But when a radio-frequency quantum is absorbed by the exciton the total spin of the electrons becomes ± 1, so that the inverse optical transition from the exciton state to the ground state is forbidden. Thus excitons that have absorbed radio-frequency quanta have considerably longer lifetimes against radiative deexcitation than ordinary excitons.

In an entirely analogous way the lifetime in the excited state is also lengthened for local electron centers that have absorbed radio-frequency quanta.

[1] M. F. Deigen, J. Exptl. Theoret. Phys. (U.S.S.R.) **26**, 293 (1954); Укр. физ. журн. (Ukr. Phys. Journal) **1**, 245 (1956).

[2] C. A. Hutchison, Jr. and R. C. Pastor, Phys. Rev. **81**, 282 (1951); Revs. Mod. Phys. **25**, 285 (1953). M. A. Garstens and A. H. Ryan, Phys. Rev. **81**, 888 (1951). Levinthal, Rogers, and Ogg, Phys. Rev. **83**, 182 (1951).

[3] G. E. Pikus and Iu. A. Firsov, J. Tech. Phys. (U.S.S.R.) **25**, 2381 (1955).

[4] J. Kaplan and C. Kittel, J. Chem. Phys. **21**, 1429 (1953).

[5] R. A. Ogg, Jr., J. Chem. Phys. **14**, 295 (1946).

Translated by W. H. Furry
129

SOVIET PHYSICS JETP VOLUME 34 (7), NUMBER 3 SEPTEMBER, 1958

PLANE PROBLEMS IN MAGNETOHYDRODYNAMICS

G. S. GOLITSYN

Moscow State University

Submitted to JETP editor September 21, 1957

J. Exptl. Theoret. Phys. (U.S.S.R.) **34**, 688-693 (March, 1958)

The conditions for potential motion in magnetohydrodynamics are deduced and such motions are investigated. The investigations reduce to the usual hydrodynamical problems. The Prandtl-Mayer problem for a conducting gas in a magnetic field and its generalizations and applications are studied in detail.

IT is of interest to ascertain what problems of magnetohydrodynamics can be solved by classical methods. In this paper only potential motions are studied.

We write the set of equations of magnetohydrodynamics for the case of an ideally conducting medium in the form

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\nabla)\mathbf{v} + \frac{1}{\rho}(\nabla p + [\mathbf{h}\times\operatorname{curl}\mathbf{h}]) = 0,$$

$$\frac{\partial \ln\rho}{\partial t} + \mathbf{v}\nabla\ln\rho + \operatorname{div}\mathbf{v} = 0, \qquad \frac{\partial \mathbf{h}}{\partial t} = \operatorname{curl}[\mathbf{v}\times\mathbf{h}],$$

$$\frac{\partial}{\partial t}\left(\frac{p}{\rho^\gamma}\right) + \mathbf{v}\nabla\left(\frac{p}{\rho^\gamma}\right) = 0. \qquad (1)$$

Here we use the notation $\mathbf{H} = \sqrt{4\pi}\,\mathbf{h}$. The remaining notation is standard. We limit ourselves here to adiabatic processes only.

We seek the conditions of potential flow. For this, employing a well known vector identity, we write the Euler equation in the form

$$\frac{\partial \mathbf{v}}{\partial t} + \nabla\frac{v^2}{2} - [\mathbf{v}\times\operatorname{curl}\mathbf{v}] + \frac{1}{\rho}(\nabla p + [\mathbf{h}\times\operatorname{curl}\mathbf{h}]) = 0$$

and apply the curl operator:

$$\frac{\partial}{\partial t}\operatorname{curl}\mathbf{v} = \operatorname{curl}[\mathbf{v}\times\operatorname{curl}\mathbf{v}] - \operatorname{curl}\left\{\frac{\nabla p}{\rho} + \frac{1}{\rho}[\mathbf{h}\times\operatorname{curl}\mathbf{h}]\right\} \qquad (2)$$

