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Translated by D. ter Haar
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SOVIET PHYSICS JETP VOLUME 34 (7), NUMBER 4 OCTOBER, 1958

QUANTUM THEORY OF THE HIGH FREQUENCY CONDUCTIVITY OF METALS

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Submitted to JETP editor November 26, 1957

J. Exptl. Theoret. Phys. (U.S.S.R.) **34**, 969-983 (April, 1958)

A quantum theory of conductivity is developed for metals placed in a high frequency electromagnetic field and a constant magnetic field. The dispersion law of the conduction electrons and the manner in which they are reflected from the surface are assumed to be arbitrary. It is shown that the amplitude of the quantum oscillations in the high frequency case is in general considerably larger than in the static case. The quantum oscillations considered here have not yet been observed experimentally.

1. INTRODUCTION

As is well known, in developing an electronic theory of the conductivity of metals it is possible, to a high degree of accuracy, to limit oneself to a semi-classical investigation which does not take account of the quantization of the energy levels of the conduction electrons. This possibility is related to the fact that, for all real cases, the level splitting $\Delta\epsilon$ is considerably smaller than the limiting Fermi energy ϵ_0 of the electrons. In order to have $\Delta\epsilon \sim \epsilon_0$ it would be necessary to have a magnetic field $H \sim \epsilon_0/\mu \sim 10^9$ oersted, or a metallic sample of width $d \sim \hbar/\sqrt{2m^*\epsilon_0} \sim 10^{-8}$ cm (m^* is the effective mass of an electron and $\mu = e\hbar/m^*c$).

However a semi-classical investigation does not permit one to look into an important effect generally absent in the classical case — purely quantum-mechanical oscillations of the conductivity. At the same time the study of these oscillations is of considerable interest, particularly because it gives a convenient method of reconstructing the form of the Fermi surface from experimental data.[1-3]

The papers of I. M. Lifshitz and Kosevich[2,3] appear to be the only ones in which diamagnetic oscillations of the static conductivity of bulk metal in a constant magnetic field were arrived at in a consistent manner. The essential assumption in their papers was that the current density in the metal was isotropic, which permitted them to regard the statistical operator as not depending explicitly on the coordinates.

In the present work a theory is developed for the general case in which there is spatial anisotropy due to a non-stationary electric field. It is assumed that the anisotropy is substantial, that is, that its characteristic dimension — the skin depth δ — is small in comparison with the Larmor radius r and with the electron mean-free-path l (the so-called anomalous skin-effect), so that the relation between the current density $\mathbf{j}$ and the electric field intensity $\mathbf{E}$ is an integral. At helium temperatures, where the quantum oscillations are observed, this is valid already for meter waves.

The study of this case is of special interest because the amplitude of the quantum oscillations of the resistivity tensor turns out, generally speaking, to be considerably greater (by a factor of $\epsilon_0/\mu H$) than in the static case.

At the same time, specific intrinsic difficulties

arise in the non-uniform case, connected mostly with the fact that it is not clear in what way one can correctly set up the quantum-mechanical problem of determining the energy spectrum and the matrix elements for a finite sample of metal, for which, in the classical case, the reflection of the electrons from the walls is diffuse. The problem can be solved because it turns out that only those electrons which do not collide with the surface make an important contribution to the quantum-mechanical correction to the classical current density.

2. THE COMPLETE SYSTEM OF EQUATIONS FOR THE PROBLEM

The complete system of equations for determining the electrical conductivity of a metal consists of Maxwell's equations

$$\operatorname{curl} \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{H}}{\partial t}; \quad \operatorname{curl} \mathbf{H} = \frac{4\pi}{c}\mathbf{j} \tag{2.1}$$

and the kinetic equation for the statistical operator $\hat{f}$

$$\frac{\partial \hat{f}}{\partial t} + \frac{i}{\hbar}[\hat{\mathcal{H}}, \hat{f}] + \tilde{W}\hat{f}/t_0 = 0. \tag{2.2}$$

Here $\mathbf{E}$ and $\mathbf{H}$ are the electric and magnetic field intensities; $\mathbf{j}$ is the current density; $\hat{\mathcal{H}}$ is the Hamiltonian; $\tilde{W}\hat{f}/t_0$ determines the change of the statistical operator as a result of collisions; and t_0 is the characteristic relaxation time.

