iting values for their cross sections.

Similar considerations hold also for K mesons. This means that at sufficiently high energies (of the order of 10^{10} ev, if electromagnetic interactions are not taken into account) the K^+, K^-, K^0, and $\overline{K}^0$ mesons should all have the same total cross section. Similarly, the π^+ and π^- cross sections should also approach equality as $E \to \infty$.[7] If we apply our result to hyperons, we obtain the following.

(1) The Λ and anti-Λ (i.e., the $\overline{\Lambda}$) have the same cross section $\sigma_\Lambda(\infty)$.

(2) The Σ^+, Σ^-, Σ^0, $\overline{\Sigma}^+$, $\overline{\Sigma}^-$, and $\overline{\Sigma}^0$ cross sections approach the common value $\sigma_\Sigma(\infty)$ as $E \to \infty$.

(3) The Ξ^-, Ξ^0, $\overline{\Xi}^-$, $\overline{\Xi}^0$ have the same cross section $\sigma_\Xi(\infty)$.

We note also that from (2), (2′), and (9) it follows that at large energies the main contributions to $D_+(E)$ and $D_-(E)$ are proportional to E and differ only in sign. This together with (9) shows that the differential cross sections for elastic scattering of nucleons and antinucleons by nucleons through the angle zero approach equality as $E \to \infty$.

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CALCULATION OF COORDINATE PROBABILITIES BY GIBBS METHOD

V. B. MAGALINSKII and Ia. P. TERLETSKII

Moscow State University

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Gibbs' statistical method is used to derive general formula which permits one to determine fully the stationary probability density as well as the transition probability density in a nonstationary process for an arbitrary generalized coordinate, provided the behavior of the mean value of the latter is known in the presence of (or after turning on) additional forces acting in the direction of this coordinate.

IT is well known that, using general methods of statistical mechanics, we can derive exact relationships which enable us to reduce a calculation of fluctuations and correlations of various quantities (among them time correlations) to a determination of average values of these quantities in the presence of (or after turning on) additional constant forces.[1-4]

It was shown in Ref. 5 that all the principal momenta determined in fluctuation theory, as well as in the theory of Brownian motion, can be computed by this method.

It will be shown below that not only momenta but also the corresponding coordinate probability densities can be exactly determined, provided one knows the behavior of these coordinates in the presence of additional constant forces or forces which were turned on at the initial instant.

1. STATIONARY PROBABILITY DENSITY OF A GIVEN VALUE OF A COORDINATE

Let X be the aggregate of all the canonical variables of the system, $q(X)$ some generalized coordinate (from now on we confine ourselves to the one-dimensional case to simplify the derivation), and $w(X)$ the stationary probability density in phase space. Then the probability density for $q(X)$ to have a given value q is

$$W(q) = \int_{(X)} \delta\{q - q(X)\} w(X)\, dX = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{i\xi q}\int_{(X)} e^{-i\xi q(X)} w(X)\, dX\, d\xi. \qquad (1)$$

In the case of a canonical distribution we have $w(X) = \exp\{\Psi - H(X)/\Theta\}$, and consequently

$$\int_{(X)} e^{-i\xi q(X)} w(X)\, dX = \int_{(X)} \exp\{[\Psi - H(X) - aq(X)]/\Theta\}\, dX = e^{(\Psi - \Psi(a))/\Theta} = e^{-\Delta\Psi(a)/\Theta}, \qquad (2)$$

where $a = i\xi\Theta$, and Ψ and $\Psi(a)$ are determined from the conditions

$$e^{-\Psi/\Theta} = \int_{(X)} e^{-H(X)/\Theta}\, dX,$$

$$e^{-\Psi(a)/\Theta} = \int_{(X)} \exp\{-[H(X) + aq(X)]/\Theta\}\, dX. \qquad (3)$$

Obviously Ψ has the meaning of the free energy of the original system, and $\Psi(a)$ has the meaning of the free energy of the system in the presence of an additional constant force $-a$ acting in the direction of the coordinate $q(X)$. Thus, according to (1) and (2)

$$W(q) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \exp\left\{i\xi q - \frac{\Delta\Psi(a)}{\Theta}\right\} d\xi. \qquad (4)$$

This formula makes it possible to compute the sought probability density $W(q)$, provided we know $\Delta\Psi(a)$ as a function of the force $a = i\xi\Theta$.

