are the thermal conductivity and Thomson coefficients.

It was shown in Ref. 13 that if the collision operator was a δ-function in the energy, then for each of the components of the tensors of conductivity and thermal conductivity, the Wiedmann-Franz law is satisfied. The calculations carried out in Ref. 13 are not connected with the concrete form of σ_{ik}, μ_{ik}, κ_{ik}, and therefore we can make direct use of the results of Ref. 13 and write down at once:

$$\sigma_{ik} = \sigma_{ik}(\zeta_0; d); \quad \varkappa_{ik} = {}^1/_3\pi^2 k^2 T \sigma_{ik};$$
$$\mu_{ik} = \frac{\pi^2 k^2 T}{3e}\left(\rho_{ip}\frac{\partial\sigma_{pk}}{\partial\zeta_0} - 2\rho_{pi}\frac{\partial\sigma_{kp}}{\partial\zeta_0}\right). \qquad (6.5)$$

Here $\zeta_0 = \zeta(0)$ is the chemical potential of the electron gas at absolute zero (the limiting Fermi energy), k is Boltzmann's constant. Thus, the tensors κ_{ik} and μ_{ik} are expressed in terms of the tensor σ_{ik} studied above.

In conclusion, I take this opportunity to thank M. Ia. Azbel', E. S. Borovik, B. G. Lazarev, and I. M. Lifshitz for useful discussions and criticism of the results of the research.

[1] K. Fuchs, Proc. Cambr. Phil. Soc. **34**, 100 (1938).

[2] M. I. Kaganov and M. Ia. Azbel', J. Exptl. Theoret. Phys. (U.S.S.R.) **27**, 762 (1954).

[3] R. Englman E. H. Sondheimer, Proc. Phys. Soc. (London) **B69**, 449 (1956).

[4] R. B. Dingle, Proc. Roy. Soc. (London) **A201**, 545 (1950).

[5] D. K. Macdonald and K. Sarginson, Proc. Roy. Soc. (London) **A203**, 223 (1950).

[6] R. G. Chambers, Proc. Roy. Soc. (London) **A202**, 378 (1950).

[7] M. Ia. Azbel', Dokl. Akad. Nauk SSSR **99**, 519 (1954).

[7a] E. H. Sondheimer, Phys. Rev. **80**, 401 (1950).

[8] E. Königsberg, Phys. Rev. **91**, 8 (1953).

[9] E. S. Borovik and B. G. Lazarev, J. Exptl. Theoret. Phys. (U.S.S.R.) **21**, 857 (1951); Dokl. Akad. Nauk SSSR **62**, 611 (1948).

[10] E. R. Andrew, Proc. Phys. Soc. (London) **A62**, 77 (1949).

[11] Lifshitz, Azbel', and Kaganov, J. Exptl. Theoret. Phys. (U.S.S.R.) **31**, 63 (1956), Soviet Phys. JETP **4**, 41 (1957).

[12] B. Shoenberg, Phil. Trans. **A245**, 1 (1952).

[13] Azbel', Kaganov, and Lifshitz, J. Exptl. Theoret. Phys. (U.S.S.R.) **32**, 1188 (1957); Soviet Phys. JETP **5**, 967 (1957).

Translated by R. T. Beyer
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STATIONARY CONVECTIVE FLOW OF AN ELASTICALLY CONDUCTING LIQUID BETWEEN PARALLEL PLATES IN A MAGNETIC FIELD

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A study is made of the stationary convection of an electrically conducting liquid in the space between two parallel plates, heated to different temperatures, in the presence of a magnetic field. The distribution of velocity, temperature, and induced fields are found, and the convective heat flow is calculated.

IT is well known that currents are induced in a conducting liquid which moves in a magnetic field. The interaction of these currents with the magnetic field is the cause of the various magnetohydrodynamic effects which have been intensively studied in recent years. The magnetic field will, of course, also have an effect on the convective flow of an electrically conducting liquid. As an example, we may cite the increased stability of the equilibrium of an electrically conducting liquid heated from below when a magnetic field is applied.[1,2] In the present paper we consider the stationary convective motion of a conducting liquid located in a magnetic field, in the space between two parallel plates heated to different temperatures.