The relation between the current density $\mathbf{j}$ and the statistical operator $\hat{f}$ is given in the quasi-classical approximation by the equation*

$$\mathbf{j}(\mathbf{R}) = e \operatorname{Sp} \hat{f}\, \mathbf{v} \cdot \delta(\mathbf{R} - \hat{\mathbf{r}}). \tag{2.3}$$

Here $\mathbf{R}$ is the radius vector, considered as a c-number in contrast to the operator $\hat{\mathbf{r}} = -i\hbar\partial/\partial\mathbf{P}$; $\hat{\mathbf{v}}$ is the velocity operator, corresponding to the classical quantity $\mathbf{v} = \partial\epsilon/\partial\mathbf{P}$; ϵ is the energy; $\mathbf{P}$ is the generalized quasi-momentum of a conduction electron [by "conduction electron" is meant the corresponding quasi-particle with the dispersion law $\epsilon = \epsilon(\mathbf{p})$]; and $\mathbf{p} = \mathbf{P} - e\mathbf{A}/c$ is the kinematic quasi-momentum.

Equation (2.3) can be obtained from the definition of $\mathbf{j}$ with the aid of a variational principle:

$$\delta\bar{\mathcal{H}} = -\frac{1}{c}\int j \delta \mathbf{A} dV, \tag{2.4}$$

where $\hat{\mathcal{H}}$ is the classical Hamiltonian and $\delta\mathbf{A}$ is the variation of the vector potential $\mathbf{A}(\mathbf{R})$; the integral is taken over all space.

In the operation, $\delta\hat{\mathcal{H}}$, it is understood that in the quasi-classical approximation, to an accuracy of order $\hbar^2$, any Hermitian operator is uniquely determined by its classical analogue, and any method of symmetrization leads to a unique result. The latter assertion is easily shown from the consideration that, to an accuracy of order $\hbar^2$,

$$[\hat{a}, \hat{b}] = -i\hbar[a, b],$$

where $[a, b]$ is the classical Poisson bracket.

This one-to-one correspondence between Hermitian operators in the quasi-classical case and the corresponding classical quantities will be used repeatedly in what follows.

We now transform Eq. (2.2). In order to do this, we first of all linearize it with respect to the high frequency field, setting

$$\hat{f} = f^0(\hat{E}) + \hat{f}', \quad \hat{E} = \varepsilon(\hat{\mathbf{p}}) + U = \hat{\mathcal{H}} - e\varphi, \tag{2.5}$$

where φ is the scalar potential, $\hat{\mathbf{p}}$ the kinematic momentum, and U the potential energy (equal to zero inside the bulk metal and to infinity outside it, if emission phenomena are ignored). In the linear approximation in the electric field we obtain

$$\frac{\partial \hat{f}'}{\partial t} + \frac{i}{\hbar}[\hat{E}^0, \hat{f}'] + \tilde{W}\hat{f}'/t_0 = -\frac{d}{dt} f^0(\hat{E}),$$
$$\hat{\mathcal{H}} = \hat{E}^0 + \hat{\mathcal{H}}',$$

or, in terms of matrix elements,

$$\partial f'_{kk'}/\partial t + \frac{i}{\hbar}(\varepsilon_k - \varepsilon_{k'}) f'_{kk'} + (\tilde{W}\hat{f}')_{kk'}/t_0 = -df^0_{kk'}(\hat{E})/dt. \tag{2.6}$$

Here $\hat{E}^0$ and ϵ_k designate the Hamiltonian and the energy levels in the absence of a varying field, and k stands for the total collection of quantum numbers.

We now transform the right side of equation (2.6), supposing that all classical quantities except $\mathbf{f}^0(\mathbf{E})$ have an even dependence on magnetic field, so that their matrix elements can be calculated quasi-classically, and that the matrix elements of $f^0(E^0)$ are known.