[$\operatorname{curl}(\nabla p/\rho) = \operatorname{curl}\nabla i = 0$, where i is the specific enthalpy]. Equation (2) is satisfied identically for $\operatorname{curl}\mathbf{v} = 0$ if only

$$\operatorname{curl}\left\{\frac{1}{\rho}[\mathbf{h}\times\operatorname{curl}\mathbf{h}]\right\} = \frac{1}{\rho}\operatorname{curl}[\mathbf{h}\times\operatorname{curl}\mathbf{h}] + \left[\nabla\left(\frac{1}{\rho}\right)\times[\mathbf{h}\times\operatorname{curl}\mathbf{h}]\right] = 0.$$

This equation is valid in two cases: first, where $\mathbf{h}\parallel\operatorname{curl}\mathbf{h}$, i.e., for the so-called force-free fields;[1] second, when

$$[\mathbf{h}\,\mathrm{curl}\,\mathbf{h}] = \nabla\Phi \parallel \nabla(1/\rho), \tag{3}$$

where Φ is some function.

We establish the connection between the field and the density for the case of motion in the plane to which the field is everywhere perpendicular. We write the continuity and field equations in the form

$$\partial\rho/\partial t + \rho\,\mathrm{div}\,\mathbf{v} + \mathbf{v}\nabla\rho = 0,$$

$$\partial\mathbf{h}/\partial t + \mathbf{h}\,\mathrm{div}\,\mathbf{v} + (\mathbf{v}\nabla)\,\mathbf{h} = 0. \tag{4}$$

In writing down the latter equation, we have used the fact that $\mathrm{div}\,\mathbf{h} = 0$ and $(\mathbf{h}\nabla)\,\mathbf{v} = 0$, because the field has only a z component, which often has no velocity.

Equations (4) are identical relative to ρ and h; therefore, as a consequence of the adiabaticity of the motion under consideration, we can write

$$h/\rho = b = \mathrm{const}. \tag{5}$$

It is easy to see that for the plane motions being considered, Eq. (3) is satisfied; hence $\Phi = h^2/2$. We also obtain exactly this same case later.

In all the remaining cases, with the exception of the one dimensional, the motion will be vortical. In particular, any three dimensional motion and any motion in the case of a finite conductivity will be vortical, since then h is a function not only of ρ but also of the time and the coordinates.

Upon observance of condition (3), all the classical theorems on vortices are satisfied.

If the fluid is incompressible and the motion is potential, then the velocity potential φ satisfies Laplace's equation. There is nothing essentially new in this case in comparison with ordinary hydrodynamics.

Plane magnetohydrodynamic problems also reduce to the corresponding ordinary gasdynamic problems if the motion is isentropic. We make the following transformation:

$$\frac{1}{\rho}\left(\nabla p + [\mathbf{h}\times\mathrm{curl}\,\mathbf{h}]\right) = \frac{1}{\rho}\nabla\left(p + \frac{h^2}{2}\right)$$
$$= \frac{1}{\rho}\frac{d}{d\rho}\left(p + \frac{h^2}{2}\right)\nabla\rho = c_m^2\,\nabla\ln\rho.$$

The set (1) reduces to two equations

$$\frac{\partial\mathbf{v}}{\partial t} + (\mathbf{v}\nabla)\,\mathbf{v} + c_m^2\,\nabla\ln\rho = 0,$$
$$\frac{\partial\ln\rho}{\partial t} + \mathbf{v}\nabla\ln\rho + \mathrm{div}\,\mathbf{v} = 0, \tag{6}$$

which differ from the usual equations of gas dynamics only in the replacement of the sound velocity c by the effective sound velocity in our medium c_m. The results of Staniukovich[2] and many others can be transformed without any change.

As an example, let us consider a class of motions which does not occur in ordinary gas dynamics; these are the simple, nonstationary generalization of the Prandtl-Mayer motion.[3] We also consider in detail the problem of the motion of a gas close to the base line — the Prandtl-Mayer problem.

If the motion is non-isentropic, then c_m is a function of ρ. It is appropriate to transform to the independent variables t, x, $z = \ln\rho$.

