are not introduced artificially or without any reasonable basis, as is done in many works on ordinary quantum field theory. One may hope that these operators will make it possible to eliminate the difficulties associated with divergences in field theory.

In conclusion, I express my gratitude to Professor Iu. M. Shirokov for discussing the results of the present work.

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CIRCULAR WAVES IN AN ELECTRON-ION BEAM

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We consider circular waves in an uncompensated electron-ion beam in a cylindrical waveguide with perfectly conducting walls. The magnetic field produced by the beam current is assumed to be very strong. We treat the problem qualitatively, without an exact solution of the differential equations. It is shown that the beam is stable with respect to the oscillations being considered, and the natural frequency bands are found. The electromagnetic field is mapped.

1. The stationary state of an electron-ion beam has been examined by Bennett[1] and Budker.[2]

In the present article we consider the oscillations of an electron-ion beam in a cylindrical waveguide of radius R. We shall treat circular oscillations, which means that we assume the electromagnetic field, as well as the electron and ion densities and velocities, to be independent of z, and the dependence of these quantities on r, φ, and the time t to be given by

$$F(r, \varphi, t) = f(r)\, e^{i(\omega t - \mu\varphi)}. \qquad (1)$$

The conclusions we reach can be extended to perturbations of a more general form, namely

$$F(r, \varphi, z, t) = f(r)\, e^{i(\omega t - \mu\varphi - \gamma z)},$$

so long as the condition $\gamma R \ll 1$ is satisfied.

The amplitude of vibrations is considered small, and the equations are linearized. The problem is solved in the hydrodynamic approximation; the electrons and ions have different temperatures, which are constant in space and time.

We assume also that the magnetic field produced by the beam current is so strong that the inequalities

$$|eH_{\varphi 0}(r)/mc\omega| \gg 1, \quad |eH_{\varphi 0}(r)/Mc\omega| \gg 1 \qquad (2)$$

are fulfilled at all points within the waveguide. Here $H_{\varphi 0}(r)$ is the magnetic field strength, ω is the frequency

$$m = m_0/\sqrt{1-\beta_e^2},\ M = M_0/\sqrt{1-\beta_i^2};\ \beta_e = v_{0e}/c,\ \beta_i = v_{0i}/c,$$

m_0 and M_0 are the electron and ion rest masses, and v_{0e} and v_{0i} are the electron and ion velocities in the stationary state. Since we are dealing with a strong magnetic field, the variable components v_{er}, v_{ez}, v_{ir}, and v_{iz} of the electron and ion velocities are small.

We shall show below that the beam is stable with respect to the oscillations we are considering, and that the μth natural frequency lies in the interval $\mu v_{iT}/R < \omega < \mu c/R$, where v_{iT} is the thermal velocity of the ions, given by $v_{iT} = \sqrt{T_i/M}$. Here T_i is the ion temperature in energy units. The thermal velocity of the ions is assumed to be less than that of the electrons, given by $v_{eT} = \sqrt{T_e/m}$.

We shall show also that at the distance $r = \mu v_{iT}/\omega$ from the beam axis the vibration amplitude of the ions has a sharp maximum. This occurs because at this radius the thermal velocity of the ions and the phase velocity of the wave are equal. A similar resonance takes place for electrons at $r = \mu v_{eT}/\omega$, so long as this value of r lies in the interval $0 < r < R$.

2. In the stationary state the electron and ion densities n_{0e} and n_{0i}, the magnetic field $H_{\varphi 0}$, and the electric field E_{r0} are of the form[1,2]

$$\begin{aligned} n_{0e}(r) &= 2[T_e(1-\beta_i^2) + T_i(1-\beta_i\beta_e)]/\pi e^2(\beta_e-\beta_i)^2 r_0^2(1+r^2/r_0^2)^2, \\ n_{0i}(r) &= 2[T_i(1-\beta_e^2) + T_e(1-\beta_e\beta_i)]/\pi e^2(\beta_e-\beta_i)^2 r_0^2(1+r^2/r_0^2)^2, \\ H_{\varphi 0}(r) &= 2\pi e[n_{0i}(0)\beta_i - n_{0e}(0)\beta_e]\, r/(1+r^2/r_0^2), \\ E_{r0}(r) &= 2\pi e[n_{0i}(0) - n_{0e}(0)]\, r/(1+r^2/r_0^2), \end{aligned} \tag{3}$$