According to (3), the mean value of the coordinate $q(X)$ over the ensemble with an additional force $-a$ is $\overline{q}^a = \partial\Psi(a)/\partial a$, hence

$$\Delta\Psi(a) = \int_0^a \overline{q}^\alpha d\alpha. \qquad (5)$$

Formula (5) enables us to determine $\Delta\Psi(a)$ from empirical data which give the dependence of the mean coordinate on the external force a in an isothermal process.

Let us illustrate what we said above using two simple examples.

(a) Coordinate probability of a harmonic oscillator. The equilibrium condition for an oscillator in the presence of an additional external α has the form

$$k\overline{q}^\alpha + \alpha = 0,$$

hence $\overline{q}^\alpha = -\alpha/k$ (k is the elastic coefficient). According to (5)

$$\Delta\Psi(a) = \int_0^a \overline{q}^\alpha d\alpha = -a^2/2k = (\Theta^2/2k)\,\xi^2. \qquad (5')$$

Substituting (5′) into (4) and integrating we get

$$W(q) = \sqrt{\frac{k}{2\pi\Theta}}\exp\left(-\frac{kq^2}{2\Theta}\right),$$

i.e., the Boltzmann distribution.

(b) Probability of volume V occupied by an ideal gas. In this case $q = V$, and $\overline{V}^\alpha$ is found from the equation of state $\overline{V}^\alpha = N\Theta/(p+\alpha)$

$$\Delta\Psi(\alpha) = \int_0^a \overline{V}^\alpha d\alpha = N\Theta\ln\left(1 + \frac{a}{p}\right) = N\Theta\ln\left(1 + i\frac{\overline{V}}{N}\xi\right),$$

where $\overline{V} = N\Theta/p$ is the equilibrium value of the volume. Finally we find

$$W(V) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{e^{iV\xi}}{(1 + i\overline{V}\xi/N)^N}\, d\xi = \frac{1}{V}\left(N\frac{V}{\overline{V}}\right)^N \frac{\exp(-NV/\overline{V})}{(N-1)!}. \qquad (4'')$$

In the case of small fluctuations ($|V - \overline{V}|/\overline{V} \ll 1$), (4″) becomes a Gaussian distribution.

2. TRANSITION PROBABILITY DENSITY

Let $W(q, t; q_0, t_0)$ be the probability density of transition of the system from the point q_0 at the instant t_0 to the point q at the instant t, and let $W_0(q_0)$ be the probability density of the initial value of the coordinate. Further, let us introduce the notation:

$$W_1(q, t;\ q_0, t_0) = W(q, t;\ q_0, t_0)\, W_0(q_0). \qquad (6)$$

The function $W(q, t; q_0, t_0)$ is uniquely determined by its characteristic function $\varphi_1(\xi, t; \eta, t_0)$ so that

$$W_1(q, t; q_0, t_0) = (2\pi)^{-2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{i\xi q + i\eta q_0}\varphi_1(\xi, t; \eta, t_0)\, d\xi\, d\eta. \qquad (7)$$

The characteristic function φ_1 is in turn determined if all the moments of the quantities q and q_0 are known, namely:

$$\varphi_1(\xi, t; \eta, t_0) = \sum_{j,k} \overline{q_t^j q_0^k} (-i\xi)^j (-i\eta)^k / j!k!, \quad (8)$$

where

$$\overline{q_t^j q_0^k} = \iint W_1(q, t; q_0, t_0)\, q^j q_0^k \, dq\, dq_0. \quad (9)$$

To compute these momenta, we make use of the general statistical method of Gibbs.

If we let X^0 be the aggregate of all the canonical variables at the initial instant, and X^t be the canonical variable of the same phase points at time t, then

$$\overline{q_t^j q_0^k} = \int_{(X_0)} q^j(X^t)\, q^k(X^0)\, w(X^0)\, dX^0, \quad (10)$$

where $w(X^0)$ is the probability density of the initial phase X^0 and X^t are assumed to be expressed in terms of X^0, t, and t_0 in accordance with the canonical equations of mechanics with the Hamiltonian function $H(X): X^t = X(X^0, t_0, t)$. In the case of the Gibbs canonical distribution

$$w(X^0) = \exp\{\Psi_0 - H(X^0)/\Theta\}.$$

Let us introduce (so far, purely formally) an auxiliary function $Z(\alpha, t; \beta, t_0)$, which we define as

$$Z(\alpha, t; \beta, t_0) = \int_{(X^0)} \exp\{-[H(X^0) + \alpha q(X^t) + \beta q(X^0)]/\Theta\}\, dX^0. \quad (11)$$