1. The current flowing in a medium moving with velocity $\mathbf{v}$ is equal to

$$\mathbf{j} = \sigma\left(\mathbf{E} + \frac{1}{c}\mathbf{v}\times\mathbf{B}\right), \tag{1}$$

where σ is the electrical conductivity, and $\mathbf{E}$ and $\mathbf{B}$ are the field strengths. The nature of the fields and material motions are to be determined from the equations of motion for the medium (in our case these are the equations of convection) and from Maxwell's equations for the fields in the medium:

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{v} = -\frac{1}{\rho}\nabla p + \nu\nabla^2\mathbf{v} + g\beta\gamma T + \frac{1}{\rho c}\mathbf{j}\times\mathbf{B}; \tag{2}$$

$$\partial T/\partial t + \mathbf{v}\nabla T = \chi\nabla^2 T; \tag{3}$$

$$\operatorname{div}\mathbf{v} = 0; \tag{4}$$

$$\operatorname{curl}\mathbf{H} = \frac{4\pi}{c}\mathbf{j}, \quad \operatorname{div}\mathbf{B} = 0; \tag{5}$$

$$\operatorname{curl}\mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}, \quad \operatorname{div}\mathbf{D} = 4\pi\rho_e. \tag{6}$$

Here p is the convective pressure, T is the temperature, ρ is the density of the liquid, g is the acceleration of gravity, ν is the kinematic viscosity, β is the thermal diffusivity, χ is the thermal conductivity, $\boldsymbol{\gamma}$ is a unit vector directed vertically upward, and ρ_e is the space charge density. In accordance with the usual assumptions, we have neglected displacement currents in the equation for curl H, and the viscous and Joule dissipations in the heat transfer equation.

By eliminating the electric field strength and current density from Maxwell's equations and Eq. (1), we obtain

$$\partial\mathbf{H}/\partial t + \operatorname{curl}(\mathbf{H}\times\mathbf{v}) = \lambda\nabla^2\mathbf{H}, \quad \lambda = c^2/4\pi\mu\sigma. \tag{7}$$

Substituting the expression $(c/4\pi)\operatorname{curl}\mathbf{H}$ for the current density, we can write two of the terms in the equation of motion (2) in the form

$$-\frac{1}{\rho}\nabla p + \frac{1}{\rho c}\mathbf{j}\times\mathbf{B} = -\frac{1}{\rho}\nabla\left(p + \frac{\mu H^2}{8\pi}\right) + \frac{\mu}{4\pi\rho}\mathbf{H}\cdot\nabla\mathbf{H};$$

Here the gradient symbol operates on the total pressure (convective plus magnetic).

We now introduce dimensionless variables. Denote by 2d the distance between the parallel plates, by 2Θ the constant temperature difference between them, and take d and Θ as the new units of length and temperature. As the unit of field strength we choose the value of the constant, uniform, external field H_0. The units of time, velocity, and pressure are chosen as d^2/ν, ν/d, and $\rho\nu^2/d^2$. In terms of these dimensionless variables, the equations take the form

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{v} = -\nabla\left(p + \frac{M^2}{P_m}\frac{H^2}{2}\right) + \nabla^2\mathbf{v} + G\gamma T + \frac{M^2}{P_m}\mathbf{H}\cdot\nabla\mathbf{H}; \tag{8}$$

$$\frac{\partial T}{\partial t} + \mathbf{v}\nabla T = \frac{1}{P}\nabla^2 T; \tag{9}$$

$$\frac{\partial \mathbf{H}}{\partial t} + \operatorname{curl}(\mathbf{H}\times\mathbf{v}) = \frac{1}{P_m}\nabla^2\mathbf{H}; \tag{10}$$

$$\operatorname{div}\mathbf{v} = 0, \quad \operatorname{div}\mathbf{H} = 0. \tag{11}$$

Four dimensionless parameters have been introduced into these equations: $G = g\beta\Theta d^3/\nu^2$, the Grasshof number; $P = \nu/\chi$, the Prandtl number; $M = (B_0 d/c)\sqrt{\sigma/\eta}$, the Hartmann number (here $B_0 = \mu H_0$, and $\eta = \rho\nu$ is the absolute viscosity); and $P_m = \nu/\lambda$.

