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ENERGY SPECTRUM OF ELECTRONS IN OPEN PERIODIC TRAJECTORIES

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In Ref. 1 we have given the equation of motion for an electron which obeys the dispersion relation $E(\mathbf{k}) = \Sigma A_n e^{i\mathbf{kn}}$ in a magnetic field ($H = H_z$):

$$\sum_{\mathbf{n}} A_{\mathbf{n}} \exp\left\{i\,(k_2 n_2 + k_3 n_3) - i\frac{n_1 n_2}{2\alpha_0^2}\right\} f\left(k_2 - \frac{n_1}{\alpha_0^2}\right) = Ef(k_2). \quad (1)$$

It is assumed that $\epsilon = \sqrt{a_1 a_2}/\alpha_0 \ll 1$ ($\alpha_0^2 = \hbar \times c/eH$; the a_i are the lattice constants). In Ref. 2 solutions of this equation were investigated for open periodic trajectories. In the present note we verify the assertion made in Ref. 2 concerning the exponential smallness of the breaks in the continuous energy spectrum in the case in which the function $\kappa_1 = \kappa_1(k_2)$ (the equation of the trajectory, i.e., the intersections of the surface $E(\mathbf{k})$ = const. with the plane k_3 = const.) is analytic.

It will be assumed that κ_1 is large enough everywhere (the other cases are considered in Ref. 2) so that the quasi-classical approximation can be used, that is, we write

$$f(k_2) = \exp\left\{\frac{i}{\varepsilon^2}\varphi_1 + \varphi_2 + \frac{\varepsilon^2}{i}\varphi_3 + \left(\frac{\varepsilon^2}{i}\right)^2\varphi_4 + \cdots\right\}. \quad (2)$$

Solution of this equation is much more difficult than solution of the Schrödinger equation because Eq. (1) is a difference rather than a differential equation. However, the general properties of the solution are the same in both cases. The following expressions are obtained in the first four approximations:

$$\varphi_1 = -\int \varkappa_1 dx; \quad \varphi_2 = -{}^1/_2 \ln P;$$

$$\varphi_3 = \int\left\{-\left(\frac{P'Q}{P}\right)'\frac{1}{4P} + \frac{QP'^2}{8P^3} + \frac{Q''}{8P} + \frac{R\varkappa_1''}{24P}\right\}dx;$$

$$\varphi_4 = \frac{\varphi_3' Q}{2P} - \frac{1}{24P}\left\{6R(\varphi_2'^2 + \varphi_2'') + 3R'\varphi_2' + 3RP'\varphi_2' P^{-1} - 3R\varphi_2'' + \frac{1}{2}R'' + \frac{1}{2}\frac{R'P'}{P} + \frac{RP'^2}{P^2} - \frac{RP''}{2P}\right\}. \quad (3)$$

Here we have introduced the notation

$$P = \partial E/\partial\varkappa_1, \quad Q = \partial^2 E/\partial\varkappa_1^2, \quad R = \partial^3 E/\partial\varkappa_1^3, \quad (4)$$

where κ_1 and k_2 are measured in dimensionless units (by κ_1 we are to understand $\kappa_1 a_1$ and by x we are to understand $k_2 a_2$).

Just as in the Schrödinger equation, the quantities φ'_{2n} are total derivatives and since P, Q, R, are periodic in k_2, this same property is characteristic of the even approximations $\varphi_2, \varphi_4, \ldots$ Consequently, if Eq. (2) is written in the form

$$f(k_2) = \rho e^{i\varphi}, \quad (5)$$

the modulus ρ will be a periodic function of k_2 (while the phase φ is an integral of a periodic function and does not change sign).

When displaced by one period, (5) should be multiplied by e^{ip}, where e^{ip} is ± 1 at the boundary of the allowed energy intervals. This locates the discontinuity at once. Keeping the first two approximations φ_1 and φ_2 (corresponding to the usual quasi-classical analysis), we have:

$$f(k_2) = P^{-1/2}\exp\left\{-i\varepsilon^{-2}\int_0^x \varkappa_1 dx\right\}.$$