By simple differentiation we find

$$\overline{q_t^j q_0^k} = \frac{1}{Z_0}(-\Theta)^{j+k}\left[\frac{\partial^{j+k} Z(\alpha, t; \beta, t_0)}{\partial\alpha^j \partial\beta^k}\right]_{\alpha=\beta=0}, \quad (12)$$

where

$$Z_0 \equiv Z(0, t; 0, t_0) = \int_{(X^0)} e^{-H(X^0)/\Theta} dX^0 = e^{-\Psi_0/\Theta}. \quad (13)$$

Substituting (12) into (8) we find the characteristic function

$$\varphi_1(\xi, t; \eta, t_0) = \frac{1}{Z_0}\sum_{j,k}\left[\frac{\partial^{j+k} Z(\alpha, t; \beta, t_0)}{\partial\alpha^j\partial\beta^k}\right]_{\alpha=\beta=0} \times \frac{(i\xi\Theta)^j (i\eta\Theta)^k}{j!k!} = \frac{1}{Z_0} Z(a, t; b, t_0), \quad (14)$$

where the notation $i\xi\Theta = a$, $i\eta\Theta = b$ is introduced. We now determine $Z(\alpha, t; \beta, t_0)$. Writing

$$Z(\alpha, t; \beta, t_0) = \exp\{-\Psi(\alpha, t; \beta, t_0)/\Theta\}, \quad (13')$$

we have by virtue of (11)

$$\int_{(X_0)} \exp\{[\Psi(\alpha, t; \beta, t_0) - H(X^0) - \alpha q(X^t) - \beta q(X^0)]/\Theta\}\, dX^0 = 1 \quad (15)$$

for any t.

Differentiating (15) with respect to time we get

$$\partial\Psi(\alpha, t; \beta, t_0)/\partial t = \alpha \dot{\overline{q}}^{\alpha,\beta}, \quad (16)$$

$$\Psi(\alpha, t; \beta, t_0) = \alpha(\overline{q}_t^{\alpha,\beta} - \overline{q}_0^{\alpha,\beta}) + \Psi(\alpha, t_0; \beta, t_0),$$

where

$$\overline{q}_t^{\alpha,\beta} = \int_{(X^0)} q(X^t) \exp\{[\Psi(\alpha, t; \beta, t_0) - H(X^0) - \alpha q(X^t) - \beta q(X^0)]/\Theta\}\, dX^0. \quad (17)$$

Quantity $\overline{q}_t^{\alpha,\beta}$ can be interpreted as a macroscopic coordinate which is obtained as a result of averaging of $q(X)$ over the ensemble separated from the original equilibrium ensemble by the following process:

(a) Before the time $t = t_0$, the system was acted on by a constant force $-(\alpha + \beta)$ so that the system was in equilibrium, having an equilibrium coordinate $\overline{q}_0^{\alpha+\beta}$ and a corresponding free energy

$$\Psi(\alpha, t; \beta, t_0) = \Psi_0 + \int_0^{\alpha+\beta} \overline{q}^\lambda d\lambda \quad (18)$$

[see (5)].

(b) Starting from the time t_0, the system continues to be acted on only by the constant force $-\alpha$; as a result, the system comes out of equilibrium and undergoes a transition.

Thus $\overline{q}_t^{\alpha,\beta}$ can be interpreted as an average displacement of the coordinate $q(X)$ during the time $t - t_0$ under the action of a constant force $-\alpha$. According to (a), the initial value of the coordinate is $\overline{q}_0^{\alpha,\beta} = \overline{q}_0^{\alpha+\beta}$.

Substituting (13), (13′), (14), (16), and (18) into (7), and taking (6) into account, we obtain the final formula for the transition probability density:

$$W(q, t; q_0, t_0) \quad (19)$$
$$= \frac{1}{(2\pi)^2 W_0(q_0)} \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \exp\left\{i\xi q + i\eta q_0 - \frac{\Delta\Psi(a, t; b, t_0)}{\Theta}\right\} d\xi\, d\eta,$$

where

$$\Delta\Psi(a, t; b, t_0) = a(\overline{q}_t^{a,b} - \overline{q}_0^{a+b}) + \int_0^{a+b} \overline{q}_0^\lambda d\lambda. \quad (20)$$

The initial probability density is found from formulae (4) and (5).

