21, 1330 (1951).

[3] V. L. Ginzburg, J. Exptl. Theoret. Phys. (U.S.S.R.) **34**, 113 (1958), Soviet Phys. JETP **7**, 78 (1958).

[4] J. A. Stratton, Electromagnetic Theory, McGraw-Hill, 1941.

Translated by D. ter Haar
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APPLICATION OF THE METHODS OF QUANTUM FIELD THEORY TO A SYSTEM OF BOSONS

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It is shown that the techniques of quantum field theory can be applied to a system of many bosons. The Dyson equation for the one-particle Green's function is derived. Properties of the condensed phase in a system of interacting bosons are investigated.

1. INTRODUCTION

In recent years Green's functions have been widely used[1] in quantum field theory, and in particular in quantum electrodynamics. This has made possible the development of methods[2] which escape from ordinary perturbation theory. The method of Green's functions has also been shown* to be applicable to many-body problems. In such problems the one-particle Green's function determines the essential characteristics of the system, the energy spectrum, the momentum distribution of particles in the ground state, etc.[3]

The present paper develops the method of Green's functions for a system consisting of a large number N of interacting bosons. The special feature of this system is the presence in the ground state of a large number of particles with momentum $p = 0$ (condensed phase), which prevent the usual methods of quantum field theory from being applied. We find that for large N the usual technique of Feynman graphs can be used for the particles with $p \neq 0$, while the condensed phase (we show that it does not disappear when interactions are introduced) can be considered as a kind of external field.

The Green's function is expressed in terms of three effective potentials Σ_{ik}, describing pair-production, pair-annihilation and scattering, and in terms of a chemical potential μ. This is the analog of Dyson's equation in electrodynamics.[4,1] Some approximation must be made in the calculation of Σ_{ik} and μ. If these quantities are computed by perturbation theory, the quasi-particle spectrum of Bogoliubov[5] is obtained. In the following paper[6] we evaluate Σ_{ik} and μ in the limit of low density.

*Private communication from A. B. Migdal.

2. STATEMENT OF THE PROBLEM. FEYNMAN GRAPHS

We consider a system of N spinless bosons with mass $m = 1$, enclosed in a volume V. We suppose N and V become infinite, the density $N/V = n$ remaining finite. A summation over discrete momenta is then replaced by an integral according to the rule

$$\sum_{p} \to (2\pi)^{-3} V \int dp.$$

The Hamiltonian of the system is $H = H_0 + H_1$, where

$$H_0 = \frac{1}{2}\int \nabla\Psi^+(x)\,\nabla\Psi(x)\,dx = \sum_p \varepsilon_p^0 a_p^+ a_p;\quad \varepsilon_p^0 = \frac{p^2}{2}, \tag{2.1}$$

$$H_1 = \frac{1}{2}\int \Psi^+(x)\,\Psi^+(x')\,U(x-x')\,\Psi(x')\,\Psi(x)\,dx\,dx' = \frac{1}{2V}\sum_{pp'q} U_q a_p^+ a_{p'}^+ a_{p'-q} a_{p+q}. \tag{2.2}$$

The units are chosen so that $\hbar = 1$. $U(\mathbf{x}-\mathbf{x}')$ is the interaction between a pair of particles, $U_{\mathbf{q}} = \int e^{-i\mathbf{q}\mathbf{x}} U(\mathbf{x})\,d\mathbf{x}$ is its Fourier transform, and

$$\Psi = V^{-1/2}\sum_{\mathbf{p}} e^{ipx} a_{\mathbf{p}}, \quad \Psi^{+} = V^{-1/2}\sum_{\mathbf{p}} e^{-ipx} a_{\mathbf{p}}^{+},$$

where $a_{\mathbf{p}}$ and $a_{\mathbf{p}}^{+}$ are the usual boson operators with the commutation law $[a_{\mathbf{p}}, a_{\mathbf{p}'}^{+}] = \delta_{\mathbf{p}\mathbf{p}'}$.

The one-particle Green's function may be defined in two equivalent ways. In terms of Heisenberg-representation operators we may write

$$iG(x-x') = \langle \Phi_0^N,\ T\{\Psi(x)\Psi^{+}(x')\}\Phi_0^N\rangle, \tag{2.3}$$

with the expectation value taken in the ground-state of the N interacting particles. In terms of interaction-representation operators we may write

$$iG(x-x') = \langle T\{\Psi(x)\Psi^{+}(x')S\}\rangle / \langle S\rangle, \tag{2.4}$$

with the expectation value taken in the ground-state of the non-interacting particles, which has all the particles in the condensed phase so that $N_{\mathbf{p}\neq 0} = 0$, $N_0 = N$. The S-matrix for this system has the form

$$S = T\left\{\exp\left(-\frac{i}{2}\int d^4x_1 d^4x_2 U\right.\right.$$
$$\left.\left.(1-2)\Psi^{+}(1)\Psi^{+}(2)\Psi(2)\Psi(1)\right)\right\}, \tag{2.5}$$

where we have written for convenience $U(1-2) = U(\mathbf{x}_1-\mathbf{x}_2)\delta(t_1-t_2)$. Here and henceforth $x, \ldots, p$ are four-vectors, and $px = \mathbf{p}\mathbf{x} - p_0x_0$. The definition (2.3) is convenient for relating G to physical quantities, while Eq. (2.4) is convenient for calculations.

In the numerator of Eq. (2.4) we expand the S-matrix in a series, each term of which is a T-product of a certain number of factors Ψ and Ψ^{+}. A T-product can be expressed by standard methods[7] as a sum of normal products in which some of the factors Ψ and Ψ^{+} have been paired. In quantum electrodynamics the vacuum expectation value of every term which contains an unpaired annihilation operator vanishes from this sum. The surviving terms, which contain only pairs of Ψ and Ψ^{+}, are represented by certain Feynman graphs. In our case the expectation value is taken in a state containing N particles with momentum $\mathbf{p} = 0$. The expectation value of an N-product containing a_0 does not vanish, and the usual method of constructing graphs is not applicable.

Because of the special role of the state with $\mathbf{p} = 0$, it is convenient to separate the operators a_0 and a_0^{+} from Ψ and Ψ^{+}. Thus we write

$$\Psi = \Psi' + a_0/\sqrt{V}; \quad \Psi^{+} = \Psi'^{+} + a_0^{+}/\sqrt{V}. \tag{2.6}$$

The Green's function (2.4) is also divided into two parts. The uncondensed particles give

$$iG'(x-x') = \langle T\{\Psi'(x)\Psi'^{+}(x')S\}\rangle / \langle S\rangle \tag{2.7}$$

while the Green's function of the condensed phase, a function of $(t-t')$ only, is

$$iG_0(t-t') = \langle T\{a_0(t)a_0^{+}(t')S\}\rangle / V\langle S\rangle. \tag{2.8}$$

The two functions are not independently determined, since the S-matrix appears in the definition of both and itself contains both Ψ' and a_0 operators. We shall prove later that when N is large the usual method of Feynman graphs can be adapted to the calculation of G', the condensed phase behaving just like an external field.

We divide the operations T and $\langle \ldots \rangle$ into two successive operations, the first acting only upon Ψ' and Ψ'^{+}, the second acting only upon a_0 and a_0^{+}. Thus

$$T = T^0T', \quad \langle \ldots \rangle = \langle\langle \ldots \rangle'\rangle^0,$$

where T^0 and $\langle \ldots \rangle^0$ act on a_0 and a_0^{+}.