Setting $u = u(z)$ and $v = v(z)$, and writing out Eq. (6) in terms of components, we get

$$u_z(uy_x - y_t + v) - c_m^2 y_x = 0,$$
$$v_z(uy_x - y_t + v) + c_m^2 = 0,$$
$$uy_x - y_t + v + v_z + u_z y_x = 0. \tag{7}$$

Let

$$y = xf_1(z) - tf_2(z) + f_3(z), \tag{8}$$

where f_1, f_2, f_3 are arbitrary functions. Introducing the function

$$F = uf_1 + f_2 + v,$$

we can put the set (7) in another form:

$$F - f_1 c_m^2\frac{dz}{du} = 0, \qquad F + c_m^2\frac{dz}{dv} = 0,$$
$$F + \frac{dv}{dz} + f_1\frac{du}{dz} = 0.$$

From this it is possible to obtain the following relations:

$$du = \pm f_1 c_m dz / \sqrt{1 + f_1^2}, \quad dv = \mp c_m dz / \sqrt{1 + f_1^2},$$
$$f_2 = \pm c_m\sqrt{1 + f_1^2} - uf_1 - v. \tag{9}$$

From the first two equations of (9), we can obtain

$$(du)^2 + (dv)^2 = (c_m d\ln\rho)^2, \tag{10}$$

which is the relation along the characteristics.

If $f_2 \equiv 0$, then the flow will be stationary. The corresponding solution will be of the form of the general solution of Prandtl-Mayer. Then

$$y = xf_1 + f_3. \tag{11}$$

If u and v are given on any line $u = \bar{u}(z)$, $v = \bar{v}(z)$, then we also have

$$f_1 = \frac{\bar{u}\,\bar{v} \pm [(\bar{u}\,\bar{v})^2 - (\bar{u}^2 - c_m^2)(\bar{v}^2 - c_m^2)]^{1/2}}{c_m^2 - \bar{u}^2}. \tag{12}$$

If $f_3 \equiv 0$ also, then we obtain the Prandtl-Mayer solution which describes the rarefaction wave. The generalized solution (11) corresponds to the stationary simple wave.[3] Formula (11) determines a family of characteristics — a pencil of straight lines. These straight lines intersect at each point of the line of flow at an angle equal to the angle of the perturbation, the sine of which is equal to $c_m/|v|$. Along these lines, all the quantities remain constant.

Construction of a simple wave for supersonic flow around a given profile is carried out exactly the same as for ordinary gasdynamics.[3]

In view of its importance and great physical interest, the Prandtl-Mayer problem will now be considered in more detail.

Let the field be perpendicular to the plane of motion. The assumptions relative to the remaining quantities are the same as for the classical statement of the problem.[3]

For stationary motion of the gas near the base line, we can assume that all the quantities depend only on the angle φ, and, writing down Eq. (6) in polar coordinates, we obtain

$$\frac{dv_r}{d\varphi} - v_\varphi = 0, \qquad v_\varphi \frac{dv_\varphi}{d\varphi} + v_r v_\varphi + c_m^2 \frac{d \ln \rho}{d\varphi} = 0,$$

$$v_\varphi \frac{d \ln \rho}{d\varphi} + \frac{dv_\varphi}{d\varphi} + v_r = 0. \tag{13}$$

From the two previous equations, we can obtain the equality

$$(dv_\varphi / d\varphi + v_r)(1 - v_\varphi^2 / c_m^2) = 0.$$

Equating the first bracket to zero, we obtain the solution which describes the homogeneous flow. Equating the second bracket to zero, we obtain $v_\varphi = \pm c_m$. This solution will describe the rarefaction wave. Insofar as this is actually the case, we can verify by carrying out considerations similar to those of the classical problem. We only remark that the field disappears along with the density. We write out the principal results:

$$\frac{1}{2}(v_\varphi^2 + v_r^2) + \int c_m^2 d \ln \rho = i_{0m} = \text{const}; \quad v_\varphi = c_m, \tag{14}$$

$$\varphi = -\int \frac{d(\rho c_m)}{\rho v_r} = \int \frac{dv_r}{v_\varphi} = \int \frac{d\sqrt{2(i_{0m} - i_m) - c_m^2}}{c_m}, \tag{15}$$