In the same linear approximation, which is the only one considered in the present work (it being perfectly clear that non-linear effects are negligibly small for all real cases), we have in the quasi-classical approximation

$$df^0_{kk'}(\hat{E})/dt = \frac{d}{dt}\{f^0(\hat{E}) - f^0(\hat{E}^0)\}_{kk'}$$
$$= \frac{d}{dt}\frac{1}{\varepsilon_k - \varepsilon_{k'}}[\hat{E}^0, f^0(\hat{E}) - f^0(\hat{E}^0)]_{kk'}$$
$$= \frac{1}{\varepsilon_k - \varepsilon_{k'}}\frac{d}{dt}[\hat{E}, f^0(\hat{E}) - f^0(\hat{E}^0)]_{kk'}$$

*This formula could also be obtained from the quantum-mechanical equation $\mathbf{j}(\mathbf{R}) = (\Psi^*\hat{\mathbf{p}}\Psi + \Psi\hat{\mathbf{p}}^*\Psi^*)/m$ (Ψ is the wave function), by replacing $\Psi^*(\mathbf{R})\Psi(\mathbf{R}')$ by $\mathbf{f}(\mathbf{R}, \mathbf{R}')$ and $\mathbf{p}/m$ by $\mathbf{v}$ for an arbitrary law of dispersion.

$$= \frac{1}{\varepsilon_k - \varepsilon_{k'}} \frac{d}{dt} [f^0(\hat{E}^0),\ \hat{E}]_{kk'}$$

$$= \frac{f^0(\varepsilon_k) - f^0(\varepsilon_{k'})}{\varepsilon_k - \varepsilon_{k'}} \left(\frac{d\hat{E}}{dt}\right)_{kk'} = \frac{f^0(\varepsilon_k) - f^0(\varepsilon_{k'})}{\varepsilon_k - \varepsilon_{k'}} (e\hat{\mathbf{v}}\cdot\mathbf{E})_{kk'}.$$

(In these calculations, allowance has been made for the above-mentioned correspondence in the quasi-classical approximation between the operators and their classical analogues).

Thus the complete system of equations has the form

$$\operatorname{curl}\mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{H}}{\partial t};\ \operatorname{curl}\mathbf{H} = \frac{4\pi}{c}\mathbf{j};$$

$$\frac{\partial f'_{kk'}}{\partial t} + \frac{i}{\hbar}(\varepsilon_k - \varepsilon_{k'}) f'_{kk'} + (\widetilde{W}f')_{kk'}/t_0$$

$$= -\frac{f^0(\varepsilon_k) - f^0(\varepsilon_{k'})}{\varepsilon_k - \varepsilon_{k'}} (e\mathbf{v}\cdot\mathbf{E})_{kk'}; \qquad (2.7)$$

$$\mathbf{j}(\mathbf{R}) = e f'_{kk'} [\hat{\mathbf{v}}\delta(\hat{\mathbf{R}} - \hat{\mathbf{r}})]_{kk'}. \qquad (2.8)$$

In what follows it will be shown that, under the conditions of the anomalous skin effect ($\delta \ll r, l$), the ratio appearing in the right side of this equation can be replaced by $\partial f^0/\partial\epsilon_k$ for the electrons making the principal contribution to the quantum-mechanical correction. If it is only collisions with impurities which play an essential role, then for these electrons

$$f'_{kk'} = -\frac{\partial f^0}{\partial \varepsilon_k} e\psi_{kk'}.$$

Taking this circumstance into account, we write the current formally as $\mathbf{f}'_{kk'}$ in the form

$$f'_{kk'} = -\frac{f^0(\varepsilon_k) - f^0(\varepsilon_{k'})}{\varepsilon_k - \varepsilon_{k'}} e\psi_{kk'}, \qquad (2.9)$$

where ψ satisfies the classical equation

$$(d\psi/dt)_{\text{field}} + \widetilde{W}\psi/t_0 = \mathbf{v}\cdot\mathbf{E}, \qquad (2.10)$$

which was solved in Ref. 4 for classical boundary conditions. Finally, in the general case, (2.9) and (2.10) will hold with $\widetilde{W} \equiv 1$; this case is also convenient for making all necessary estimates.