The boundary conditions for Eqs. (8) to (11) will be established later.

2. Let us consider stationary convection in the space between vertical parallel plates, when an external magnetic field is applied perpendicular to the plates. We shall locate the origin of coordinates midway between the plates. The x axis is directed normal to the plates, in the direction of the colder plate; the z axis is vertically upward; and the y axis is perpendicular to the x and z axes (see Fig. 1).

If the dimensions of the plates are sufficiently large compared to the distance between them, it is possible to find an exact solution of Eqs. (8) to (11) which will describe the stationary motion at all points except near the edges of the plates. For this type of motion, (a) the velocity $\mathbf{v}$ is everywhere parallel to the z axis, (b) the temperature T depends only on x, (c) the field vector $\mathbf{H}$ is always in the xz plane, i.e., $H_y = 0$, (d) all quantities are independent of y (i.e., the problem is two-dimensional), and (e) all quantities except the pressure are independent of z.

Let us now find the profiles of temperature, velocity, and field strength for this system. From the assumptions which have been made as to the type of motion we will have, instead of Eq. (9), $d^2T/dx^2 = 0$, i.e., the temperature profile is linear. If we take the zero of our temperature scale to be the value calculated for $x = 0$, then the boundary conditions for the temperature equation will be $T(-1) = 1$; $T(1) = -1$; and accordingly,

$$T = -x. \qquad (12)$$

We now find the velocities and field strengths. From (11) it follows that $v_z = v(x)$ and $dH_x/dx = 0$; i.e., the component of magnetic field normal to the plates is a constant, and is obviously equal to the external field. Let us assume for the sake of definiteness that the direction of the external field coincides with the positive sense of the x axis. Recalling our choice of unit field strength, we have

$$H_x = 1. \qquad (13)$$

The x component of Eq. (8) is

$$\frac{\partial}{\partial x}\left(p + \frac{M^2}{P_m}\frac{H^2}{2}\right) = 0, \qquad (14)$$

i.e., the total pressure depends only upon z. Let us now consider the z component of Eq. (8)

$$\frac{d}{dz}\left(p + \frac{M^2}{P_m}\frac{H^2}{2}\right) = v'' + GT + \frac{M^2}{P_m}H_z'$$

(the primes represent differentiation with respect to x). Since the right-hand side is a function of x only, and the left-hand side is a function only of z, we may separate the variables, denoting the separation constant by C, and writing

$$v'' + GT + (M^2/P_m)H_z' = C. \qquad (15)$$

The z component of Eq. (10) gives another equation connecting v and H_z:

$$H_z''/P_m = -v', \qquad (16)$$

whence

$$H_z'/P_m = -v + C_1. \qquad (17)$$

By substituting (12) and (17) into (15) we obtain an equation for v:

$$v'' - M^2 v = G(x + A), \quad A = (C - C_1 M^2)/G. \qquad (18)$$

The solution of (18) will contain two constants of integration which must be determined from the boundary conditions $v(-1) = v(1) = 0$. In order to determine the constant A we must know the flow of liquid through a cross-section. If the channel is closed at the top and bottom, then the liquid will circulate, rising near the warm plate and sinking near the cold one. In this case, obviously

$$\int_{-1}^{1} v\,dx = 0.$$

If A and the constants of integration are determined for this case, we obtain the velocity profile

$$v = \frac{G}{M^2}\left(\frac{\sinh Mx}{\sinh M} - x\right). \qquad (19)$$

To find the induced field H_z we substitute v from (19) into (16), integrate twice, and determine the constants of integration from the boundary conditions

$$H_z(-1) = H_z(1) = 0.$$

As a result we obtain

$$H_z = \frac{GP_m}{M^2}\left(\frac{x^2 - 1}{2} - \frac{\cosh Mx - \cosh M}{M \sinh M}\right). \qquad (20)$$

(Had the external field been in the negative x direction, H_z would have been of the opposite sign.)