We now drop the prime from G' and write Eq. (2.7) in the form

$$iG(x-x') = \langle T^0\{\mathfrak{G}(x-x')\}\rangle^0 / \langle S\rangle, \tag{2.9}$$

with

$$\mathfrak{G}(x-x') = \langle T'\{\Psi'(x)\Psi'^{+}(x')S\}\rangle'. \tag{2.10}$$

Eq. (2.10) has the same structure as the numerator of Eq. (2.7), but the operators a_0, a_0^{+} occurring in S are now to be treated as parameters. The expectation value in Eq. (2.10) is taken in the ground state of the operators Ψ', Ψ'^{+}. This equivalent to a vacuum expectation value, and so the usual formalism of Feynman graphs can be used for calculating $\mathfrak{G}$.

We represent the potential $-iU(1-2)$ by a dotted line joining the points 1 and 2. The pair of operators $\dot{\Psi}'(1)\ \dot{\Psi}'^{+}(2) = iG^{(0)}(1-2)$ is represented by a continuous line directed from 2 to 1. From the form of the interaction Hamiltonian (2.2) it follows that every graph contributing to Eq. (2.10) is a combination of the eight elementary graphs shown in Fig. 1. These correspond to the various terms which appear in Eq. (2.2) after the substitution (2.6). A missing continuous line (incomplete vertex) corresponds to a factor $(a_0/\sqrt{V})$ or $(a_0^{+}/\sqrt{V})$. Fig. 2 shows an example of one graph which appears in $\mathfrak{G}(x_1-x_2)$, corresponding to the integral

$$\mathfrak{M}_2(x_1;\ x_2) = i^2\int G^{(0)}(1-3)U$$
$$\times (3-4)\,G^{(0)}(3-5)\,G^{(0)}(4-6)\,U(5-6) \tag{2.11}$$
$$\times\, G^{(0)}(6-2)\,V^{-1}a_0^{+}(t_4)\,a_0(t_5)\,d^4x_3 d^4x_4 d^4x_5 d^4x_6.$$

Let $\mathfrak{M}(x; x')$ be any graph contributing to Eq. (2.10) and not containing disconnected parts or vacuum loops. Together with $\mathfrak{M}$ we may consider all graphs differing from $\mathfrak{M}$ by the addition of vacuum loops. The totality of such graphs gives $\mathfrak{M}$ multiplied by a factor which is just the vacuum expectation value of the S matrix, namely $\langle S\rangle'$ in this case, since we are taking matrix elements only of Ψ' and Ψ'^{+}. Thus the inclusion of vacuum loops changes $\mathfrak{M}$ into

$$\mathfrak{M}(x; x')\langle S\rangle'. \tag{2.12}$$

In quantum electrodynamics the factor $\langle S\rangle$ cancels the denominator of Eq. (2.4), so that we can ignore the vacuum loops and merely omit this denominator. In our case, as we shall see later, the factor $\langle S\rangle'$ has a real significance.

Eq. (2.12) substituted into Eq. (2.9) gives

$$\langle T^0\{\mathfrak{M}(x; x')\langle S\rangle'\}\rangle^0/\langle S\rangle, \tag{2.13}$$

where the operation T^0 acts on the factors a_0, a_0^+ occurring in $\mathfrak{M}$ and in $\langle S\rangle'$. Suppose that $\mathfrak{M}$ contains m pairs of operators a_0, a_0^+. Then

$$\mathfrak{M}(x; x') = V^{-m}\int M(x; x'; t_1 \ldots t_m; t_1' \ldots t_m')\, a_0(t_1)\ldots$$

$$\ldots a_0(t_m)\, a_0^+(t_1')\ldots a_0^+(t_m')\,(dt)\,(dt'),$$

and Eq. (2.13) becomes

$$\int MiG_0(t_1 \ldots t_m; t_1' \ldots t_m')\,(dt)\,(dt'), \tag{2.14}$$

where

$$iG_0(t_1 \ldots t_m; t_1' \ldots t_m')$$

$$= \langle T\{a_0(t_1)\ldots a_0^+(t_m')\,S\}\rangle/V^m\langle S\rangle \tag{2.15}$$

is the m-particle Green's function of the condensed phase, Eq. (2.8) being the special case m = 1.

The graphs for the Green's function (2.9) thus coincide with the graphs for $\mathfrak{G}$, only the factors $(a_0a_0^+/V)$ in the integrals are replaced by the corresponding Green's function of the condensed phase. For example, in the integral (2.11), the factor $(a_0^+(t_4)\,a_0(t_5)/V)$ is replaced by $iG_0(t_5 - t_4)$. We need not consider graphs with disconnected parts, since these are already included in G_0. The problem is therefore reduced to the determination of the Green's functions G_0 of the condensed phase.

3. THE GREEN'S FUNCTIONS OF THE CONDENSED PHASE

We write the m-particle Green's function (2.15) of the condensed phase in the form

$$iG_0(t_1 \ldots t_m; t_1' \ldots t_m')$$

$$= \frac{1}{V^m\langle S\rangle}\langle T^0\{a_0(t_1)\ldots a_0(t_m)\,a_0^+(t_1')\ldots$$

$$\ldots a_0^+(t_m')\langle S\rangle'\}\rangle^0. \tag{3.1}$$

The quantity $\langle S\rangle'$ is the sum of contributions from all vacuum loops. If λ is the sum of contributions from all connected vacuum loops, then[8] the sum of contributions from all pairs of connected loops is $(\lambda^2/2!)$, the sum of contributions from all triples is $(\lambda^3/3!)$, and so on. Therefore $\langle S\rangle' = e^{\lambda}$. In our case λ is a functional of $a_0a_0^+$, and is proportional to the volume V if we take $(a_0a_0^+/V)$ to be finite. This can be seen by considering any vacuum loop as obtained from a graph with two free ends, carrying momenta $\mathbf{p}$ and $\mathbf{p}'$, by setting $\mathbf{p} = \mathbf{p}' = 0$. The graph with free ends gives a contribution proportional to $\delta(\mathbf{p}-\mathbf{p}') \sim (2\pi)^{-3}\,V\delta_{\mathbf{pp}'}$, and so the vacuum loop becomes proportional to V. Therefore $\lambda = V\sigma$, where σ is a finite functional of $(a_0a_0^+/V)$, and

$$\langle S\rangle' = e^{V\sigma}. \tag{3.2}$$

The commutator of a_0 and a_0^+ is unity, and is small compared with their product which is of order N. At first glance it would seem that the order of factors a_0, a_0^+ was unimportant, and that the T-product in Eq. (3.1) could be omitted. But one must remember that the T^0 in Eq. (3.1) links the product $a_0 \ldots a_0^+$ with the quantity $e^{V\sigma}$, which contains all powers of the volume and hence may compensate for the smallness of the commutator of a_0 with a_0^+. Only after disentangling $a_0 \ldots a_0^+$ from the T-product may we neglect the commutators. We observe that a_0 and a_0^+ commute with H_0 given by Eq. (2.1), so these operators are independent of time in the interaction representation. The arguments of the $a_0(t)$ and $a_0^+(t')$ in Eq. (3.1) are only ordering symbols for the operation of T^0. After carrying out the T-ordering we may consider a_0 and a_0^+ as time-independent.