As has already been shown,[4] for the anomalous skin effect ($\delta \ll r, l$) it is always possible, at any temperature, to introduce a time of free flight and to set $\widetilde{W} \equiv 1$, taking $t_0 = t_0(\mathbf{p})$.

In terms of the variables E^0, t_1 (the period of revolution of the electron in its orbit), p_z (the projection of the quasi-momentum in the direction z of the constant magnetic field), and ζ (the coordinate in the direction normal to the surface of the metal), we obtain, after setting $\psi = \psi_\omega e^{i\omega t}$ and $\mathbf{E} = \mathbf{E}_\omega e^{i\omega t}$:

$$i\omega\psi_\omega + v_\zeta \frac{\partial\psi_\omega}{\partial\zeta} + \frac{\partial\psi_\omega}{\partial t_1} + \frac{\psi_\omega}{t_0} = \mathbf{v}\cdot\mathbf{E}_\omega. \qquad (2.11)$$

Hence

$$\psi_\omega = \int_\alpha^{t_1} \exp\left(-\frac{t_1 - t'_1}{t_0^*}\right) \mathbf{v}(t'_1)\cdot\mathbf{E}_\omega\left(\zeta - \int_{t'_1}^{t_1} v_\zeta dt_2\right) dt'_1; \qquad (2.12)$$

$$1/t_0^* = 1/t_0 + i\omega. \qquad (2.13)$$

Here we have chosen that solution of Eq. (2.11) which is finite and periodic in t_1. In what follows, j_ω, E_ω, and ψ_ω will always be understood and the index ω will be dropped. The quantity α is the solution of the equation

$$\int_\alpha^{t_1} v_\zeta dt_2 = \zeta. \qquad (2.14)$$

which is closest to, but smaller than, the quantity t_1. In case there is no root it must be assumed $\alpha = -\infty$. Thus

$$\mathbf{j}(Z) = -e^2 \sum_{k,l} \frac{f^0(\varepsilon_{k+l}) - f^0(\varepsilon_k)}{\varepsilon_{k+l} - \varepsilon_k} \left[\int_{\alpha(\zeta,t_1)}^{\infty} \exp\left\{-\frac{t_1 - t'_1}{t_0^*}\right\} v(t'_1) \times \mathbf{E}\left(\hat{\zeta} - \int_{t'_1}^{t_1} v_\zeta dt_2\right) dt'_1\right]_l [v\delta(Z - \hat{\zeta})]_{-l}, \qquad (2.15)$$

where it has been assumed that the quasi-classical matrix elements depend essentially only on the difference of the quantum numbers, and the following notation has been introduced:

$$\Phi_{kk'} \equiv \Phi_{k-k'}(k).$$

Consequently the problem reduces to the calculation of the quasi-classical matrix elements.

If the surface of the metal could be considered as a geometrical plane, on one side of which (inside the metal) the potential energy were zero, apart from emission phenomena, and on the other side infinite, so that there would be a correspondence with the classical case of the specular reflection of electrons from the surface, than finding the quasi-classical energy levels and matrix elements would reduce to the solution of the classical problem. This case is studied in the following paragraph; it is also the only one which has been considered in the literature to date.[5,6]

In itself, specular reflection is of purely academic interest. Actually the reflection of electrons from the surface of a metal is diffuse or nearly so in the classical case (see for instance, Refs. 7, 8), because the potential energy increases from zero to infinity over distances of the order of the interatomic spacing; and, over just such distances, unavoidable distortions of every kind, non-uniformities of the surface of the metal, make the law of the increase of the potential energy a random function. It is thus very difficult simply to formulate correctly the quantum-mechanical problem of determining the electronic energy spectrum even for diffuse reflection. Furthermore, even for mirror

reflection, the task of actually calculating the energy spectrum and the matrix elements is extremely involved, since it requires finding the adiabatic invariants and the angular variables for the motion of electrons in a magnetic field in a potential well.

In Section 4, however, it will be shown that in the case which is of interest to us, that of the anomalous skin effect, only those electrons which do not have any forward motion, do not collide with the surface of the metal, and whose spectrum coincides with the spectrum of electrons in unlimited space, make an essential contribution in the quantum-mechanical correction to the classical current density. Thus the character of the reflection seems to be unimportant for determining the quantum contribution to the impedance, so we are able to obtain the quantum-mechanical formula for the total surface impedance of a metal.