where r_0 is the effective radius of the beam. All the other components of the electric and magnetic fields vanish. If the beam axis is made to coincide with the axis of the waveguide, Eqs. (3) for the electric and magnetic fields satisfy the boundary conditions $\mathbf{E}\times\mathbf{n} = 0$ and $\mathbf{H}\cdot\mathbf{n} = 0$ on the surface of the waveguide. Therefore the state of the electron-ion beam will not change if it is in a cylindrical waveguide of any radius R whatsoever.

If we use $\mathbf{E}$ and $\mathbf{H}$ to denote the variable parts of the electric and magnetic fields, and $v_{e\varphi}$, $v_{i\varphi}$, n_e, and n_i to denote the variable components of the electron and ion velocities and densities, then Maxwell's equations, the equations of motion, and the continuity equations become (the constant components cancel out)

$$\begin{aligned} &-\frac{\partial H_z}{\partial r} = \frac{1}{c}\frac{\partial E_\varphi}{\partial t} - \frac{4\pi e}{c}(n_{0e}+n_e)v_{e\varphi} + \frac{4\pi e}{c}(n_{0i}+n_i)v_{i\varphi}, \\ &\frac{\partial(rE_\varphi)}{r\,\partial r} - \frac{\partial E_r}{r\partial\varphi} = -\frac{1}{c}\frac{\partial H_z}{\partial t}, \quad \frac{\partial(rE_r)}{r\partial r} + \frac{\partial E_\varphi}{r\partial\varphi} = 4\pi e(n_i - n_e), \\ &m\left(\frac{\partial v_{e\varphi}}{\partial t} + v_{e\varphi}\frac{\partial v_{e\varphi}}{r\partial\varphi}\right) = -eE_\varphi - \frac{T_e}{n_{0e}+n_e}\frac{\partial(n_{0e}+n_e)}{r\partial\varphi}, \\ &M\left(\frac{\partial v_{i\varphi}}{\partial t} + v_{i\varphi}\frac{\partial v_{i\varphi}}{r\partial\varphi}\right) = eE_\varphi - \frac{T_i}{n_{0i}+n_i}\frac{\partial(n_{0i}+n_i)}{r\partial\varphi}, \\ &\frac{\partial n_e}{\partial t} + \frac{\partial[(n_{0e}+n_e)v_{e\varphi}]}{r\partial\varphi} = 0, \quad \frac{\partial n_i}{\partial t} + \frac{\partial[(n_{0i}+n_i)v_{i\varphi}]}{r\partial\varphi} = 0, \\ &H_r = H_\varphi = E_z = 0. \end{aligned} \tag{4}$$

In the continuity equations we have dropped terms containing v_{er} and v_{ir}, since they are small compared with $v_{e\varphi}$ and $v_{i\varphi}$. We shall not write out those equations which contain products of a large quantity $H_{\varphi 0}$, n_{0e}, or n_{0i} by a small one v_{0r}, v_{ez}, v_{ir}, or v_{iz}. These equations can be used to determine the velocities v_{er}, v_{ez}, v_{ir}, and v_{iz} in the second approximation.