The disentangling of $a_0 \ldots a_0^+$ from the T-product is done by means of the following theorem. Let $B(a_0a_0^+/V)$ and $\sigma(a_0a_0^+/V)$ be any functionals of $(a_0a_0^+/V)$, which is considered as a finite quantity. The "disentangling rule"

$$T^0\{B(a_0a_0^+/V)\,e^{V\sigma}\} = B(AA^+)\,T^0\{e^{V\sigma}\}. \tag{3.3}$$

holds with an error of order $(1/V)$. The quantities A and A^+ are defined by the integral equations

$$A(t) = C(AA^+) + \int dt'\,\theta(t-t')\,\delta\sigma(AA^+)/\delta A^+(t'),$$

$$A^+(t) = C^+(AA^+) + \int dt'\,\theta(t'-t)\,\delta\sigma(AA^+)/\delta A(t'), \tag{3.4}$$

where $\theta(t-t')$ is the contribution from a factor-pair (a_0, a_0^+),

$$\theta(t-t') = \dot{a}_0(t)\,\dot{a}_0^+(t') = \begin{cases} 1 & \text{for } t > t' \\ 0 & \text{for } t < t', \end{cases} \tag{3.5}$$

and C, C^+ are time-independent functionals defined by the quadratic equations

$$C^2 + C\int \frac{\delta\sigma(AA^+)}{\delta A^+(t)}\,dt = \frac{a_0^2}{V};$$

$$C^{+2} + C^+\int \frac{\delta\sigma(AA^+)}{\delta A(t)}\,dt = \frac{a_0^{+2}}{V}. \tag{3.6}$$

A proof of this theorem is given in the Appendix.

Applying Eq. (3.3) to (3.1), we obtain

$$iG_0(t_1 \ldots; \ldots t_m) = \langle A(t_1) \ldots A^+(t'_m)\rangle^0.$$

The denominator of Eq. (3.1) cancels against $\langle T^0\langle S\rangle'\rangle^0 = \langle S\rangle$. When the expectation value of the product $(A \ldots A^+)$ is taken, we may with an error of order $(1/V)$ replace all factors a_0, a_0^+ by $\sqrt{N}$. Let K and K^+ denote the result of making this replacement in A and A^+. Then

$$iG_0(t_1 \ldots t_m;\, t'_1 \ldots t'_m)$$
$$= K(t_1) \ldots K(t_m)\,K^+(t'_1) \ldots K(t'_m), \tag{3.7}$$

holds, with K and K′ given according to Eq. (3.4) by the integral equations

$$K(t) = \bar{C} + \int dt'\theta(t-t')\,\frac{\delta\sigma(KK^+)}{\delta K^+(t')},$$

$$K^+(t) = \bar{C}^+ + \int dt'\theta(t'-t)\,\frac{\delta\sigma(KK^+)}{\delta K(t')}, \tag{3.8}$$

and with $\bar{C}$ and $\bar{C}^+$ defined by

$$\bar{C}^2 + \bar{C}\int \frac{\delta\sigma(KK^+)}{\delta K^+(t)}\,dt = \frac{N}{V};$$

$$\bar{C}^{+2} + \bar{C}^+\int \frac{\delta\sigma(KK^+)}{\delta K(t)}\,dt = \frac{N}{V}. \tag{3.9}$$

Eq. (3.7) shows that the Green's functions of the condensed phase are products of factors, each factor being a function of one time variable. The physical meaning of this result may be clarified by the following qualitative argument. For simplicity we consider the one-particle function for non-interacting particles $iG_0^{(0)}(t-t') = \langle a_0(t)\,a_0^+(t')\rangle/V$. It describes the propagation of a particle from t′ to t. If there was originally a vacuum, then this process can proceed only by creating a particle at time t′ and annihilating it at the later time t, which is represented by the factor-pairing $\mathring{a}_0\mathring{a}_0^+ = \theta$. In this case $G^{(0)}$ coincides with the factor-pairing, as is the case in electrodynamics. But if the process occurs in the presence of N particles of the same type, the created and absorbed particles may be different. In this case the propagation of a particle from t′ to t is composed of two processes, the creation of an extra particle in the condensed phase at time t′, and the absorption of one particle from the condensed phase at time t. The time sequence of these two events is immaterial, to order N^{-1}, if N is large. The processes are therefore independent. These arguments are valid also for the exact function G_0. It is also a product of two factors K(t) and $K^+(t')$, describing the two independent processes of emission and absorption of a particle at the two corresponding times.

We consider in greater detail the one-particle function of the condensed phase

$$iG_0(t-t') = K(t)\,K^+(t'). \tag{3.10}$$

The left side is a function of the difference $(t-t')$. The right side is a product of functions of t and t′. Therefore K(t) and $K^+(t')$ must be exponentials

$$K(t) = \sqrt{n_0}\,e^{-i\mu t}; \quad K^+(t) = \sqrt{n_0}\,e^{i\mu t} \tag{3.11}$$

so that

$$iG_0(t-t') = n_0 e^{-i\mu(t-t')}. \tag{3.12}$$

To understand the physical meaning of the quantities n_0 and μ, we go back to the definition (2.3) of G_0

$$iG_0(t-t') = \langle\Phi_0^N,\, T\{a_0(t)\,a_0^+(t')\}\,\Phi_0^N\rangle/V. \tag{3.13}$$

Putting $t' = t$ in Eq. (3.13) we find

$$iG_0(0) = \langle\Phi_0^N,\, a_0^+a_0\Phi_0^N\rangle/V = \bar{N}_0/V. \tag{3.14}$$

Comparing Eq. (3.14) with (3.12), we see that $n_0 = (\bar{N}_0/V)$ is the mean density of particles in the condensed phase.

Next, suppose for definiteness $t > t'$, and write Eq. (3.13) in the form

$$iG_0(t-t') = \frac{1}{V}\langle\Phi_0^N a_0(t)\,\Phi_0^{N+1}\rangle\langle\Phi_0^{N+1}a_0^+(t')\,\Phi_0^N\rangle$$
$$+ \frac{1}{V}\sum_{s\neq 0}\langle\Phi_0^N a_0(t)\,\Phi_s^{N+1}\rangle\langle\Phi_s^{N+1}a_0^+(t')\,\Phi_0^N\rangle,$$

Separating out the time dependence of the Heisenberg operators, this expression becomes

$$iG_0(t-t') = \frac{1}{V}\exp\{-i(E_0^{N+1}-E_0^N)(t-t')\}\langle\Phi_0^N a_0\Phi_0^{N+1}\rangle\langle\Phi_0^{N+1}a_0^+\Phi_0^N\rangle$$
$$+\frac{1}{V}\sum_{s\neq 0}\exp\{-i(E_s^{N+1}-E_0^N)(t-t')\}\langle\Phi_0^N a_0\Phi_s^{N+1}\rangle\langle\Phi_s^{N+1}a_0^+\Phi_0^N\rangle. \quad (3.15)$$

We compare the exact Eq. (3.15) with the approximation (3.12) which is valid as $N\to\infty$, and conclude that the second term in Eq. (3.15) must vanish as $N\to\infty$. Comparison of the time dependence of the first term in Eq. (3.15) with that of Eq. (3.12) then shows that μ is the chemical potential of the system,

$$\mu = E_0^{N+1} - E_0^N \approx \partial E_0^N/\partial N. \quad (3.16)$$

The parameters n_0 and μ which appear in K and K^+ can be calculated in principle by solving Eq. (3.8). In practice this is very difficult. The trouble is that, in calculating the vacuum loops which contribute to σ, one has first to integrate over a finite time interval $(-T, T)$, so that the parameter T appears in Eq. (3.8). One may pass to the limit $T\to\infty$ in the solutions, but not in the equations. Thus it is incorrect to use in $\sigma(KK^+)$ the limiting expressions (3.11) for K and K^+. One has instead to solve the nonlinear equations (3.8) directly.