3. SOLUTION OF THE PROBLEM FOR SPECULAR REFLECTION OF ELECTRONS FROM A METALLIC SURFACE

For mirror reflection of electrons from a surface it is convenient to take as quantum numbers (k) the quantum numbers n_1, n_2, n_3 corresponding to the adiabatic invariants I_1, I_2, I_3 (in this section we will for generality treat an arbitrary finite sample of metal). The sets of numbers (n_1, n_2, n_3) and (I_1, I_2, I_3) will for convenience be designated $\mathbf{n}$ and $\mathbf{I}$.

The energy spectrum is obtained from the classical function $E^0(\mathbf{I})$ by replacing $\mathbf{I}$ by $(\mathbf{n}+\boldsymbol{\gamma})h$ (where n_i is integral[6] and $\gamma_i < 1$) and by adding a spin component $\pm e\hbar H/m_0 c$, where m_0 is the mass of a free electron.

In this paper we will ignore the spin component altogether, for the sole purpose of simplifying the writing, and, in conformity with this, apart from the summation over $\mathbf{n}$ in the equation for the current density, will write down spin two:

$$\mathbf{j}(\mathbf{R}) = 2ef'_{\mathbf{nn'}}\,[\hat{\mathbf{v}}\delta(\mathbf{R}-\hat{\mathbf{r}})]_{\mathbf{n'n}}. \qquad (3.1)$$

It is possible to neglect the spin component because in general the effect mass which enters into μ is not the same as the mass m_0 of a free electron.

In terms of these variables, the quasi-classical matrix elements $\Phi_{\mathbf{nn'}}$ represent the Fourier components of the corresponding physical quantity $\Phi(\mathbf{w}, \mathbf{I})$ with respect to the angular variables:*

$$\Phi_{\mathbf{nn'}} \equiv \Phi_{\mathbf{n}-\mathbf{n'}}(\mathbf{n})$$

$$= \int_0^1\int_0^1\int_0^1 e^{-2\pi i(\mathbf{n}-\mathbf{n'})\mathbf{w}}\,\Phi(\mathbf{w}, (\mathbf{n}+\boldsymbol{\gamma})h)\,dw_1\,dw_2\,dw_3. \qquad (3.2)$$

In order to make calculations one naturally has to know $\Phi(\mathbf{w}, \mathbf{I})$ for all values of the angular variables. For electrons colliding with the surface in the classical case, one has to write down the trajectories taking account of their reflection from the surface. It can be shown that such a description of the trajectories is equivalent to a suitable continuation of the function Φ in the region outside the metal analogous to that developed in Ref. 10, to which, as it was found there, the reflection conditions of the electrons from the metallic surface reduce.

For the choice of quantum numbers which has been made, the final transition to the classical case is particularly graphic.

As seen from (3.2) and (2.7), the equations determining the current density in the quasi-classical approximation have the form

$$\mathbf{j} = 2e\sum_{\mathbf{n}}\sum_{\mathbf{k}} f'_{\mathbf{k}}(\mathbf{n})\,[\mathbf{v}\delta(\mathbf{R}-\hat{\mathbf{r}})]_{-\mathbf{k}}; \qquad (3.3)$$

$$\frac{\partial f'_{\mathbf{k}}(\mathbf{n})}{\partial t} + 2\pi i\,\frac{\partial E^0}{\partial \mathbf{I}_{\mathbf{n}}}\,\mathbf{k}f'_{\mathbf{k}}(\mathbf{n}) + \sum_{\mathbf{k'},\,\mathbf{n'}} W_{\mathbf{k},-\mathbf{k'}}(\mathbf{I}_{\mathbf{n}}, \mathbf{I}_{\mathbf{n'}})\,f'_{\mathbf{k'}}(\mathbf{n'})\,\Delta\mathbf{I}_{\mathbf{n'}}$$
$$= -\frac{f^0(\varepsilon_{\mathbf{n}+\mathbf{k}}) - f^0(\varepsilon_{\mathbf{n}})}{\varepsilon_{\mathbf{n}+\mathbf{k}} - \varepsilon_{\mathbf{n}}}\,(e\mathbf{v}\mathbf{E})_{\mathbf{k}}; \quad \Delta\mathbf{I}_{\mathbf{n}} = \Delta I_{n_1}\Delta I_{n_2}\Delta I_{n_3} = h^3. \qquad (3.4)$$