where

$$i_m = \int c_m^2 d \ln \rho = \frac{c^2}{\gamma - 1} + \frac{h^2}{\rho} = \frac{c^2}{\gamma - 1} + b^2 \rho$$

has the meaning of a generalized enthalpy. We transform the equation for i_m and c_m with the aid of the Poisson adiabatic

$$\rho c^{-2/(\gamma-1)} = A = \text{const}.$$

Then

$$i_m = \frac{c^2}{\gamma - 1} + Ab^2 c^{2/(\gamma-1)} = \frac{c^2}{\gamma - 1} + \eta \frac{c^{2/(\gamma-1)}}{c_0^{2(2-\gamma)/(\gamma-1)}},$$

$$c_m^2 = c^2 + Ab^2 c^{2/(\gamma-1)} = c^2 + \eta \frac{c^{2/(\gamma-1)}}{c_0^{2(2-\gamma)/(\gamma-1)}},$$

$$\eta = Ab^2 c_0^{2(2-\gamma)/(\gamma-1)}; \tag{16}$$

c_0 is the sound velocity where the gas is at rest. The parameter η, which characterizes the field, has the simple physical meaning of the square of the ratio of the Alfven velocity $V_0 = h_0/\sqrt{\rho_0}$ to the sound velocity $c_0 = \sqrt{\gamma p_0/\rho_0}$, determined where $v = 0$.

We return to the computation of the integral (15). We introduce the independent variable $x = c/c_0$ and, carrying out differentiation with respect to x in the integrand, and taking (6) into account, we get

$$\varphi = -\frac{1}{\gamma - 1} \tag{17}$$

$$\times \int \frac{[\gamma + 1 + 3\eta x^{2(2-\gamma)/(\gamma-1)}]\, dx}{(1 + \eta x^{2(2-\gamma)/(\gamma-1)}) \left[\frac{2}{\gamma - 1} - \frac{\gamma + 1}{\gamma - 1} x^2 + \eta (2 - 3x^{2/(\gamma-1)})\right]}$$

This integral is taken in two limiting cases $\eta = 0$ and $\eta \to \infty$. In the first case we obtain the well known result:

$$x = \sqrt{\frac{2}{\gamma + 1}} \cos \sqrt{\frac{\gamma - 1}{\gamma + 1}}\, \varphi. \tag{18}$$

In the other limiting case, taking the integral and solving for x, we obtain

$$x = \left(\frac{2}{3} \cos^2 \frac{\varphi}{\sqrt{3}}\right)^{(\gamma-1)/2}. \tag{19}$$

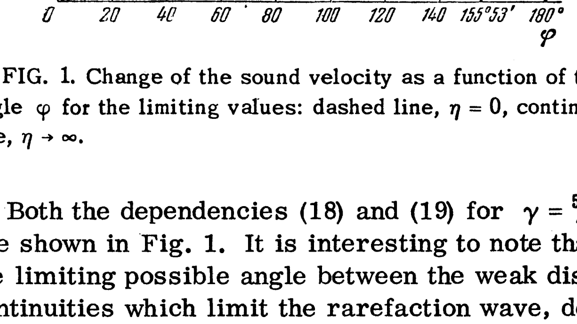

FIG. 1. Change of the sound velocity as a function of the angle φ for the limiting values: dashed line, $\eta = 0$, continuous line, $\eta \to \infty$.

Both the dependencies (18) and (19) for $\gamma = 5/3$ are shown in Fig. 1. It is interesting to note that the limiting possible angle between the weak discontinuities which limit the rarefaction wave, decrease from $(\pi/2)\sqrt{(\gamma+1)(\gamma-1)}$ (220° 27′ for $\gamma = 1.4$ and 180° for $\gamma = 5/3$) up to $\pi\sqrt{3/2} = 155° 53'$. Consequently, the maximum possible turning angle of the vector velocity in the rarefaction wave decreases to 65° 53′ against 130° 27′ or 90° in the absence of a field.