We can obtain from Eq. (3.8) one relation between the quantities n_0 and μ. Differentiating Eq. (3.8) with respect to t, and remembering that $d\theta(t-t')/dt = \delta(t-t')$, we obtain the differential equations

$$dK/dt = \delta\sigma(KK^+)/\delta K^+(t);$$
$$dK^+/dt = -\delta\sigma(KK^+)/\delta K(t). \quad (3.17)$$

Let σ be expanded in a series

$$\sigma(KK^+) = -i\sum_{(W)}\frac{1}{m}\int W_m(t_1'\ldots t_m'; t_1\ldots t_m)K^+(t_1')\ldots$$
$$\ldots K^+(t_m')K(t_1)\ldots K(t_m)(dt)(dt'), \quad (3.18)$$

in which each term corresponds to a certain vacuum loop with m pairs of incomplete vertices, and the sum is taken over all such loops. The "vacuum amplitudes" W_m are functions of only $(2m-1)$ variables (time differences), so that the limiting values of K and K^+ would give an infinite result when substituted into Eq. (3.18). If Eq. (3.18) is varied with respect to K^+, one integration disappears, and the result becomes finite. The Fourier transform of $W_m(t'; t)$ may be written

$$W_m(\omega_1'\ldots;\ldots\omega_m)\,\delta\Big(\sum\omega - \sum\omega'\Big)$$
$$= \int W_m(t'; t)\,e^{i(\omega' t') - i(\omega t)}(dt)(dt'),$$

Then Eq. (3.11) and (3.18) give

$$\frac{\delta\sigma(KK^+)}{\delta K^+(t)}$$
$$= -i\sqrt{n_0}\sum_{(W)}n_0^{m-1}W_m(\mu\ldots;\ldots\mu)\,e^{-i\mu t}. \quad (3.19)$$

Substituting Eq. (3.19) into (3.17) and using Eq. (3.11), we obtain the desired relation

$$\mu = \sum_{(W)} n_0^{m-1}W_m(\mu\ldots\mu;\mu\ldots\mu). \quad (3.20)$$

The summation here extends over the various vacuum loops which contribute to the quantity $(\delta\sigma/\delta K^+(t))$ according to Eq. (3.19). Each such loop is to be taken with unit weight, remembering that there is one special incomplete vertex t, at which the variation with respect to K^+ was taken. Two loops are to be counted as different if they have the same geometrical structure and differ only in the position of the special vertex.

Equation (3.20) may be considered as an equation for $\mu(n_0)$. There is one free parameter in the problem, the total particle number N or the density n. Thus μ and n_0 ought to be expressible in terms of n. However n does not appear explicitly in the equation. It is thus convenient to consider n_0 instead of n as the free parameter, and to express all other quantities as functions of n_0. The connection between n_0 and n can be found after the problem is solved. From this standpoint, Eq. (3.20) completely determines K and K^+ and consequently all the Green's functions of the condensed phase. We might also solve the problem with two free parameters μ and n_0, and only consider the connection between them in the final result.[6] But this procedure would considerably increase the mathematical difficulties.

4. PROPERTIES OF THE CONDENSED PHASE

The form of the functions G_0 of the condensed phase leads to some deductions concerning the properties of the condensed phase in a system of interacting particles.

In the absence of interaction, the momentum distribution of particles in the ground state is $\delta(\mathbf{p}) = (2\pi)^{-3}V\delta_{\mathbf{p}0}$. When interactions are introduced the distribution is smeared out. In principle two possibilities are open. Either the term in $\delta(\mathbf{p})$ completely disappears and the distribution

becomes continuous (there is no condensed phase), or a term in $\delta(\mathbf{p})$ remains and the state $\mathbf{p} = 0$ is still exceptional (there is a condensed phase). In the first case all average occupation numbers $\overline{N}_{\mathbf{p}}$ are finite, and $\overline{N}_{\mathbf{p}} \to \overline{N}_0$ as $\mathbf{p} \to 0$. In the second case $\overline{N}_{\mathbf{p}\neq 0}$ is finite but $\overline{N}_0 \sim V$.

The neglect of the second term in Eq. (3.15) is equivalent to the assumption that a_0, operating on the ground state Φ_0^N, does not excite the system, or in symbols

$$a_0\Phi_0^N \approx (N_0)^{1/2}\Phi_0^{N-1}. \tag{4.1}$$

This assumption seems at first glance strange. A change in the number of particles with $\mathbf{p} = 0$, disturbing the stationary relation between the occupation numbers, must excite the system. If N_0 were finite, a change of it by one unit would change the state appreciably, but if $N_0 \sim V$ this change will practically not disturb the ground state. Equation (4.1) supports the second alternative. Therefore the introduction of interactions never causes the condensed phase to disappear entirely.

We next examine the problem of the fluctuation of the number of particles in the condensed phase. The quantity N_0 does not have an exact value in the state Φ_0^N. We expand Φ_0^N into eigenstates of the operator N_0. The expansion may be written

$$\Phi_0^N = \sum_{m=0}^{N} C_{N-m}^N \varphi_0^{N-m} \chi_m^N, \tag{4.2}$$

where $\varphi_0^{N_0}$ is a function only of the occupation number of the condensed phase, while χ_m^N depends on the other variables. χ_m^N describes a state of m particles with momenta distributed over all values $\mathbf{p} \neq 0$. It is a superposition of states with definite occupation numbers for the momenta $\mathbf{p} \neq 0$. The coefficients in this superposition depend on the upper index N. The normalization of χ_m^N is given by $\langle \chi_m^N \chi_{m'}^N \rangle \delta_{mm'}$.

Equation (4.1), with the orthogonality of $\varphi_0^{N_0}$ and $\varphi_0^{N_0'}$ for $N_0 \neq N_0'$, now gives the result

$$\langle \Phi_0^{N-1} a_0 \Phi_0^N \rangle = \sum_{m=0}^{N-1} \sqrt{N-m}\,(C_{N-m-1}^{N-1})^* C_{N-m}^N \langle \chi_m^{N-1} \chi_m^N \rangle \tag{4.3}$$

We assume that C_{N-m}^N and χ_m^N are smooth functions of N, so that

$$C_{N-m-1}^{N-1} \approx C_{N-m}^N - \partial C_{N-m}^N / \partial N = C_{N-m}^N \{1 + O(N^{-1})\};$$
$$\chi_m^{N-1} \approx \chi_m^N - \partial \chi_m^N / \partial N = \chi_m^N \{1 + O(N^{-1})\}.$$

Then Eq. (4.3) becomes

$$\langle \Phi_0^{N-1} a_0 \Phi_0^N \rangle = \sum_{m=0}^{N-1} \sqrt{N-m}\,|C_{N-m}^N|^2 \{1 + O(N^{-1})\}.$$

The sum here is simply $N_0^{1/2}$, so that

$$\langle \Phi_0^{N-1} a_0 \Phi_0^N \rangle = \overline{N_0^{1/2}} \{1 + O(N^{-1})\}. \tag{4.4}$$

A similar expression naturally holds also for $\langle \Phi_0^{N+1} a_0^+ \Phi_0^N \rangle$.

We estimate the sum in Eq. (3.15) after setting $t = t'$. Using Eq. (4.4) and (3.14), we find

$$\sum_{s\neq 0} \langle \Phi_0^N a_0 \Phi_s^{N+1} \rangle \langle \Phi_s^{N+1} a_0^+ \Phi_0^N \rangle \approx \overline{N}_0 - (\overline{N_0^{1/2}})^2, \tag{4.5}$$

from which it is clear that the sum is connected with the magnitude of the fluctuations in the number of particles in the condensed phase. From the fact that the sum is negligible as $N \to \infty$, we conclude

$$[\overline{N}_0 - (\overline{N_0^{1/2}})^2] / \overline{N}_0 \to \infty \text{ for } N \to \infty, \tag{4.6}$$

Thus the fluctuations in the number of particles in the condensed phase are relatively small.

5. GREEN'S FUNCTION FOR A PARTICLE WITH $\mathbf{p} \neq 0$

The expressions obtained in Sec. 3 for the functions of the condensed phase allow us to reformulate the rules which were described in Sec. 2 for the construction of graphs.