When we linearize Eqs. (4) and make use of (1), we arrive at

$$-\frac{dH_z}{dr} = \frac{i\omega}{c}E_\varphi - \frac{4\pi e n_{0e}}{c}v_{e\varphi} + \frac{4\pi e n_{0i}}{c}v_{i\varphi}, \tag{5}$$

$$\frac{d(rE_\varphi)}{r\,dr} + \frac{i\mu}{r}E_r = -\frac{i\omega}{c}H_z, \tag{6}$$

$$\frac{d(rE_r)}{rdr} - \frac{i\mu}{r}E_\varphi = 4\pi e(n_i - n_e), \tag{7}$$

$$im\omega v_{e\varphi} = -eE_\varphi + i\mu T_e n_e/n_{0e}r, \tag{8}$$

$$iM\omega v_{i\varphi} = eE_\varphi + i\mu T_i n_i/n_{0i}r, \tag{9}$$

$$\omega n_e = (\mu/r)\,n_{0e}v_{e\varphi}, \tag{10}$$

$$\omega n_i = (\mu/r)\,n_{0i}v_{i\varphi}. \tag{11}$$

From these equations we eliminate the electron and ion densities, obtaining

$$\begin{aligned} v_{e\varphi} &= i\omega r^2 eE_\varphi/(\omega^2r^2 - \mu^2v_{eT}^2), \\ v_{i\varphi} &= -i\omega r^2 eE_\varphi/(\omega^2r^2 - \mu^2v_{iT}^2). \end{aligned} \tag{12}$$

Inserting (12) into Eqs. (5) – (7), we arrive at

$$\begin{aligned} -dH_z/dr &= (i\omega/c)\,q(r)\,E_\varphi, \\ d(rE_r)/dr &= i\mu q(r)\,E_\varphi, \end{aligned} \tag{13}$$

where

$$q(r) = 1 + \Omega_e^2(r)\,r^2/(\mu^2v_{eT}^2 - \omega^2r^2) + \Omega_i^2(r)\,r^2/(\mu^2v_{iT}^2 - \omega^2r^2), \tag{14}$$

$$\Omega_e^2(r) = 4\pi e^2 n_{0e}(r)/m, \quad \Omega_i^2 = 4\pi e^2 n_{0i}(r)/M. \tag{15}$$

Eliminating $q(r)$ from (13), and integrating over r we have

$$E_r = -(\mu c/\omega r)\,H_z.$$

Inserting this expression into (6), we have

$$H_z = -\frac{i\omega cr}{\mu^2c^2 - \omega^2r^2}\frac{d(rE_\varphi)}{dr}. \tag{16}$$

Finally, eliminating E_r and H_z from (6), (15), and (16), we obtain the differential equation

$$\frac{d}{dr}\left[p(r)\frac{d(rE_\varphi)}{dr}\right] - q(r)\,E_\varphi = 0, \quad p(r) = \frac{c^2r}{\mu^2c^2 - \omega^2r^2} \tag{17}$$

for E_φ, with the boundary conditions[3]

$$\begin{aligned} &E_\varphi(0) \text{ finite}, \quad E_\varphi(R) = 0 \quad \text{if} \quad \mu = 1, \\ &E_\varphi(0) = 0, \qquad E_\varphi(R) = 0 \quad \text{if} \quad \mu \geqslant 2. \end{aligned} \tag{18}$$

The frequency ω is an eigenvalue of the differential operator of Eq. (17) with the boundary conditions (18).

We note that $E_\varphi(r)$ must not only satisfy the boundary conditions (18), but must also be finite in the interval $0 \le r \le R$. In addition, $v_{e\varphi}(r)$, $v_{i\varphi}(r)$, and $H_z(r)$, which are related to $E_\varphi(r)$ by (12) and (16), must also be finite in this interval.

If all the eigenvalues ω are real, the beam is stable. If, however, one or more of the eigenvalues are complex, the beam is unstable.

Since Eq. (17) cannot be solved exactly, we shall treat it in a qualitative way.

We shall first show that ω is always real. Let us assume the contrary, writing $\omega = \omega_0 + i\eta$, where $\eta \neq 0$. Then the imaginary parts of p and q are given by

$$\operatorname{Im} p = 2c^2 r\omega_0\eta/\{[\mu^2c^2 - (\omega_0^2 - \eta^2)r^2]^2 + 4\omega_0^2\eta^2 r^4\},$$

$$\operatorname{Im} q = \frac{2\Omega_e^2 r\omega_0\eta}{[\mu^2 v_{eT}^2 - (\omega_0^2 - \eta^2)r^2]^2 + 4\omega_0^2\eta^2 r^2} + \frac{2\Omega_i^2 r\omega_0\eta}{[\mu^2 v_{iT}^2 - (\omega_0^2 - \eta^2)r^2]^2 + 4\omega_0^2\eta^2 r^4},$$

from which it follows that Im p and Im q always have the same sign.