The classical kinetic equation for the distribution function $\mathbf{f'}$

$$\frac{\partial f'}{\partial t} + [E^0, f'] + \widetilde{W}f'/t_0 = -\frac{\partial f^0}{\partial \varepsilon}\,e\mathbf{v}\cdot\mathbf{E}$$

is written in terms of the canonical variables $\mathbf{I}$, $\mathbf{w}$ in the form

$$\frac{\partial f'}{\partial t} + \frac{\partial E^0}{\partial \mathbf{I}}\frac{\partial f'}{\partial \mathbf{w}} + \frac{\widetilde{W}f'}{t_0} = -\frac{\partial f^0}{\partial \varepsilon}\,e\mathbf{v}\cdot\mathbf{E}.$$

(Here we make use of the fact that E^0 depends only on $\mathbf{I}$, and consequently $\partial E^0/d\mathbf{w} = 0$). Expanding all functions of $\mathbf{w}$ as Fourier series, we obtain

$$f'(\mathbf{w}, \mathbf{I}) = \sum_{\mathbf{k}} f'_{\mathbf{k}}(\mathbf{I})\,e^{2\pi i\mathbf{k}\mathbf{w}};$$

$$\frac{\partial f'_{\mathbf{k}}(\mathbf{I})}{\partial t} + 2\pi i\,\frac{\partial E^0}{\partial \mathbf{I}}\,\mathbf{k}f'_{\mathbf{k}}(\mathbf{I})$$
$$+ \frac{1}{t_0}\sum_{\mathbf{k'}}\int W_{\mathbf{k},-\mathbf{k'}}(\mathbf{I}, \mathbf{I'})\,f'_{\mathbf{k'}}(\mathbf{I'})\,d\mathbf{I'} = -\frac{\partial f^0}{\partial \varepsilon}\,(e\mathbf{v}\mathbf{E})_{\mathbf{k}},$$

which differs from (3.4) only in that $\mathbf{I}$ is considered not as a discrete but as a continuous variable.

We now write down the current density in terms of these same variables. Making use of the fact that the Jacobian of the transformation from one set of canonical variables $(\mathbf{r}, \mathbf{P})$ to another $(\mathbf{I}, \mathbf{w})$ is equal to unity (see, for example, Ref. 11), we have

*This formula is a natural extension of the formula developed in Ref. 9, considering that $\dot{\mathbf{w}} = \partial E^0/\partial \mathbf{I}$.

$$\mathbf{j} = \frac{2e}{h^3}\int \mathbf{v} f' d\mathbf{P} = \frac{2e}{h^3}\int \mathbf{v} f' \delta(\mathbf{R}-\mathbf{r})\, d\mathbf{P} d\mathbf{r}$$
$$= \frac{2e}{h^3}\int \mathbf{v} f' \delta(\mathbf{R}-\mathbf{r})\, d\mathbf{I} dw = \frac{2e}{h^3}\int [\mathbf{v} f' \delta(\mathbf{R}-\mathbf{r})]_{k=0}\, d\mathbf{I}$$
$$= \frac{2e}{h^3}\int d\mathbf{I} \sum_k f'_k(\mathbf{I})\, [\delta(\mathbf{R}-\mathbf{r})\, \mathbf{v}]_{-k},$$

which corresponds to (3.3).

The final transition to the classical equations is now quite apparent. Thus the solution of the quantum mechanical problem for the case of mirror reflection reduces completely to the solution of the classical problem.

4. SOLUTION OF THE PROBLEM FOR AN ARBITRARY LAW OF REFLECTION OF ELECTRONS FROM THE METALLIC SURFACE

As is well known, for the anomalous skin effect, when the smallest characteristic dimension is the penetration depth, the classical magnitude of the surface impedance depends on the way in which electrons are reflected from the surface of the metal. It follows that one must take the boundary conditions at the surface into account in solving the problem.