The condition $e^{ip} = \pm 1 = e^{i\pi n}$ obviously means:

$$S = 2\pi\alpha_0^{-2} n \quad (6)$$

[S is the area bounded by the curve $\kappa_1(k_2)$ in one cell]. Thus the center of the allowed interval is determined from the same relation that applies for the discrete levels in the case of closed trajectories:

$$S = 2\pi\alpha_0^{-2}(n + {}^1/_2). \quad (7)$$

The width of the discontinuities can be determined from the usual dispersion equation:

$$\cos p = \frac{f(2\pi/a_2) + f(-2\pi/a_2)}{2f(0)}, \quad (8)$$

which follows from the relation $f(k_2 \pm 2\pi/a_2) = e^{ip} f(k_2)$. Since ρ the modulus of the function in Eq. (5) is periodic, Eq. (8) is of the form

$$\cos p = \cos\{\varepsilon^{-2}\varphi_1(2\pi/a_2) - \varepsilon^2\varphi_3(2\pi/a_2) + \varepsilon^6\varphi_5(2\pi/a_2) - \ldots\}. \quad (9)$$

This equation can always be solved. It shows that the discontinuities fall off with ϵ faster than any finite power of ϵ. The results of Ref. 2 and the well-known fact that in the discontinuities in the

Mathieu equation fall off exponentially indicate that the reduction of the discontinuities with ϵ for the case of an analytic function $\kappa_1(k_2)$ is exponential.

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LATERAL DISTRIBUTION OF PHOTONS NEAR THE AXIS OF EXTENSIVE ATMOSPHERIC SHOWERS

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It is known that it is very difficult to calculate the lateral-distribution function of the soft component of extensive atmospheric showers with allowance for the cascade processes and for the ionization losses. However, at small values of r, the principal role is played by high-energy particles, for which the ionization losses can be neglected. In the works by Pomeranchuk[1] and Migdal[2] the lateral distribution function of electrons with energies greater than a given value has been calculated for small r, with the ionization losses neglected*

$$N(t,\ E,\ r) \sim 1/r^{2-s}, \tag{1}$$

and with the parameter s determined from the condition

$$-\lambda_1'(s)\,t = \ln(E_0/\beta) - \ln(R/r)$$

(the same result was obtained by Nishimura and Kamata[4]).

Let us determine the photon density corresponding to such an electron distribution. For this purpose we use one of the Landau equations (see Ref. 5):

$$\partial\Gamma(t,\ E,\ \mathbf{r},\ \theta)/\partial t + \theta\partial\Gamma(t,\ E,\ \mathbf{r},\ \theta)/\partial\mathbf{r}$$

$$= -\sigma_0\Gamma(t,E,\mathbf{r},\theta) + \int_E^\infty P(t,\ E',\ \mathbf{r},\ \theta)\,\varphi_{\mathrm{rad}}(E',\ E)\,dE', \tag{2}$$

where $\varphi_{\mathrm{rad}}(E', E)$ is the probability that an electron with energy E' will radiate a photon with energy E, r is the radius vector in the transverse plane, and θ is the projection of the direction of motion of the particles on this plane.

To solve our problem it is quite enough to put $\varphi_{\mathrm{rad}}(E', E) = 1/E$. Then, integrating over all θ, and also over the azimuth in the plane perpendicular to the axis of the shower (taking account of the symmetry of the problem in the last equation), Eq. (2) can be rewritten†

$$\partial N_\Gamma(t,\ E,\ r)/\partial t = -\sigma_0 N_\Gamma(t,\ E,\ r) + N(t,\ E,\ r)/E. \tag{3}$$

We assume for N(t, E, r) the expression given in Ref. 3

$$N(t,\ E,\ r) \approx e^{\lambda_1(s)t} E_k^{-s}\,[1-(rE/E_k)^{2-s}]/r^{2-s}(2-s).$$

A solution of Eq. (3), with boundary conditions at t = 0, $N_B = 0$, is

$$N_\Gamma(t,\ E,\ r) \approx \frac{e^{\lambda_1(s)t} - e^{-\sigma_0 t}}{\lambda_1(s)+\sigma_0}\,\frac{E_k^{-s}[1-(rE/E_k)^{2-s}]}{r^{2-s}(2-s)E}.$$

This expression is correct for $E_k/E_0 < r < E/E_k$.

Let us now determine the ratio $\dfrac{N_1(t, > E, r)}{N(t, E, r)}$, where $N_1(t, > E, r)$ is the number of photons with energies greater than the given value. Taking it into account that at a fixed value of r the particle energy E cannot be greater than E_k/r (see Ref. 3), we obtain the following value for N_1:

$$N_1 \sim \ln(E_k/rE)/r^{2-s}. \tag{4}$$

Let us note that in the derivation of Eq. (3) we used the condition $rE/E_k \ll 1$, which determines the boundary of applicability of formula (1). Thus, we obtain

$$N_1/N \sim \ln(E_k/rE). \tag{5}$$

*The symbols used here are the same as in the book by Belen'kii.(Ref. 3).

†It can be shown that for our problem it is possible to neglect the term containing the derivative with respect to r.