Every graph is a combination of eight elementary graphs (Fig. 1). Every incomplete vertex carries a factor $K(t) = \sqrt{n_0}\, e^{-i\mu t}$ corresponding to a missing incoming continuous line, or a factor $K^+(t) = \sqrt{n_0}\, e^{i\mu t}$ corresponding to a missing outgoing line. These factors mean that the interaction

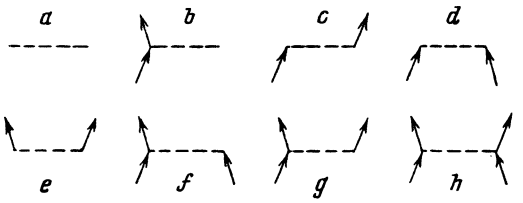

FIG. 1

involves the absorption or the emission of a particle of energy μ in the condensed phase. We may draw a wavy line corresponding to every incoming or outgoing particle of the condensed phase. All such lines have free ends. In analogy with quantum electrodynamics, we may say that the condensed phase behaves like an external field with frequency μ.

Consider the general structure of a graph which contributes to the Green's function (2.7). Every

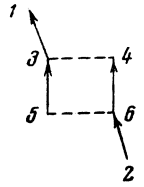

FIG. 2

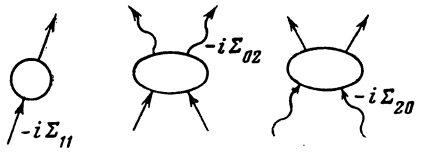

FIG. 3

graph contributing to G has the form of a chain consisting of separate irreducible parts connected to each other by only one continuous line. There are only three types of irreducible parts (i.e., parts which cannot be separated into pieces joined by only one continuous line). The three types differ in the number of outgoing and incoming continuous lines (Fig. 3). The sums of the contributions from all irreducible parts of each type we call respectively $-i\Sigma_{11}$, $-i\Sigma_{02}$, $-i\Sigma_{20}$. Σ_{11} describes processes in which the number of particles out of the condensed phase is conserved. Σ_{02} and Σ_{20} describe the absorption and emission of two particles out of the condensed phase; in these processes two particles in the condensed phase must be simultaneously emitted or absorbed, and their energy 2μ must be taken into account. In the momentum representation, $\Sigma_{11}(p_1; p_2)$ contains a factor $\delta(p_1 - p_2)$, while $\Sigma_{02}(p_1 p_2)$ and $\Sigma_{20}(p_1 p_2)$ contain* $\delta(p_1 + p_2 - 2\mu)$. Henceforth we shall assume momentum conservation in the arguments of the functions Σ_{ik}, representing the quantities which multiply the δ-functions by the notations

$$\Sigma_{11}(p;\ p) \equiv \Sigma_{11}(p);\ \Sigma_{02}(p+\mu, -p+\mu) \equiv \Sigma_{02}(p+\mu);$$
$$\Sigma_{20}(p+\mu, -p+\mu) \equiv \Sigma_{20}(p+\mu). \qquad (5.1)$$

The functions Σ_{ik} are characteristic of the particle interactions, and we may call them the effective potentials of the pair interaction.

Besides the Green's function G we introduce an auxiliary quantity $\hat{G}$, consisting of the sum of contributions from graphs with two ingoing lines. Graphs contributing to G have one ingoing and one outgoing. Figure 4 shows some of the graphs which contribute to $\hat{G}$. The quantity $\hat{G}$ describes the transition of two particles into the condensed phase. In momentum representation we write $\hat{G}(p+\mu)$ when the ingoing lines carry momenta $(p+\mu)$ and $(-p+\mu)$.

There are two equations, analogous to the Dyson equation in electrodynamics,[5,1] for the functions G and $\hat{G}$,

$$G(p+\mu) = G^{(0)}(p+\mu) + G^{(0)}(p+\mu)\Sigma_{11}(p+\mu)G(p+\mu)$$
$$+ G^{(0)}(p+\mu)\Sigma_{20}(p+\mu)\hat{G}(p+\mu),$$
$$\hat{G}(p+\mu) = G^{(0)}(-p+\mu)\Sigma_{11}(-p+\mu)\hat{G}(p+\mu)$$
$$+ G^{(0)}(-p+\mu)\Sigma_{02}(p+\mu)G(p+\mu). \qquad (5.2)$$

The structure of these equations is illustrated graphically by Fig. 5 and does not need any further explanation. Solving the system (5.2) for G and $\hat{G}$, we find

$$G(p+\mu) = (G^{(0)-1} - \Sigma_{11})^{-}\{(G^{(0)-1} - \Sigma_{11})^{+}(G^{(0)-1} - \Sigma_{11})^{-} - \Sigma_{20}\Sigma_{02}\}^{-1},$$
$$\hat{G}(p+\mu) = \Sigma_{02}\{(G^{(0)-1} - \Sigma_{11})^{+}(G^{(0)-1} - \Sigma_{11})^{-} - \Sigma_{20}\Sigma_{02}\}^{-1}, \qquad (5.3)$$

where the suffixes $\pm$ indicate the values $(\pm p + \mu)$ of the arguments. Equation (5.3) for G may be written in the usual form of a Dyson equation

$$G^{-1} = G^{(0)-1} - \Sigma,$$
$$\Sigma(p+\mu) = \Sigma_{11}(p+\mu)$$
$$+ \Sigma_{20}\Sigma_{02}/[G^{(0)-1}(-p+\mu) - \Sigma_{11}(-p+\mu)]. \qquad (5.4)$$

Into Eq. (5.3) we substitute the explicit form of the free-particle Green's function,

$$G^{(0)-1}(p) = p^0 - \varepsilon_p^0 + i\delta; \qquad (\varepsilon_p^0 = p^2/2;\ \delta \to +0), \qquad (5.5)$$

and obtain for G and $\hat{G}$ the expressions

$$G(p+\mu) = \frac{p^0 + \varepsilon_p^0 + \Sigma_{11}^{-} - \mu}{[p^0 - (\Sigma_{11}^{+} - \Sigma_{11}^{-})/2]^2 - [\varepsilon_p^0 + (\Sigma_{11}^{+} + \Sigma_{11}^{-})/2 - \mu]^2 + \Sigma_{20}\Sigma_{02}},$$
$$(5.6)$$

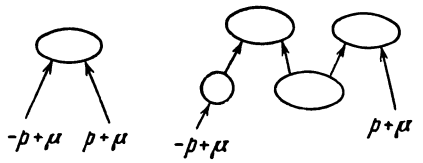

FIG. 4

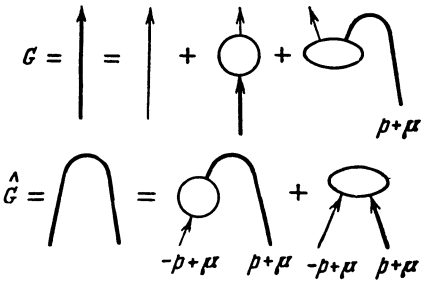

FIG. 5

*Here μ represents a 4-vector having only its fourth component non-zero.

$$\hat{G}(p+\mu) = \frac{-\Sigma_{02}}{[p^0-(\Sigma_{11}^+-\Sigma_{11}^-)/2]^2-[\varepsilon_p^0+(\Sigma_{11}^++\Sigma_{11}^-)/2-\mu]^2+\Sigma_{20}\Sigma_{02}}. \quad (5.7)$$

Equation (5.6) determines the Green's function in terms of the effective potentials Σ_{ik} and the chemical potential μ of the system. Equations for Σ_{ik} and μ cannot be obtained in so general a form. To calculate these quantities we have to use approximate methods to sum over series of graphs. In Sec. 7 we shall calculate Σ_{ik} and μ by perturbation theory. In the following paper[6] we develop an approximation in which the density is considered as a small parameter.