Equations (12), (19), and (20) provide the solution of the given problem.

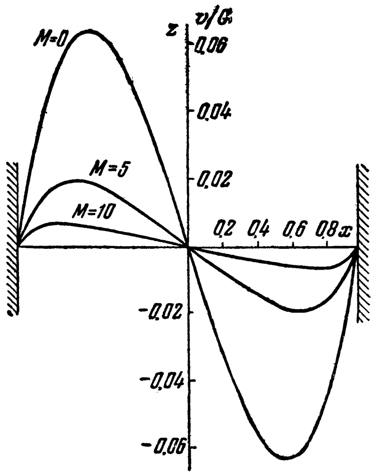

Fig. 1.

3. Figure 1 shows the velocity profiles for $M = 0$, 5, and 10. As an example, we may note that for mercury $\sigma = 0.945 \times 10^{16}$ sec^{-1} and $\eta = 1.55 \times 10^{-2}$ poise; consequently, $M = 0.026\,B_0 d$. Therefore if $d = 1$ cm, a value of $M = 10$ corresponds to a field B_0 of the order of 400 gauss. In the absence of a field ($M = 0$), Eq. (19) gives the velocity profile

$$v = Gx(x^2 - 1)/6. \qquad (21)$$

As the field increases, it is evident from Fig. 1 that the flow rapidly decreases. In addition, an unusual boundary layer appears in the flow pattern; a thin layer develops near the walls in which the velocity gradient is large. (The occurrence of a boundary layer in the flow of liquids in a magnetic field was observed by Hartmann[3] while he was studying the effect of a field on Poiseuille flow.) If the thickness of the boundary layer δ is defined as the distance from the wall to the point where the velocity is a maximum, then for large values of the Hartmann number M (in prac-

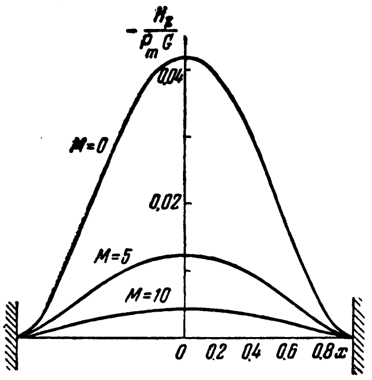

Fig. 2.

tice, when $M > 5$), we have from Eq. (19)

$$\delta = (\ln M) / M. \tag{22}$$

The distribution of induced magnetic field over the cross-section is shown in Fig. 2. It must be noted that H_z in Eq. (20) and in Fig. 2 is the ratio of the induced (longitudinal) to the external (transverse) field. As $M \to 0$ this ratio approaches a limiting value; for large values of the Hartmann number, H_z as determined from Eq. (20) diminishes in proportion to M^{-2}. Thus, for small external fields H_0 the induced field increases proportionally to H_0, while for large H_0 it decreases as $1/H_0$. A maximum is attained at a field corresponding to $M = 3$. Note also that for moderate values of the Grasshof number the induced field is very much smaller than the external field, since the parameter P_m which enters into formula (20) has the value 10^{-7} for mercury, for example.

The induced field is equal to $\mathbf{j} = (c/4\pi)\,\text{curl}\,\mathbf{H}$, and therefore the only non-zero component of the current density is j_y, which is proportional to dH_z/dx. The current density therefore varies in the same way as the velocity does over the cross-section.

We can also find the upward flow of heat due to convection, which is equal to

$$Q_M = c_p \rho \int_{-d}^{d} vT dx, \tag{23}$$

per unit of length in the direction of the y axis, where c_p is the specific heat of the liquid, and v and T are the dimensional velocity and temperature. Evaluation of this formula gives

$$\frac{Q_M}{Q_0} = \frac{45}{M^2}\left(\frac{1}{3} - \frac{\coth M}{M} + \frac{1}{M^2}\right), \tag{24}$$

where Q_0 is the flow in the absence of magnetic field and is equal to

$$Q_0 = 2c_p \rho g \beta \Theta^2 d^3 / 45\nu. \tag{25}$$

The ratio Q_M/Q_0 decreases monotonically from 1 to 0 as the number M increases.