Calculations in the intermediate case $0 < \eta < \infty$ are carried out for $\gamma = 5/3$. Then

$$\varphi = -\int \frac{(4 + 4.5\eta x)\, dx}{\sqrt{(1 + \eta x)\,[3 - 4x^2 + \eta\,(2 - 3x^3)]}}\,. \qquad (17')$$

The question arises as to the limits of change of x. The upper limit ought to be zero — the boundary with a vacuum. The lower limit is determined by the requirement of the reality of the integrand, i.e., the root x_0 of the polynomial in square brackets. Upon change in η over the infinite range, x_0 changes from $\sqrt{3/4} = 0.86603$ to $(2/3)^{1/3} = 0.87358$; i.e., the change is very small. However, since x_0 determines the limit of integration, then the error ϵ in the determination of x_0 itself gives an error of the order of $\sqrt{\epsilon}$ in the integration of our expression; therefore, it is necessary to determine x_0 with a sufficiently great degree of accuracy. We give a table of the dependence on η of the root x_0 by the equation

$$3 - 4x^2 + \eta(2 - 3x^3) = 0,$$

where η was computed from the given x_0.

η	x_0	η	x_0
0	0.86603	1.9560	0.8710
0.15061	0.8670	3.8332	0.8720
0.35963	0.8680	12.177	0.8730
0.65965	0.8690	94.115	0.8735
1.12694	0.8700	∞	0.87358

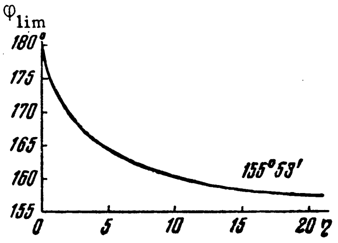

FIG. 2. Dependence on $\eta = H_0^2/4\pi\,\rho_0 c_0^2$ of the limiting possible angle between weak discontinuities.

The integral (17′) can, in principle, be expressed by elementary and elliptic functions; however, the method of numerical integration is much less difficult: from 0 to 0.72 the integral is computed by Simpson's rule;[4] thereafter, the integrand is represented in the form $(x_0 - x)^{-1/2} h(x)$ and $h(x)$ is expanded in a Taylor series, after which the integration becomes elementary. For calculation with accuracy up to 5 places, it is always sufficient to find only four terms in the expansion. It is not possible to apply numerical methods directly to the calculation of the entire integral, since it is not a proper one. The results of the integration are given in Fig. 2.

Finally, when the dependence $c/c_0 = x = x(\varphi)$ is found, it is of no difficulty to find other quantities as functions of the angle φ. Thus, for $\gamma = 5/3$,

$$\frac{\rho}{\rho_0} = \left(\frac{c}{c_0}\right)^{2/(\gamma-1)} = x^3;\quad \frac{p}{p_0} = \left(\frac{c}{c_0}\right)^{2\gamma/(\gamma-1)} = x^5;$$

$$\frac{c_m}{c_{m_0}} = x\sqrt{\frac{1+\eta x}{1+\eta}}\,;\quad \frac{v}{c_0} = \sqrt{3\,(1 - x^2) + 2\eta\,(1 - x^3)};$$

$$\chi = \varphi + \arctan\left[x\left(\frac{1+\eta x}{3 - 4x^2 + \eta\,(2 - 3x^3)}\right)^{1/2}\right].$$

Graphs are given in Figs. 3 and 4 for these quantities for $\eta = 1.127$. The dashes show the change in the corresponding quantities in the absence of field.

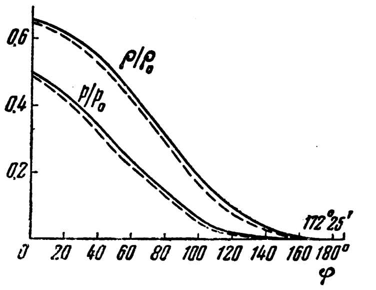

FIG. 3. Change of the density and pressure in a rarefaction wave in dependence on the angle φ for $\eta = 1.12694$.