6. CONNECTION BETWEEN THE GREEN'S FUNCTION AND PROPERTIES OF THE SYSTEM

The energy E_0 of the ground state is the expectation value of the Hamiltonian (2.1), (2.2) in the state Φ_0^N,

$$E_0 = \langle \Phi_0^N H \Phi_0^N \rangle = \sum_p \varepsilon_p^0 \langle a_p^+ a_p \rangle + \frac{1}{2V} \sum_{pp'q} U_q \langle a_p^+ a_{p'}^+ a_{p'-q} a_{p+q} \rangle. \quad (6.1)$$

The last term in Eq. (6.1) is connected with the Green's function G. Consider G in the (p, t) representation, i.e., in the momentum representation for the space-components only. Taking the expectation value in the state Φ_0^N, Eq. (2.3) becomes

$$iG(\mathbf{p};\ t-t') = \langle T\{a_p(t)\, a_p^+(t')\}\rangle, \qquad (\mathbf{p}\neq 0), \quad (6.2)$$

from which it is easy to deduce

$$(i\partial/\partial t - \varepsilon_p^0)\, G(\mathbf{p};\ t-t') = \delta(t-t') + R(\mathbf{p};\ t-t'), \quad (6.3)$$

with

$$R(\mathbf{p};\ t-t') = -\frac{i}{V}\sum_{p'q} U_q \langle T\{a_{p'}^+(t)\, a_{p'-q}(t)\, a_{p+q}(t) a_p^+(t')\}\rangle. \quad (6.4)$$

We multiply Eq. (6.3) by $e^{ip^0(t-t')}$ and integrate with respect to t. Then using Eq. (5.5) we obtain

$$G^{(0)-1}(p)\, G(p) = 1 + R(p),$$

This, with the definition (5.4) of Σ, gives immediately

$$R(p) = \Sigma(p)\, G(p), \qquad (\mathbf{p}\neq 0). \quad (6.5)$$

On the other hand, Eq. (6.4) shows that $R(\mathbf{p}; -0)$ is related to the last sum in Eq. (6.1), namely

$$\frac{1}{2V}\sum_{pp'q} U_q \langle a_p^+ a_{p'}^+ a_{p'-q} a_{p+q}\rangle = \frac{i}{2}\sum_p R(\mathbf{p};\ -0). \quad (6.6)$$

The expression (6.5) for R holds only when $\mathbf{p} \neq 0$. For $\mathbf{p} = 0$ the left side of Eq. (6.2) is $iVG_0(t-t')$ according to Eq. (3.13). Instead of Eq. (6.3) we have in this case

$$i\frac{\partial}{\partial t} G_0(t-t') = \frac{1}{V}\delta(t-t') + \frac{1}{V} R(0;\ t-t'). \quad (6.7)$$

We neglect the δ-function since it is of order V^{-1}, and use Eq. (3.12) for G_0. This gives

$$R(0;\ -0) = i\left[\frac{\partial}{\partial\tau} G_0(\tau)\right]_{\tau=0} = -i\mu n_0. \quad (6.8)$$

Equations (6.6) and (6.8) bring the expression (6.1) for E_0 into the form

$$E_0 = \sum_p \varepsilon_p^0 \langle a_p^+ a_p\rangle + \frac{i}{2}\sum_{p\neq 0} R(\mathbf{p};\ -0) + \frac{1}{2}\mu n_0. \quad (6.9)$$

From Eq. (6.2) we find

$$\overline{N}_p = \langle a_p^+ a_p\rangle = iG(\mathbf{p};\ -0) = i\int G(p)\, dp^0/2\pi \quad (6.10)$$

Using Eq. (6.5) and (6.10), and passing from summation to integration in Eq. (6.9), we obtain the following expression* for the ground-state energy E_0,

$$E_0/V = i\int [\varepsilon_p^0 + {}^1/_2\Sigma(p)]\, G(p)\, d^4p/(2\pi)^4 + \mu n_0/2 \quad (6.11)$$

The p_0-integration is to be taken with a small detour into the upper half-plane.

The quantities Σ and G depend parametrically upon μ and n_0, supposing that Eq. (3.20) has not been used in order to eliminate one of these parameters. Therefore Eq. (6.11) gives a relation between (E_0/V), μ, and n_0. There are two further relations between these quantities. First there is the definition of the chemical potential μ,

$$\mu = \frac{\partial E_0}{\partial N} = \frac{\partial}{\partial n}\left(\frac{E_0}{V}\right), \quad (6.12)$$

and second there is the condition that the total number of particles is conserved, which by Eq. (6.10) can be written in the form

$$n = n_0 + i\int G(p)\, d^4p/(2\pi)^4. \quad (6.13)$$

Equations (6.11), (6.12), and (6.13) determine (E_0/V), μ, and n_0 in terms of the density n, or determine any three of these quantities in terms of the fourth. The relation (3.20) which we found earlier does not give any new information; it is

*V. M. Galitskii informed me that a similar relation exists for Fermi systems.

satisfied identically when Eq. (6.11), (6.12), and (6.13) hold.

7. PERTURBATION THEORY APPROXIMATION TO Σ_{ik} AND μ

In the first order of perturbation theory, the graphs which contribute to Σ_{ik} are the elementary graphs shown in Fig. 1. Graphs b and c refer to Σ_{11}, d to Σ_{02} and e to Σ_{20}. These graphs give the contributions

$$\Sigma_{02} = \Sigma_{20} = n_0 U_p; \qquad \Sigma_{11}^{+} = n_0 (U_0 + U_p). \tag{7.1}$$

In first approximation the only vacuum loop is the elementary graph a of Fig. 1. Thus Eq. (3.20) gives for μ the value

$$\mu = n_0 U_0. \tag{7.2}$$

Inserting Eq. (7.1) and (7.2) into the expression (5.6) for the Green's function, we find

$$G(p + \mu) = p^0 + \varepsilon_p^0 + n_0 U_p / (p^{0\,2} - \varepsilon_p^{0\,2} - 2 n_0 U_p \varepsilon_p^0 + i\delta). \tag{7.3}$$

The Green's function $G(p + \mu)$ has a pole at a value $p_0(\mathbf{p})$ which defines the energy of an elementary excitation of the system[3] (quasi-particle). Equation (7.3) gives for the quasi-particle energy

$$\varepsilon_p = \sqrt{\varepsilon_p^{0\,2} + 2 n_0 U_p \varepsilon_p^0}. \tag{7.4}$$

Substituting Eq. (7.3) into (6.10), we obtain the mean occupation number of the ground state,

$$\begin{aligned} \overline{N}_p &= (-\varepsilon_p + \varepsilon_p^0 + n_0 U_p) / 2\varepsilon_p \\ &= (n_0 U_p)^2 / 2\varepsilon_p (\varepsilon_p + \varepsilon_p^0 + n_0 U_p). \end{aligned} \tag{7.5}$$

Equations (7.4) and (7.5) coincide with the results of the well-known work of Bogoliubov.[5]

APPENDIX. PROOF OF THEOREM (3.3)

We use the method of Wick[9] to transform the T-product in Eq. (3.3), which may be written symbolically

$$T\{Be^{V\sigma}\} = N\{e^{\Delta} B e^{V\sigma}\}, \tag{A.1}$$

where Δ is an operator which changes a pair $a_0 a_0^{+}$ into its replacement (3.5),

$$\begin{aligned} \Delta &= \frac{1}{V} \int dt dt' \theta(t - t') \frac{\delta^2}{\delta\alpha(t)\, \delta\alpha^{+}(t')}; \\ (\alpha &= a_0 / \sqrt{V}; \quad \alpha^{+} = a_0^{+} / \sqrt{V}). \end{aligned} \tag{A.2}$$

The proof of the theorem proceeds in two stages: (1) pulling B out across the operator e^{Δ}, and (2) a final disentangling of the N-product.