We may, on the other hand, by introducing the new variable $u = rE_\varphi$, write (17) in the form

$$(pu')' - qu/r = 0 \tag{19}$$

with the boundary conditions

$$u(0) = u(R) = 0. \tag{20}$$

Multiplying (19) by u^*, the complex conjugate of u, integrating over r from zero to R, and then taking the imaginary part, we arrive at

$$\int_0^R \operatorname{Im} p\,|u'|^2\,dr + \int_0^R \operatorname{Im} q\,|u|^2\,dr = 0. \tag{21}$$

Since $|u'|^2$ and $|u|^2$ are both positive, this equation cannot hold if Im p and Im q have the same signs. Thus all the eigenvalues ω are real, and the beam is stable with respect to the oscillations under consideration.

To examine (17) further, let us eliminate from it the first derivative. To do this, we make the substitution

$$y = cr\sqrt{r}\,E_\varphi/\sqrt{\mu^2c^2 - \omega^2r^2}. \tag{22}$$

Then (17) becomes

$$d^2y/dr^2 + Q(r)\,y = 0, \tag{23}$$

where

$$Q(r) = \frac{\mu^4c^4 - 10\mu^2c^2\omega^2r^2 - 3\omega^4r^4}{4r^2(\mu^2c^2 - \omega^2r^2)^2} - \frac{\mu^2c^2 - \omega^2r^2}{c^2r^2}\,q(r); \tag{24}$$

q(r) is given by (14), and y(r) satisfies the boundary conditions

$$y(0) = y(R) = 0. \tag{25}$$

We shall first show that the frequencies ω are no greater than $\mu c/R$. To this end, let us consider the behavior of $E_\varphi(r)$, $v_{i\varphi}(r)$, $v_{e\varphi}(r)$, and $H_z(r)$ in the interval $0 \le r \le R$. Equation (23) has four singular points, namely $r = 0$, $\mu \times v_{iT}/\omega$, $\mu v_{eT}/\omega$, and $\mu c/\omega$. It can be shown that $E_\varphi(r)$, since it satisfies the boundary conditions (18), remains finite at these points. Expressions (12) and (16) show, however, that at $r = \mu v_{iT}/\omega$ and $r = \mu v_{eT}/\omega$ the velocities $v_{i\varphi}$ and $v_{e\varphi}$, respectively, become infinite, and that at $r = \mu c/\omega$ the magnetic field H_z becomes infinite.

The first two infinities are fictitious because the linearization of (27) in the neighborhoods of $r = \mu v_{iT}/\omega$ and $r = \mu v_{eT}/\omega$ is invalid. To show this, consider the nonlinear equations (4). We shall attempt to find a solution of these equations in which the desired functions contain the time t and the angle φ only in the combination $\xi = \omega t - \mu\varphi$. It follows from Eqs. (4) that at $r = \mu \times v_{iT}/\omega$

$$2v_{i\varphi}\,dv_{i\varphi}/d\xi = -(ev_{iT}/M\omega)\,E_\varphi. \tag{26}$$

This equation indicates that $v_{i\varphi}$ is of the same order as $\sqrt{E_\varphi}$. Since E_φ is small, $v_{i\varphi}$ is also small, but of lower order. In the linear approximation this means that $v_{i\varphi}$ becomes infinite if E_φ remains finite. Thus at the point $r = \mu \times v_{iT}/\omega$ the amplitude of the ion vibrations increases sharply. At the point $r = \mu v_{eT}/\omega$ there will take place a similar resonance in the electron vibrations.

