\omega^2 = \omega_L^2 + 3r_d^2 k^2, \quad \sigma_k = \omega_L \sqrt{\frac{\pi}{8}} (kr_d)^{-3} e^{-k^2 r_d^2/2}$$

where r_d is the Debye radius.

The corresponding formulas for (35) when $\omega_k/k \gg \sqrt{\kappa T/m}$ are

$$\omega_k^2 = \omega_L^2 + c^2 k^2; \quad \sigma_k = \frac{\pi\omega_L^2}{2kn} (\overline{N}_{p_x})_{p_x} = \omega_k/k. \quad (36)$$

It will become clear from what follows that the magnitude of σ_k is associated with the imaginary part of the dielectric constant for transverse oscillations. Since $N_{p_x} = 0$ for $p_x > mc$ it follows from (36) that $\sigma_k = 0$ when $\omega_k/k > c$, i.e., the damping vanishes for waves whose phase velocity is greater than the speed of light. These waves cannot be excited by thermal motion and thus do no contribute to thermal fluctuations. This fact limits the wavelengths of random transverse oscillations. For example, oscillations with the frequency $\omega_k \approx \omega_L$ which contribute to thermal oscillations can have only wave numbers for which $k \geq \omega_L/c = 1/\delta$, where $\delta = (mc^2/4\pi e^2 \overline{n})^{1/2}$. These results can be obtained in a more consistent manner by using the relativistic equation for $N_{\mathbf{qp}}(t)$.

4. In this section we shall derive expressions for the space-time correlation function of the vector potentials $\mathbf{A}(\mathbf{r}, \tau) = \overline{\mathbf{A}(\mathbf{q}', t')\, \mathbf{A}(\mathbf{q}, t)}$, by means of which we can obtain correlation functions for the electric and magnetic field strengths. We shall then establish the correspondence of these results with the expressions for the correlation functions of extraneous currents and electric and magnetic fields that were obtained phenomenologically by Leontovich and Rytov.[12,1] Multiplying (21) by $\mathbf{A}(\mathbf{q}', t')$, averaging, expanding into a Fourier integral with respect to $\mathbf{r}$, and taking the Laplace transform with respect to τ, we obtain for the functions

$$\mathbf{S}_{\mathbf{k}s}(\mathbf{p}) = \int_0^\infty \int \overline{\mathbf{A}(\mathbf{q}', t')\, \delta N_{\mathbf{q}\mathbf{p}}^{(\mathbf{r})}(t)}\, e^{-s\tau + i\mathbf{k}\mathbf{r}}\, d\tau\, d\mathbf{r}, \quad A_{\mathbf{k}s},$$

the following equations

*Expressions for the dielectric constant of a plasma, based on equations involving a self-consistent field, were studied by Gertsenshtein.[11]

$$\left(s - \frac{i\mathbf{kp}}{m}\right)S_{\mathbf{k}s}(\mathbf{p}) - \frac{es}{mc}A_{\mathbf{k}s}\frac{\partial \bar{N}_{\mathbf{p}}}{\partial \mathbf{p}} = S_{\mathbf{k}}(\tau = 0, \mathbf{p}) - \frac{e}{mc}\overline{\{\mathbf{A}(\mathbf{q}', t')\dot{\mathbf{A}}_i(\mathbf{q}, t)\}}_{\mathbf{k}, \tau=0}\frac{\partial \bar{N}_{\mathbf{p}}}{\partial \mathbf{p}}, \quad (37)$$

$$\left(k^2 + \frac{s^2}{c^2}\right)A_{\mathbf{k}s} = \frac{4\pi e}{mc}\int \mathbf{p}S_{\mathbf{k}s}(\mathbf{p})\,d\mathbf{p} + \frac{s}{c^2}A_{\mathbf{k}}(\tau = 0) + \frac{1}{c^2}\overline{(\mathbf{A}(\mathbf{q}', t')\dot{\mathbf{A}}(\mathbf{q}, t))}_{\mathbf{k}, \tau=0}. \quad (38)$$