(1) The result of the first stage can be formulated as follows. With an error of order V^{-1} we have

$$e^{\Delta}\{B(\alpha;\ \alpha^{+})\, e^{V\sigma}\} = B(\beta;\ \beta^{+})\, e^{\Delta}\{e^{V\sigma}\}, \tag{A.3}$$

where β and β^{+} are defined by the equations

$$\begin{aligned} \beta(t) &= \alpha + \int dt' \theta(t - t')\, \delta\sigma\, (\beta\beta^{+}) / \delta\beta^{+}(t'); \\ \beta^{+}(t) &= \alpha^{+} + \int dt' \theta(t' - t)\, \delta\sigma\, (\beta\beta^{+}) / \delta\beta(t'). \end{aligned} \tag{A.4}$$

Proof. The factor V^{-1} in Δ can be compensated on the left side of Eq. (A.3) only if $e^{V\sigma}$ is involved in at least one operation of Δ. We write $\Delta = \Delta_{\sigma\sigma} + \Delta_{B\sigma} + \Delta_{\sigma B}$, where the first suffix indicates the object upon which the variation with respect to α operates, and the second suffix refers to the variation with respect to α^{+}. The result of operating with $e^{\Delta\sigma\sigma}$ can be written

$$e^{\Delta_{\sigma\sigma}}\{e^{V\sigma}\} = e^{V\sigma'} \tag{A.5}$$

It will be shown later that σ' is independent of V. Equation (A.5) gives

$$e^{\Delta}\{Be^{V\sigma}\} = \exp(\Delta_{B\sigma} + \Delta_{\sigma B})\{Be^{V\sigma'}\}. \tag{A.6}$$

We let $e^{\Delta B\sigma}$ operate first on $e^{V\sigma'}$. From Eq. (A.2) we obtain

$$\Delta_{B\sigma} e^{V\sigma'} = e^{V\sigma'} \int dt dt' \theta(t - t') \frac{\delta\sigma'}{\delta\alpha^{+}(t')} \left[\frac{\delta}{\delta\alpha(t)}\right]_B \equiv e^{V\sigma'} D_B. \tag{A.7}$$

Successive application of Eq. (A.7) gives $(\Delta_{B\sigma})^k \times e^{V\sigma'} = e^{V\sigma'} (D_B)^k$, since the operators $\Delta_{B\sigma}$ produce variations only in the exponential. Therefore

$$e^{\Delta_{B\sigma}} e^{V\sigma'} = e^{V\sigma'} e^{D_B}. \tag{A.8}$$

From (A.7) it is clear that e^{D_B} is a displacement operator, displacing $\alpha(t)$ by the quantity

$$\alpha_1(t) = \int dt' \theta(t - t')\, \delta\sigma' / \delta\alpha^{+}(t'),$$

Therefore

$$\begin{aligned} e^{\Delta_{B\sigma}}\{B(\alpha;\ \alpha^{+})\, e^{V\sigma'}\} &= e^{V\sigma'} e^{D_B} B(\alpha;\ \alpha^{+}) \\ &= e^{V\sigma'} B(\alpha + \alpha_1;\ \alpha^{+}). \end{aligned} \tag{A.9}$$

An analogous result holds for the operator $e^{\Delta\sigma B}$, which displaces $\alpha^{+}(t)$. We obtain finally from Eq. (A.6)

$$e^{\Delta}\{B(\alpha;\ \alpha^{+})\, e^{V\sigma}\} = B(\beta;\ \beta^{+})\, e^{V\sigma'} = B(\beta;\ \beta^{+})\, e^{\Delta}\{e^{V\sigma}\}, \tag{A.10}$$

where

$$\begin{aligned} \beta(t) &= \alpha + \int dt' \theta(t - t')\, \delta\sigma' / \delta\alpha^{+}(t'); \quad \beta^{+}(t) \\ &= \alpha^{+} + \int dt' \theta(t' - t) \delta\sigma' / \delta\alpha(t'); \end{aligned} \tag{A.11}$$

Equation (A.10) is identical with (A.3). It remains to show that Eq. (A.11) and (A.4) are identical. Varying both sides of Eq. (A.5) with respect to

$\alpha(t)$ and $\alpha^+(t)$, and using Eq. (A.10), we obtain

$$\delta\sigma'(\alpha\alpha^+)/\delta\alpha(t) = \delta\sigma(\beta\beta^+)/\delta\beta(t);$$
$$\delta\sigma'(\alpha\alpha^+)/\delta\alpha^+(t) = \delta\sigma(\beta\beta^+)/\delta\beta^+(t); \quad \text{(A.12)}$$

Equations (A.12) and (A.11) define σ'. When Eq. (A.12) is substituted into (A.11), the result is Eq. (A.4).

(2) In the second stage of the proof we may consider α and α^+ to be constant operators (see the beginning of Sec. 3). Then σ' and $B(\beta\beta^+) = B'(\alpha\alpha^+)$ are functions of α, α^+ instead of functionals. By integrating Eq. (A.12) with respect to time we obtain the connection between the functions $\sigma'(\alpha\alpha^+)$ and σ,

$$\int \frac{\delta\sigma(\beta\beta^+)}{\delta\beta(t)}\,dt = \int \frac{\delta\sigma'(\alpha\alpha^+)}{\delta\alpha(t)}\,dt = \frac{\partial\sigma'}{\partial\alpha}; \quad \int \frac{\delta\sigma(\beta\beta^+)}{\delta\beta^+(t)}\,dt = \frac{\partial\sigma'}{\partial\alpha^+}. \quad \text{(A.13)}$$

For the following argument it is important that σ' and β' depend only on $\nu = \alpha^+\alpha$. This being so, the disentangling proceeds according to the rule

$$N\{B'(\nu)\,e^{V\sigma'(\nu)}\} = B'(\bar\nu)\,N\{e^{V\sigma'}\}, \quad \text{(A.14)}$$

with $\bar\nu$ obtained from ν by the relation

$$\nu = \bar\nu[1 + \partial\sigma'(\bar\nu)/\partial\bar\nu] \equiv \bar\nu X^2 \quad \text{(A.15)}$$

We defer the proof of Eq. (A.14), and show first that Eq. (A.3) and (A.14) imply the truth of the theorem (3.3). After the infinite factor $e^{V\sigma'}$ is removed, the lack of commutativity of α and α^+ can be neglected. The substitution $\nu \to \bar\nu$ which appears in Eq. (A.14), can therefore be divided into the two substitutions $\alpha \to \bar\alpha$, $\alpha^+ \to \bar\alpha^+$, where $\alpha = \bar\alpha X$ and $\alpha^+ = \bar\alpha^+ X$ according to Eq. (A.15). After some algebra we find

$$\alpha^2 = \bar\alpha^2 + \bar\alpha\,\partial\sigma'(\bar\nu)/\partial\bar\alpha^+; \quad \alpha^{+2} = \bar\alpha^{+2} + \bar\alpha^+\partial\sigma'(\bar\nu)/\partial\bar\alpha. \quad \text{(A.16)}$$

We denote by A, A^+ the quantities into which β, β^+ are transformed under the substitution α, $\alpha^+ \to \bar\alpha$, $\bar\alpha^+$. The equations for A and A^+ are obtained from Eq. (A.4) by changing the terms outside the integrals into $\bar\alpha$ and $\bar\alpha^+$, thus

$$A(t) = \bar\alpha + \int dt'\theta(t-t')\,\delta\sigma(AA^+)/\delta A^+(t');$$
$$A^+(t) = \bar\alpha^+ + \int dt'\theta(t'-t)\,\delta\sigma(AA^+)/\delta A(t'), \quad \text{(A.17)}$$

By the definition of $B'(\nu)$, $B'(\bar\nu) = B(AA^+)$, and so Eq. (A.3) and (A.14) imply (3.3). To complete the proof of the theorem it remains to show the equivalence of Eq. (A.17) and (3.4). To do this, we substitute α, $\alpha^+ \to \bar\alpha$, $\bar\alpha^+$ in Eq. (A.13), and find the result

$$\frac{\partial\sigma'(\bar\nu)}{\partial\bar\alpha} = \int \frac{\delta\sigma(AA^+)}{\delta A(t)}\,dt; \quad \frac{\partial\sigma'(\bar\nu)}{\partial\bar\alpha^+} = \int \frac{\delta\sigma(AA^+)}{\delta A^+(t)}\,dt. \quad \text{(A.18)}$$

Equation (A.16) for $\bar\alpha$ and $\bar\alpha^+$ are identical with Eq. (3.6) by virtue of Eq. (A.18). Therefore $\bar\alpha$, $\bar\alpha^+$ are identical with C, C^+, and the theorem is proved.