Thus, if $\overline{q_t^{a,b}}$ and $\overline{q^\lambda}$ are known from the phenomenological equations of motion or from the empirical data, we can obtain an expression for $W(q,t;\ b,t_0)$ by substituting the value of $\Delta\Psi(a,t;\ b,t_0)$ found from (20) into (19) and integrating.

In this way the suggested method makes it possible to solve a much wider class of problems than can be done with the known methods of the theory of Brownian motion.

If the system is subjected to the action of forces independent of the coordinate q, the transition probability density depends only on the magnitude of the displacement $q - q_0$, i.e.,

$$W(q,\ t;\ q_0,\ t_0) = W(q - q_0;\ t - t_0).$$

Using arguments analogous to those cited above, we obtain the following formula for the transition probability density:

$$W(q - q_0;\ t - t_0) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \exp\{i\xi(q - q_0) - \Delta\Psi(a,\ t;\ t_0)/\Theta\}\, d\xi, \quad (19')$$

$$\Delta\Psi(a,\ t;\ t_0) = a\overline{\Delta q_t^a},\quad a = i\xi\Theta, \quad (20')$$

and $\overline{\Delta q_t^a}$ is the mean displacement of the coordinate $q(X)$ during the time $t - t_0$ under the action of an additional constant force $-a$.

Let us illustrate the method developed above with a simple example of a Brownian particle in the gravitational field.

According to the above, the mean displacement $\overline{\Delta q_t^a}$ which we are interested in obeys the phenomenological equation

$$m\ddot{\overline{\Delta q_t^a}} + \gamma\dot{\overline{\Delta q_t^a}} = g - a$$

with homogeneous initial conditions. (Here m is mass, γ the coefficient of friction, and g the acceleration due to gravity.) The solution of this equation for $t = 0$ is

$$\overline{\Delta q_t^a} = \frac{g-a}{m\varkappa}t - \frac{g-a}{m\varkappa^2}(1 - e^{-\varkappa t}),\quad \varkappa = \frac{\gamma}{m}. \quad (21)$$

Substituting the value of $\Delta\Psi(a,t;\ t_0)$ found from (21) and (20′) into (19′) and carrying out the integration we easily find

$$W(q - q_0;\ t) = \sqrt{\frac{m\varkappa^2}{4\pi\Theta f(t)}}\exp\left\{-\frac{m\varkappa^2[q - Q(t)]^2}{4\Theta f(t)}\right\} \quad (22)$$

Here

$$Q(t) = \frac{g}{m\varkappa}t - \frac{g}{m\varkappa^2}(1 - e^{-\varkappa t})$$

is the mean coordinate of the particle at the instant t,

$$f(t) = \varkappa t - (1 - e^{-\varkappa t}).$$

Formula (22) describes a Gaussian distribution which spreads with time; its center moves according to the laws of mechanics.

It should be mentioned that the transition probability density $W(q - q_0;\ t)$, (22), which was obtained above on the basis of general principles of statistical mechanics, also satisfies a corresponding equation of Einstein — Fokker — Planck

$$\frac{\partial W}{\partial t} = D(t)\left(\frac{\partial^2 W}{\partial q^2} - \frac{g}{\Theta}\frac{\partial W}{\partial q}\right), \quad (23)$$

being its source function. The diffusion coefficient $D(t)$ depends on time and is equal to the mean velocity of the particle when it is acted on by an additional constant external force equal to unity.

For $t \gg \kappa^{-1}$ (diffusion mode) $D(t) \to \Theta/\gamma$ and equation (23) takes on its usual form.

For $t \ll \kappa^{-1}$ (inertial mode) formula (22) gives

$$W(q - q_0;\ t)\, dq = \sqrt{\frac{m}{2\pi\Theta}}\exp\left(-\frac{mv^2}{2\Theta}\right) dv \quad (24)$$

[where $v = (q - q_0)\ t$], i.e., Maxwell distribution corresponding to the fact that before the collisions take place, the particle undergoes inertial motion.

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