If $\mathbf{S}_{\mathbf{k}s}(\mathbf{p})$ is eliminated from these equations and we perform transformations similar to those employed in deriving (32), we obtain the following equation for $A_{\mathbf{k}s}$:

$$\left\{k^2 + \frac{s^2}{c^2}\left(1 + \frac{4\pi e^2}{sm^2\varkappa T}\int\frac{N_{\mathbf{p}}p_y^2}{s - ikp_x/m}\,d\mathbf{p}\right\}A_{\mathbf{k}s}\right.$$
$$= \frac{1}{c^2}\overline{(\mathbf{A}(\mathbf{q}', t')\dot{\mathbf{A}}(\mathbf{q}, t))}_{\mathbf{k}, \tau=0}$$
$$+ \frac{s}{c^2}\left(1 + \frac{4\pi e^2}{sm^2\varkappa T}\int\frac{N_{\mathbf{p}}p_y^2}{s - ikp_x/m}\,d\mathbf{p}\right)A_{\mathbf{k}}(\tau = 0)$$
$$+ \frac{4\pi e}{mc}\int\frac{\mathbf{p}S_{\mathbf{k}s}}{s - i\mathbf{kp}/m}\,d\mathbf{p}. \quad (39)$$

When the left-hand side of (39) is equated to the corresponding part of the equation for a medium of dielectric constant ϵ we obtain

$$\varepsilon_{\mathbf{k}s} = 1 + \frac{4\pi e^2}{sm^2\varkappa T}\int\frac{\bar{N}_{\mathbf{p}}}{s - ikp_x/m}\,d\mathbf{p}. \quad (40)$$

Equation (39) can then be written as

$$\left(k^2 + \frac{s^2\varepsilon_{\mathbf{k}s}}{c^2}\right)A_{\mathbf{k}s} = \frac{s\varepsilon_{\mathbf{k}s}}{c^2}A_{\mathbf{k}}(\tau = 0) + \frac{4\pi e}{mc}\int\frac{\mathbf{p}S_{\mathbf{k}s}(\mathbf{p})}{s - i\mathbf{kp}/m}\,d\mathbf{p}$$
$$+ \frac{1}{c^2}\overline{(\mathbf{A}(\mathbf{q}', t')\dot{\mathbf{A}}(\mathbf{q}, t))}_{\mathbf{k}, \tau=0}. \quad (41)$$

This enables us to obtain

$$A(\tau, r) = \overline{\mathbf{A}(\mathbf{q}', t')\mathbf{A}(\mathbf{q}, t)} = \frac{1}{(2\pi)^4 i}\iint A_{\mathbf{k}s}e^{s\tau - i\mathbf{kr}}\,ds\,d\mathbf{k}. \quad (42)$$

Equations (41) and (42) express in general form the space-time function of the vector potentials by means of equilibrium correlation functions (correlation functions for $\tau = 0$).

We now turn to the comparison of (41) and (42) with the corresponding expressions which were obtained phenomenologically by Rytov (see also the book of Landau and Lifshitz[13]), and ascertain under what conditions is it possible to derive the expressions given by the latter for the correlation functions of extraneous random electric and magnetic fields.

We first take it into account that the dielectric constant is a complex quantity: $\epsilon_{\mathbf{k}s} = \epsilon'_{\mathbf{k}s} + i\epsilon''_{\mathbf{k}s}$. It follows from (40) that

$$\varepsilon'_{k\omega} = 1 - \frac{\omega_L^2}{\omega_L^2 + c^2k^2}, \quad \text{for } \frac{\omega_k}{k} \gg \sqrt{\frac{\varkappa T}{m}}, \quad \varepsilon''_{k\omega} = 2\sigma_k/\omega. \quad (43)$$

$\epsilon'_{\mathbf{k}\omega}$ and $\epsilon''_{\mathbf{k}\omega}$ depend generally on both ω and k, i.e., both temporal and spatial dispersion occur. The space-time correlation function diminishes quite rapidly with increasing τ and r, so that instead of a Laplace transformation it is possible to use the one-sided Fourier transformation. The coefficients of the expansion determine the energy distribution of the thermal oscillations with respect to frequencies and wave numbers. In (41) we now replace s by $i\omega$ [as was done in deriving (43)] and assume $\mathbf{A}\dot{\mathbf{A}} = 0$, $\mathbf{p}\mathbf{S}(\tau = 0, \mathbf{r}, \mathbf{p}) = 0$. Then (41) can be written as

$$(c^2k^2 - \omega_k^2\varepsilon_{\mathbf{k}\omega})A_{\mathbf{k}\omega} = i\omega\varepsilon_{\mathbf{k}\omega}A_{\mathbf{k}}(\tau = 0). \quad (44)$$