As for the point $r = \mu c/\omega$, the magnetic field H_z remains infinite at this point also in the nonlinear theory. Indeed, were we to repeat the operations which led to (16), this time on the nonlinear equations (4), we would obtain

$$\frac{\partial H_z}{\partial \xi} = \frac{\omega c r}{\mu^2c^2 - \omega^2r^2}\,\frac{\partial(rE_\varphi)}{\partial r},$$

for which it follows that H_z becomes infinite at $r = \mu c/\omega$. It can also be shown that this infinity is not removed when one takes account of collisions.

It follows from this that the point $r = \mu c/\omega$ must lie outside the interval $0 \le r \le R$. In other words, it is necessary that

$$\omega < \mu c/R.$$

This inequality determines the upper bound of the eigenvalues ω.

In order to find the lower bound, we note that Q (r), the function given by (24), will be negative over the whole interval $0 \le r \le R$ if $\omega < \mu \times v_{iT}/R$. As is known,[4] however, Eq. (23) has no solutions satisfying conditions (25) if Q (r) is negative. It is therefore necessary that $\omega > \mu \times v_{iT}/R$.

We have thus proven that the eigenvalues ω cannot be complex and must lie in the interval $\mu v_{iT}/R < \omega < \mu c/R$. In the process we have assumed the existence of eigenvalues of the operator of (23) with boundary conditions (25). In other words, we have assumed that circular waves can actually exist. In order to prove the last statement, we note that as $\omega \to \mu v_{iT}/R + 0$, the function $Q(r) \to +\infty$. From this it follows that there exists an infinite number of eigenvalues.[4]

The inequality $\omega < \mu c/R$ indicates that circular waves have low frequencies, which means that they are not magnetohydrodynamic.[5,6] We note that at the resonance point $r = \mu v_{iT}/\omega$, the phase velocity of the wave is $V = r\omega/\mu$, whose form is reminiscent of the Alfven velocity[7]

$$V = H_{\varphi 0}\left[\frac{n_{0e}\left(1-\beta_e\beta_i\right)-n_{0i}\left(1-\beta_i^2\right)}{4\pi M\left(n_{0i}\beta_i-n_{0e}\beta_e\right)^2}\,\frac{r_0^2}{2r^2}\right]^{1/2}.$$

Let us go on, finally, to the mapping of the field. From the form of p (r) and q (r) in Eq. (17) it follows that an expansion of $E_\varphi(r)$ in powers of r contains only even powers when $\mu = 1$. Therefore $E'_\varphi(0) = 0$. In order to find the behavior of $E_\varphi(r)$ in the neighborhood of $r = \mu v_{iT}/\omega$, let us make use of (23). Since $Q(r) \to -\infty$ as $r \to \mu v_{iT}/\omega - 0$, and $Q(r) \to +\infty$ as $r \to (\mu v_{iT}/\omega) + 0$, the r dependence of y [and therefore also the function $E_\varphi(r)$] is of the form shown in Fig. 1 (when $\mu v_{eT}/\omega < R$).

When $\mu v_{eT}/\omega > R$, the graph of $E_\varphi(r)$ is of the form shown in Fig. 2. Both of these graphs are for $\mu = 1$. Figures 1 and 2 show those eigen-

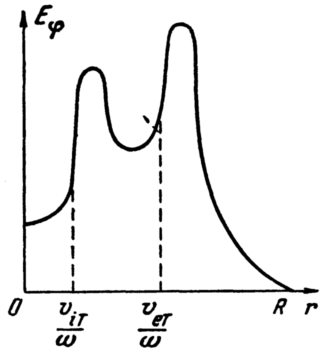

Fig. 1.

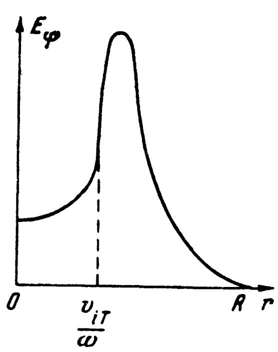

Fig. 2.

functions of the operator of (17) which belong to the largest eigenvalue ω. It can be shown that these functions have no zeros within the interval $0 < r < R$.

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