

## ENERGY SPECTRUM OF A BOSE GAS

V. M. ELEONSKII and P. S. ZYRIANOV

Polytechnic Institute of the Ural

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RECENTLY papers have appeared in the literature<sup>1,2</sup> concerning the energy spectrum of a system of Bose particles. This problem is of great interest in connection with the phenomenon of superfluidity of He<sup>4</sup> at low temperatures. It can be shown that the basic results of these papers essentially do not go beyond the framework of the method of collective interactions or the theory of perturbations in the second quantization as developed by Bogoliubov.<sup>3</sup> We investigate below the problem of finding the energy spectrum of a system of Bose particles by means of expanding the interaction kernels in series of moments. In some cases this method proves to be more convenient than the method of expanding the interaction kernels into a Fourier series.

In the second quantization the Hamiltonian of the system has the form:

$$H = \frac{\hbar^2}{2m} \int \nabla \psi^* \nabla \psi d\mathbf{r} + \frac{1}{2} \int \rho(\mathbf{r}) G(\mathbf{r} - \mathbf{r}') \rho(\mathbf{r}') d\mathbf{r} d\mathbf{r}',$$

where  $\rho(\mathbf{r}) = \psi^*(\mathbf{r})\psi(\mathbf{r})$ . Introducing the new variable  $\xi = \mathbf{r} - \mathbf{r}'$ , we rewrite the operator of the potential energy in the form

$$\frac{1}{2} \int \rho(\mathbf{r}) G(\xi) \rho(\mathbf{r} - \xi) d\mathbf{r} d\xi,$$

expanding  $\rho(\mathbf{r} - \xi)$  in a series of powers of  $\xi$  we obtain

$$H = -\frac{\hbar^2}{2m} \int \nabla \psi^* \nabla \psi d\mathbf{r} + \frac{1}{2} \sum_{n=0}^{\infty} \frac{(-n)^n}{n!} \int \rho(\mathbf{r}) G(\xi) (\xi \nabla)^n \rho(\mathbf{r}) d\mathbf{r} d\xi. \quad (1)$$

For weakly excited states one can replace in the Bose operator  $\psi$  those operators, which refer to the ground state, by the numbers  $\sqrt{\rho}$  ( $\rho$  - particle density, with the volume of the system taken as equal to unity); then

$$\psi \sim \sqrt{\rho} + \sum_{\neq 0} a_k e^{ik\mathbf{r}}, \quad (2)$$

where  $a_k$  are Bose operators. Substituting (2) in (1), we obtain, with an accuracy to terms quadratic in the operators,

$$H = H_0 + \sum_{k>0} \left( \frac{\hbar^2 k^2}{2m} + \frac{1}{2} A(k) \right) (a_k^+ a_k + a_{-k}^+ a_{-k}) + \sum_{k>0} 2 (B(k) a_k a_{-k} + C(k) a_k^+ a_{-k}^+), \quad (3)$$

where

$$H_0 = \frac{1}{2} G_0(0) \rho^2 - \rho \sum_{k>0} \left\{ \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} G_n(k) + G_0(0) \right\},$$

$$A(k) = 4G_0(0)\rho + 2\rho \sum_{n=1}^{\infty} \left\{ \frac{(-1)^n}{n!} G_n(k) + \frac{1}{n!} G_n(k) \right\},$$

$$B(k) = \rho G_0(0) + \rho \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} G_n(k), \quad C(k) = \rho G_0(0) + \rho \sum_{n=1}^{\infty} \frac{1}{n!} G_n(k), \quad G_n(k) = \int G(\xi) (ik\xi)^n d\xi.$$

In the case of the central forces  $A(k)/2 = 2B(k) = 2C(k)$ .

The quadratic expression (3), after diagonalization with respect to the central forces, gives the following eigenvalues for the energy:

$$E_k = \left\{ \left( \frac{\hbar^2 k^2}{2m} \right) \left[ \frac{\hbar^2 k^2}{2m} + 4\pi\rho \sum_{l=0}^{\infty} \frac{(-1)^l}{(2l+1)!} k^{2l} \int_0^{\infty} G(\xi) \xi^{2l+2} d\xi \right] \right\}^{1/2}. \quad (4)$$

In the case of completely hard spheres with diameter  $a$  we assume for the repulsive forces

$$G(\xi) = (\hbar^2/ma^2) \delta(\xi - a). \quad (5)$$

Then Eq. (4) gives the results of Brueckner and Sawada<sup>1</sup>

$$E(x) = \left( \frac{\hbar^2}{2ma^2} \right) x \left[ x^2 + 2\lambda^2 \frac{\sin x}{x} \right]^{1/2}, \quad (6)$$

where  $x = ka$ ,  $\lambda^2 = 8\pi\rho a^3$ .