From a comparison of (44) with the corresponding equation in Rytov's paper[1] we conclude that the right-hand side of this equation represents the coefficients in the Fourier integral for the space-time correlation function of the extraneous electric induction **D**. Separating the real part, we obtain

$$\overline{\mathbf{D}_{\mathbf{k}\omega}\mathbf{D}^*_{\mathbf{k}\omega}} = \frac{2\omega\varepsilon''_{\mathbf{k}\omega}}{c^2}A_{\mathbf{k}}(\tau = 0). \quad (45)$$

When the spatial dispersion can be neglected, (45) becomes

$$\overline{\mathbf{D}_{\mathbf{k}\omega}\mathbf{D}^*_{\mathbf{k}\omega}} = 8\pi\frac{\varepsilon^*_\omega}{\omega}\varkappa T. \quad (45')$$

We have considered that in this case $A_{\mathbf{k}}(\tau = 0) = 4\pi\kappa T/\omega^2$. From (45′) we obtain

$$\overline{\mathbf{D}_\omega(\mathbf{q})\mathbf{D}_\omega(\mathbf{q}')}/8\pi = \frac{\varepsilon''_\omega}{\omega}\varkappa T\delta(\mathbf{q} - \mathbf{q}'). \quad (46)$$

Equations (46) and (45) agree with the corresponding equations of Rytov.

From the comparison of these results we see that the microscopic approach permits us to obtain an explicit expression for the dielectric constant and to derive more general relations which are also valid when spatial dispersion is present.

By a similar comparison in a quasi-stationary approximation we obtain an expression for the space-time spectral function of extraneous currents:

$$\overline{(\mathbf{j}_{\mathbf{k}\omega}\mathbf{j}^*_{\mathbf{k}\omega})}_{\text{extr}} = 2\omega k^2\varepsilon''_{\mathbf{k}\omega}A_{\mathbf{k}}(\tau = 0)/(4\pi)^2. \quad (47)$$

When spatial dispersion can be neglected we obtain from (47)

$$\overline{(\mathbf{j}_{\omega\mathbf{k}}\mathbf{j}_{\omega\mathbf{k}})}_{\text{extr}} = 2\omega\varepsilon''_\omega\varkappa T/4\pi,$$
$$(\mathbf{j}_\omega(q)\mathbf{j}^*_\omega(\mathbf{q}'))_{\text{extr}} = 2\sigma\delta(\mathbf{q} - \mathbf{q}'). \quad (48)$$

Here $\epsilon'' = 4\pi\sigma/\omega$ and σ is the conductivity. Equation (48) agrees with the expression obtained by Leontovich and Rytov.[12]

We note once more that by the present method there is no need to introduce the concepts of "extraneous" random currents and fields, since all space-time correlation functions can be expressed in terms of simultaneous correlation functions. The correlation functions of extraneous currents and fields in the phenomenological theory are equivalent to approximate simultaneous correlation functions.

Using (9) — (11) for the random functions $\delta N_{qp}(t)$, $\mathbf{A}$, and φ, we can also obtain equations for the equilibrium correlation functions. In the approximation where third central moments are dropped, the equations for simultaneous correlation functions can be obtained from (20) and (21). Thus, for example, the equation for the correlation function $M^{(p)}(\mathbf{r}, 0, \mathbf{p}, \mathbf{p}')$ in the present approximation is obtained from (20) in the form