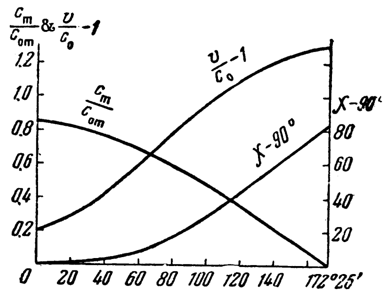

FIG. 4. Change in the magnetogasdynamical sound velocity, velocity and angle of its turn χ in dependence on the angle φ for $\eta = 1.2694$.

The results obtained can be applied without any principal difficulties to the problem of the flow angle, to the consideration of plane stationary motion.

In conclusion, I express my thanks to my director K. P. Staniukovich for suggesting the topic and for discussion of the results. I also express my gratitude to A. I. Morozov for his help in numerous suggestions for the solution of the Prandtl-Mayer problem.

[1] S. Lundquist, Arkiv. f. Fysik **5**, 297 (1952).
[2] K. P. Staniukovich, Dokl. Akad. Nauk SSSR **112**, 595 (1957).

[3] L. D. Landau and E. M. Lifshitz, Механика сплошных сред (Mechanics of Continuous Media), Secs. 101, 107, GITTL, 1954.

[4] G. M. Fikhtengol'ts, Курс дифференциального и интегрального исчисления (Course in Differential and Integral Calculus), Vol. 2, GITTL, 1951.

Translated by R. T. Beyer
130

SOVIET PHYSICS JETP VOLUME 34 (7), NUMBER 3 SEPTEMBER, 1958

FRACTIONAL PARENTAGE COEFFICIENTS FOR THE WAVE FUNCTION OF FOUR PARTICLES

G. I. ZEL'TSER

Leningrad Agricultural Institute

Submitted to JETP editor September 26, 1957

J. Exptl. Theoret. Phys. (U.S.S.R.) **34**, 694-699 (March, 1958)

General formulas are obtained for the fractional parentage expansion of type $\langle n | n-2, 2 \rangle$ for the wave function of four nucleons in j-j coupling, with inclusion of effects of isotopic spin. The normalized fractional parentage coefficients (both for nonequivalent and for equivalent particles) are expressed in terms of the Hope χ functions, i.e., essentially in terms of Wigner 9j symbols. The results can also be applied directly to the case of $\tilde{L}S$ coupling in atoms.

WHEN two-particle interactions are taken into account in the individual-particle nuclear model it turns out to be necessary to calculate the matrix elements of symmetric two-particle operators (of the type $G = \sum_{i<k} g_{ik}$) between antisymmetric states of n particles with prescribed total angular momentum and total isotopic spin. The wave functions of these states are constructed by vector composition from the functions for the individual particles. When the number n of particles is larger than two, the functions obtained by vector composition are not automatically antisymmetric, so that subsequent antisymmetrization is necessary.

In calculating the matrix elements of operators of the type G by the methods of the Racah algebra of tensor operators[1] it is convenient to possess a representation of the antisymmetric wave functions of n particles in the form of an expansion in terms of functions formed by vector composition from antisymmetric functions of the first $n-2$ particles and of the last two particles. The coefficients in this expansion are called fractional parentage coefficients of the type $\langle n | n-2, 2 \rangle$. Together with the analogous coefficients of the type $\langle n | n-1, 1 \rangle$, they were first introduced by Racah[2] for the case of equivalent electrons.

For small values of n general expressions for the fractional parentage coefficients can be obtained in terms of the Racah coefficients (Wigner 6j symbols) and more complicated invariants formed from the Clebsch-Gordan coefficients. The general expression for the coefficients $\langle 3 | 2, 1 \rangle$ for three equivalent or nonequivalent nucleons with inclusion of isotopic spin effects was given by Redlich;[3] Schwartz and de-Shalit[4] gave the formula for the case of four equivalent particles in the j-j coupling scheme. The problem of the fractional parentage coefficients $\langle 4 | 2, 2 \rangle$ is dealt with in a paper by Jahn[5] (see also the related paper of Englefield[6]). In this paper the fractional parentage expansion is indicated for a function of arbitrary symmetry (belonging to an arbitrary representation of the permutation group, but depending on only one type of