The solution derived above describes the flow in a vertical channel in the presence of a perpendicular external field. For channels inclined at an angle α to the vertical, it can easily be verified that the solution is the same as before, with $G \times \cos\alpha$ substituted for G. If the field has any arbitrary orientation with respect to the channel, its longitudinal components have no effect on the detailed motions which have been discussed.

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[1] S. Chandrasekhar, Phil. Mag. **43**, 501 (1952), **45**, 1177 (1954).

[2] E. M. Zhukovitskii, Физика металлов и металловедение (Physics of Metals and Metal Research) (in press).

[3] J. Hartmann, Kgl. Danske Vidensk., Selskab. Math.-fys. Medd. **15**, 6 (1937).

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up the crystals as infinitely long unidimensional or two-dimensional atom complexes, bound together by "small" forces of one nature, whereas in the complex itself the atoms are bound by "big" forces of another nature.

6. The difference between the typical molecular crystals (e.g., the CH_4 or C_6H_6 crystals) and the heteropolar molecular crystals (such as the NaCl, $HgCl_2$ or PbS crystals) lies: (1) in the degree of molecularity β; (2) in the nature of the forces in the molecules; (3) in the nature of intermolecular forces. The quantity β is defined as the ratio of the intramolecular energy $U^a \cong D$ (D is the energy of dissociation of the diatomic molecule into ions) to the intermolecular energy U^e per bond. For the substances for which β is given below, it is possible to take $U^e \approx 2S/l$. Example:

$\beta = 300(CH_4)$, $200(HCl)$, $22(HgCl_2)$, $10(NaCl)$ taking $l = 12$ in all four cases.

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ERRATA

Volume 5

Page	Line	Reads	Should Read
1043	Eq. (4)		$W = y^2 a_{14}^2 \sin 2\phi / 2\rho\, (a_{11} a_{44} - \alpha_{14}^2 \sin^2 3\phi)$ The coefficient k_2 equals $0.185 \times 10^{-3}\,cm^{-1}$.
1044	3 from bottom (l.h.)	$\Delta y = 2.87 \times 10^{-3}$ cm	$\Delta y = 3.18 \times 10^{-3}$ cm
	4 from top (r.h.)	$\Delta\vartheta_\Sigma = 7.2 \times 10^{-5}$ radians	$\Delta\vartheta_\Sigma = 5.9 \times 10^{-5}$ radians

Volume 6

Page	Line	Reads	Should Read
1090	4 and 5 from top	2—(d, 3n); and of the I_{53}^{127} cross section, 3—(d, 2n); 4—(d, 3n)	2—(d, 3n) on I_{53}^{127} and 3—(d, 3n); 4—(d, 3n) on Bi_{83}^{209}
1091	6 from bottom expression for determinant C(y)	ρ, γp, h, $1/\rho$	ρy_2, $\gamma p y_2$, $h y_2$, y_2/ρ
1094	7 from bottom	For $\gamma = 5/3$, μ has . . .	Here μ has . . .

Volume 7

Page	Line	Reads	Should Read
55	16 from bottom	Correct submittal date is April 5, 1957	
169	17 from bottom	Delete "Joint Institute for Nuclear Research"	
215	Table	Add: Note. Columns 2—9 give the number of counts per 10^6 monitor counts	
215	Table, column headings	1, 2, 3, 4-7, 8	1, 2, 3, 4, 8-7
312	Eq. (8)	. . . $(1 \pm \mu/2M)^2$	. . . $(1 \mp \mu/2M)^2$
313	2, r.h. col.	$\alpha_{33} = 0.235$	$a_{33} = 0.235$
692	Eq. (5)	$m_B/M_B = \ldots \mp [1 + \ldots]$	$m_B/M_B = \mp [1 + \ldots]$
461	Title	. . . Elastically Conducting	. . . Electrically Conducting