$$\frac{\mathbf{p}}{m}\frac{\partial M^{(\mathbf{p})}}{\partial \mathbf{q}} + \frac{\mathbf{p}'}{m}\frac{\partial M^{(\mathbf{p})}}{\partial \mathbf{q}'} - \frac{\partial}{\partial \mathbf{q}}\int \frac{e^2}{|\mathbf{q}-\mathbf{q}''|} M^{(\mathbf{p})}(\mathbf{q}''-\mathbf{q}', \mathbf{p}', \mathbf{p}'')\, d\mathbf{q}''\, d\mathbf{p}'' \frac{\partial \bar{N}_{\mathbf{p}}}{\partial \mathbf{p}} - \frac{\partial}{\partial \mathbf{q}'}\int \frac{e^2}{|\mathbf{q}'-\mathbf{q}''|} M^{(\mathbf{p})}(\mathbf{q}''-\mathbf{q}, \mathbf{p}, \mathbf{p}'')\, d\mathbf{q}''\, d\mathbf{p}'' \frac{\partial \bar{N}_{\mathbf{p}'}}{\partial \mathbf{p}'} = 0. \quad (49)$$

This equation agrees with the corresponding equation obtained in Bogoliubov's book[14] by expansion with respect to a plasma parameter. The correlation function which is obtained from the solution of (49) agrees with the Debye correlation function.

By means of (21) or the corresponding equations in the variables $\mathbf{q}$, $\mathbf{P}$, we can obtain equations for the equilibrium correlation functions of the rotational fields which have been considered. This is an independent problem and will be considered separately.

In conclusion we shall make two additional comments. Vector fields are characterized by a correlation tensor. For homogeneous and isotropic fields the correlation tensor is entirely determined by two scalar functions. We have obtained above expressions for the diagonal element sums of correlation tensors for the current and the vector potential. The diagonal element sum of a correlation tensor is expressed in terms of the two scalar functions which determine the tensor. A second equation that relates these functions is the Kármán condition for random rotational fields. These formulas thus enable us to obtain the correlation tensor.

The solution that has been given above is valid whenever third central moments can be neglected. This is a poor approximation when collisions between particles of the system play an important part. However, the applicability of the foregoing formulas can be extended considerably if collisions are taken into account by introducing into the right-hand side of (20) and of the first equation in (21) the terms

$$-\nu\delta N^{(\mathbf{p})}_{\mathbf{q}\mu}(t), \qquad -\nu\delta N^{(\mathbf{r})}_{\mathbf{q}\mathbf{p}}(t),$$

in which ν is the frequency of the collisions. In this approximation all of the formulas derived above remain valid when $s - i\mathbf{k}\mathbf{p}/m$ is replaced by $s + \nu - i\mathbf{k}\mathbf{p}/m$. For example, the expression for the dielectric constant becomes

$$\varepsilon_{\mathbf{k}s} = 1 + \frac{4\pi e^2}{sm^2\varkappa T}\int \frac{\bar{N}_{\mathbf{p}} p_y^2}{s + \nu - ikp_x/m}\, d\mathbf{p}.$$

The method which has been described can be applied to the study of a quantized system of particles with electromagnetic interaction and for a system of electrons interacting with lattice vibrations. It is thus possible, in particular, to obtain corresponding microscopic expressions for the relations of Callen and Welton.

[1] S. M. Rytov, Теория электрических флуктуаций и теплового излучения (Theory of Electrical Fluctuations and Thermal Radiation), 1953; Dokl. Akad. Nauk SSSR **110**, 371 (1956), Soviet Phys. "Doklady" **1**, 555 (1956).

[2] L. D. Landau and E. M. Lifshitz, J. Exptl. Theoret. Phys. (U.S.S.R.) **32**, 618 (1957), Soviet Phys. JETP **5**, 512 (1957).

[3] V. V. Tolmachev, Dokl. Akad. Nauk SSSR **112**, 842 (1957), Soviet Phys. "Doklady" **2**, 85 (1957).

[4] V. V. Tolmachev, Dokl. Akad. Nauk SSSR **113**, 301 (1957), Soviet Phys. "Doklady" **2**, 124 (1957).

[5] Iu. L. Klimontovich, Dokl. Akad. Nauk SSSR **96**, 43 (1954).

[6] Iu. L. Klimontovich, J. Exptl. Theoret. Phys. (U.S.S.R.) **33**, 982 (1957), Soviet Phys. JETP **6**, 753 (1958).