We now return to the proof of Eq. (A.14). Let L be a quantity related to σ' by the equation

$$N\{e^{V\sigma'}\} = e^{VL}. \quad \text{(A.19)}$$

We shall later express L explicitly in terms of σ' and shall verify that L is independent of V. Suppose that $B'(\nu)$ has the form

$$B'(\nu) = \sum_k b_k \nu^k, \quad \text{(A.20)}$$

Then Eq. (A.19) implies

$$N\{B'e^{V\sigma'}\} = \sum_k b_k \alpha^{+k} N\{e^{V\sigma'}\}\alpha^k = \sum_k b_k \alpha^{+k} e^{VL}\alpha^k. \quad \text{(A.21)}$$

The commutation relation $\alpha y(\nu) = (y + \frac{1}{V}\frac{\partial y}{\partial\nu})\alpha$ holds for any function $y(\nu)$. Applying it repeatedly, we find

$$\alpha^k e^y = \exp\left\{\left(1 + \frac{1}{V}\frac{\partial}{\partial\nu}\right)^k y\right\}\alpha^k. \quad \text{(A.22)}$$

We choose y to satisfy $(1 + \frac{1}{V}\frac{\partial}{\partial\nu})^k y = VL$. Then Eq. (A.22) gives the rule for pulling α^k through e^{VL},

$$\exp\{VL\}\alpha^k = \alpha^k \exp\left\{\left(1 + \frac{1}{V}\frac{\partial}{\partial\nu}\right)^{-k} VL\right\}$$
$$\approx \alpha^k \exp\left(-k\frac{\partial L}{\partial\nu}\right)\exp\{VL\}. \quad \text{(A.23)}$$

Applying Eq. (A.23) to Eq. (A.21), we obtain

$$N\{B'e^{V\sigma'}\} = \sum_k b_k (\nu e^{-\partial L/\partial\nu})^k e^{VL} = B'(\bar\nu)\,e^{VL}, \quad \text{(A.24)}$$

with

$$\bar\nu = \nu e^{-\partial L/\partial\nu}. \quad \text{(A.25)}$$

L can be determined by differentiating both sides of Eq. (A.19) with respect to α. On the left side we find

$$\frac{1}{V}\frac{\partial}{\partial\alpha}N\{e^{V\sigma'}\} = N\left\{\frac{\partial\sigma'}{\partial\alpha}e^{V\sigma'}\right\} = \alpha^+ N\left\{\frac{\partial\sigma'}{\partial\nu}e^{V\sigma'}\right\}$$

which with Eq. (A.24) and (A.19) gives

$$\frac{1}{V}\frac{\partial}{\partial\alpha}N\{e^{V\sigma'}\} = \alpha^+\frac{\partial\sigma'(\bar\nu)}{\partial\bar\nu}e^{VL}. \quad \text{(A.26)}$$

In differentiating the right side of Eq. (A.19) with respect to α, we must remember that $(\partial L/\partial\alpha)$ and L do not commute. Since $(\partial L/\partial\alpha) = \alpha^+(\partial L/\partial\nu)$, the commutation rule $L\alpha^+ = \alpha^+(L + \frac{1}{V}\frac{\partial L}{\partial\nu})$ implies

$$\frac{1}{V}\frac{\partial}{\partial\alpha}L^k = \alpha^+\left\{\left(L + \frac{1}{V}\frac{\partial L}{\partial\nu}\right)^k - L^k\right\}, \quad \text{(A.27)}$$

and hence

$$\frac{1}{V}\frac{\partial}{\partial\alpha}e^{VL}=\alpha^{+}(e^{\partial L/\partial\nu}-1)e^{VL}. \qquad (A.28)$$

By comparing Eq. (A.28) with (A.26), we obtain the desired relation between L and σ',

$$\partial\sigma'(\bar{\nu})/\partial\bar{\nu}=e^{\partial L/\partial\nu}-1. \qquad (A.29)$$

Equations (A.29) and (A.25) imply Eq. (A.15), and by Eq. (A.24) this proves Eq. (A.14).

[1] V. B. Berestetskii and A. D. Galanin, Проблемы современной физики (Problems of Modern Physics), No. 3 (1955), (introductory paper).

[2] Abrikosov, Landau, and Khalatnikov, Dokl. Akad. Akad. Nauk SSSR **95**, 497, 773 and 1177; **96**, 261 (1954).

[3] V. M. Galitskii and A. B. Migdal, J. Exptl. Theoret. Phys. (U.S.S.R.) **34**, 139 (1958); Soviet Phys. JETP **7**, 96 (1958).

[4] F. J. Dyson, Phys. Rev. **75**, 1736 (1949).

[5] N. N. Bogoliubov, Izv. Akad. Nauk SSSR, ser. fiz. **11**, 77 (1947).

[6] S. T. Beliaev, J. Exptl. Theoret. Phys. (U.S.S.R.) **34**, 433 (1958); Soviet Phys. JETP **7**, 289 (1958) (this issue).

[7] G. C. Wick, Phys. Rev. **80**, 268 (1950).

[8] R. P. Feynman, Phys. Rev. **76**, 749 (1949).

[9] S. Hori, Prog. Theoret. Phys. **7**, 578 (1952).

Translated by F. J. Dyson
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ENERGY-SPECTRUM OF A NON-IDEAL BOSE GAS

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The one-particle Green's function is calculated in a low-density approximation for a system of interacting bosons. The energy spectrum of states near to the ground state (quasi-particle spectrum is derived.

1. INTRODUCTION

IN the preceding paper[1] the method of Green's functions was developed for a system consisting of a large number of bosons. The one-particle Green's function was expressed in terms of the effective potentials Σ_{ik} of pair interactions and the chemical potential μ of the system. Approximate methods must be used to determine Σ_{ik} and μ. In the present paper we study a "gaseous" approximation, in which the density n, or the ratio between the volume occupied by particles and the total volume, is treated as a small parameter. The interaction between particles is assumed to be central and short-range, but not necessarily weak. The first two orders of approximation involve only the scattering amplitude f of a two-particle system. But in the next order (proportional to $(\sqrt{nf^3})^2$) the effects of three-particle interaction amplitudes appear, which means that practical calculations to this order are hardly possible.

From the Green's function which we calculate, we derive the energy spectrum of excitations or quasi-particles, the energy of the ground state, and also the momentum distribution of particles in the ground state.

2. ESTIMATE OF THE GRAPHS CONTRIBUTING TO THE EFFECTIVE POTENTIALS

The definition of the potentials Σ_{ik}, and the rules for constructing Feynman graphs, were described in our earlier paper,[1] which we shall call I.

We shall estimate by perturbation theory the various graphs contributing to Σ_{ik} and μ. For the Fourier transform of the potential $U(\mathbf{p}) = U_{\mathbf{p}}$,