The values of  $\lambda^2$  were determined in Ref. 1 by an integral equation. For the values of  $a$  and  $\rho$  assumed there for liquid helium, its value is close to our value of  $\lambda^2$ , since in Ref. 1 the dependence of  $\lambda^2$  on  $\rho a^3$  is almost linear. The first term of the expansion of  $\sin x/x$  in (6) leads to the results found by Lie, Huang, and Yang with the use of a pseudopotential for completely elastic spheres with diameter  $a$ .

In Refs. 1 and 2 the weak forces of attraction between the helium atoms were neglected. In the simplest case it is not difficult to take them into account by adding to (5) the potential for the forces of attraction. For the sake of simplicity we use the following expression instead of (5):

$$G(\xi) = (\hbar^2/ma^2) \delta(\xi - a) - U_0 \eta(\xi), \quad (7)$$

where

$$\eta(\xi) = \begin{cases} 0, & \xi < a \\ 1, & a < \xi < b \\ 0, & \xi > b; \end{cases}$$

$U_0$  is the depth of potential well, and  $b - a = d$  is the width of potential well. Knowing the binding energy  $E_0$  per atom of  $\text{He}^4$  at a temperature close to the absolute zero (the experiment gives for this energy the value  $\sim \kappa 7^\circ\text{K}$ , where  $\kappa$  is the Boltzmann's constant), and using the known results of quantum mechanics, a relationship can be established between the binding energy and the width and depth of the potential well. Then only one parameter (with the exception of  $\rho$  and  $a$ ) will appear in the energy spectrum of the system. This relation is expressed by the equation

$$d = d_0 \left( \frac{U_0}{2E_0} - 1 \right)^{-1/2} \tan^{-1} \left( - \sqrt{\frac{U_0}{2E_0} - 1} \right),$$

$$d_0 = \left( \frac{mE_0}{\hbar^2} \right)^{-1/2}.$$

Use of the potential (7) instead of (6) leads to the following energy spectrum:

$$E(x) = \left( \frac{\hbar^2}{2ma^2} \right) x \left[ x^2 + 2\lambda^2 \frac{\sin x}{x} - \gamma^2 x^{-3} \left( \sin \left( 1 + \frac{d}{a} \right) x - \sin x \right) + \gamma^2 \left( 1 + \frac{d}{a} \right) x^{-2} \cos \left( 1 + \frac{d}{a} \right) x - \gamma^2 x^{-2} \cos x \right]^{1/2},$$

$$\gamma^2 = 4\pi U_0 \rho a^5 m / \hbar^2. \quad (8)$$

This form of the spectrum yields a better approximation to the experimental curve  $E(x)$  for  $\text{He}^4$  than equation (6), as is shown by an analysis of the relationship (8).

Note added in proof (February 21, 1957). Taking exchange forces into account leads to a change of parameter  $\lambda^2 = 16\pi\rho a^3$ . In the limit of  $k \rightarrow 0$  this change is equivalent to the introduction of an effective mass  $m^* = m/2$ .

<sup>1</sup>K. A. Brueckner and K. Sawada, Phys. Rev. **106**, 1128 (1957).

<sup>2</sup>Lie, Huang, and Yang, Phys. Rev. **106**, 1135 (1957).

<sup>3</sup>N. N. Bogoliubov, J. Phys. (U.S.S.R.) **2**, 23 (1947).

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### THE ANTIFERROMAGNETIC ORIENTATION OF MAGNETIC MOMENTS\* IN THE ALLOY $\text{Ni}_3\text{Fe}$

M. V. DEKHTIAR

Moscow State University

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THE influence of ordering the atoms on the magnetic structure of the alloy  $\text{Ni}_3\text{Fe}$  has been studied by Shull and Wilkinson<sup>1</sup> by means of neutron diffraction. These authors discovered the effect of the magnetic state of the alloy on the intensity of the superlattice lines which are observed as a result of the atomic ordering. The intensity of these lines is weakened when the neutron diffraction is observed on a magnetized sample. On this basis, the authors have concluded that the rearrangement of the atoms is accompanied by the appearance of

a magnetic superlattice. The nature of the latter, however, remained unclear.

We have shown earlier<sup>2</sup> that the observed anomalies accompanying the variation of the magnetic properties with temperature permit the detection of structural changes in single phase alloys caused by ordering processes. In the present paper the temperature dependence of the saturation magnetization  $I_S$ , the remanent magnetization  $I_R$  and of the coercive force  $H_C$  of the alloy  $\text{Ni}_3\text{Fe}$  is investigated over a wide temperature range.

The investigated sample, with a diameter of 4.5 mm, was enclosed in a thin quartz tube, evacuated to  $10^{-4}$  mm Hg, and quenched together with it in water after two hours of annealing at  $1200^\circ\text{C}$ . The experimental points in the region from  $300$  to  $600^\circ\text{C}$  were obtained for closely spaced values of the temperature (at intervals of  $10$  to  $15^\circ\text{C}$ ) in order to determine more accurately the shape of the temperature dependence of  $I_S$ ,  $I_R$  and  $H_C$ . The sample was kept one hour at each temperature before taking a reading.