[7] A. A. Vlasov, J. Exptl. Theoret. Phys. (U.S.S.R.) **8**, 291 (1938); Теория многих частиц (Theory of Many Particles), Gostekhizdat, 1950.

[8] A. M. Iaglom, Usp. Mat. Nauk **7**, 3 (1952).

[9] A. M. Obukhov, Труды геофизического института (Trans. of Geophys. Inst.) **24**, 151 (1954).

[10] L. D. Landau, J. Exptl. Theoret. Phys. (U.S.S.R.) **16**, 574 (1946).

[11] M. E. Gertsenshtein, J. Exptl. Theoret. Phys. (U.S.S.R.) **22**, 303 (1952).

[12] M. A. Leontovich and S. M. Rytov, J. Exptl. Theoret. Phys. (U.S.S.R.) 23, 246 (1952).
[13] L. D. Landau and E. M. Lifshitz, Электродинамика сплошных сред (Electrodynamics of Continuous Media, GTTI, 1957.
[14] N. N. Bogoliubov, Проблемы динамической теории в статистической физике (Problems of Dynamic Theory in Statistical Physics), Gostekhizdat, 1948.

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STABILITY OF SHOCK WAVES IN RELATIVISTIC HYDRODYNAMICS

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Stability against small perturbations of the discontinuity surface is investigated for shock waves in an arbitrary medium, described by relativistic equations for an ideal fluid.*

1. INTRODUCTION

THE concept of an ideal fluid is applicable in two limiting cases of relativistic hydrodynamics: at sufficiently low temperatures, when the mean number of produced pairs is much smaller than the number of virtual particles, and in the ultra-relativistic case of super-high temperatures, when the mean number of pairs is much larger than the number of virtual particles. In fact, it follows from the equations†

$$\partial T_i^k / \partial x^k = 0, \qquad T_{ik} = w u_i u_k + p g_{ik} \tag{1.1}$$

that the entropy flux density satisfies the equation

$$\frac{\partial \sigma^i}{\partial x^i} = -\frac{\mu}{T}\frac{\partial n^i}{\partial x^i}, \quad \sigma^i = \sigma u^i, \ n^i = n u^i. \tag{1.2}$$

In the first case, which we shall call relativistic, the equation of continuity holds for the number of particles in zero approximation of the ratio of the mean number of pairs to the number of particles

$$\partial n^i / \partial x^i = 0. \tag{1.3}$$

In the ultra-relativistic limit the chemical potential is equal to zero in zeroth approximation of the ratio of virtual particles to the number of pairs:

$$\mu = 0. \tag{1.4}$$

In both limiting cases (and only then), the entropy is conserved:

$$\partial \sigma^i / \partial x^i = 0. \tag{1.5}$$

It should be noted that, as shown by Khalatnikov,[3] it is possible to obtain the equations for the ultra-relativistic case, (1.1) and (1.5) from Eqs. (1.1) and (1.3) of the relativistic case by a simple substitution:

$$w \to T\sigma, \ n \to \sigma, \tag{1.6}$$

and putting $\mu = 0$.* We shall make use of this result later.

*In classical hydrodynamics this problem was solved by D'iakov.[1]

†Our notation follows Ch. XV of the book by Landau and Lifshitz.[2]

*Thermodynamical relations necessary for the completeness of the system (with exception of the equation of state) remain valid after the substitution (1.6), in view of Eq. (1.4): if $\mu = 0$, n does not enter into the thermodynamical identities, and $w = T\sigma$. The equations for the ultra-relativistic case can therefore be obtained at any stage from the relativistic equations if one does not use the equation of state explicitly. If the boundary conditions are obtained directly from the equations, or if there is no condition imposed on n at the boundary, then the above procedure permits us to obtain the corresponding solution for the ultra-relativistic case from the solution of the boundary problem.

If we note that conditions at hydrodynamic discontinuities do not follow from equations of the ideal fluid, but represent additional physical requirements (following from the equations with dissipation), it becomes clear that the substitution (1.6) is applicable to tangential and is inapplicable to normal discontinuities, since n enters the boundary conditions for the latter.