As already stated, however, the nature of the surface reflection does not show up in the quantum mechanical correction in the zeroth approximation in the "anomalousness" (that is, in δ/l or δ/r, where δ is the penetration depth and l and r are the average mean-free-path and the radius of the Larmor orbit) or in the "quasi-classicalness" (that is, in $\mu H/\epsilon_0$, where μ is the Bohr magneton for a conduction electron and ϵ_0 is the limiting Fermi energy). We will now formulate this statement more precisely. We assume that the width D of the film is not too small, so that the Larmor orbit corresponding to a central cross-section $p_z = 0$ can be "fitted" into it:

$$D > d = \left| \frac{c}{eH} \int_{t_0'}^{t_0''} v_\zeta\, dt_2 \right| = \left| \frac{c}{eH} \left(\int_{t_0'}^{t_0''} v_z\, dt_2 \cos\phi + 2p_x^{\max} \sin\phi \right) \right|;$$

$$v_\zeta(t_0') = v_\zeta(t_0'') = 0; \quad v_\zeta'(t_0') > 0, \quad v_\zeta'(t_0'') < 0, \qquad (4.1)$$

where all quantities have their values at $p_z = 0$, $\epsilon = \epsilon_0$. The maximum $p_x(\epsilon_0, p_z, t_1)$ is taken with respect to the time t_1 of an orbital revolution, ϕ is the angle between the direction z of the constant magnetic field and the normal to the surface of the metal, and x lies in the plane of the surface. Then only those electrons which simultaneously satisfy the following conditions make a substantial contribution in the quantum correction to the classical current density:

1. Their average velocity of motion in the interior of the metal during the time of one orbital revolution is small: $|\bar{v}_\zeta| = |\bar{v}_z \cos\phi| \ll |\bar{v}_z| = v_0$.

2. The area S of the cross-sections $\epsilon(\mathbf{p}) = \epsilon_0$, $p_z = \text{const}$ corresponding to their orbits is near an extremum S_{ext}.

If the magnetic field is not parallel to the surface of the metal, $\phi \not\approx 90°$, then only electrons to which there correspond cross-sections close to a central one (where $p_z = 0$) satisfy both conditions. In a magnetic field parallel to the surface of the metal the first requirement for closed cross-sections is satisfied automatically, and only the second one remains.

3. In the course of their motion, the electrons d do not collide with the surface. The division of the electrons into those which do collide with the surface and those which do not can be carried out because of the first condition.

The proof of these assertions is quite simple conceptually, but exceedingly involved even for specular reflection from the surface. Thus we will limit ourselves to illustrations of the corresponding reasons for the simplest cases.

Let us demonstrate the first two assumptions for the example of an unbounded volume, where the electric field is different from zero only in the half-space $\zeta \geq 0$ (obviously this case is in itself of purely academic interest). The energy spectrum of the electrons in unbounded space has been found in Ref. 1: $E^0 = \epsilon(n, p_z)$, where the function ϵ is determined from the condition

$$S(E^0, p_z) = (n+\gamma)\, ehH/c.$$

Since ψ depends on ζ, that is, on both y and z, the matrix elements of ψ in (2.9) will be nondiagonal with respect to n and to p_z, so that from (2.8) and (2.9) we have

$$\mathbf{j} = -\int_{-\infty}^{\infty} dp_z \int_{-\infty}^{\infty} dp_z' \sum_{n=0}^{\infty} \sum_{l=-\infty}^{\infty} \times \frac{f^0(\varepsilon_{n+l,\, p_z+p_z'}) - f^0(\varepsilon_{np_z})}{\varepsilon_{n+l,\, p_z+p_z'} - \varepsilon_{np_z}} \mathbf{A}_{lp_z'}(n, p_z), \qquad (4.2)$$

where

$$\mathbf{A}_{lp_z'}(n, p_z) \sim \int_{-\infty}^{\infty} \psi_{lp_z'}(n, p_z)\, [\mathbf{v}\delta(Z-\hat{\zeta})]_{-l,\, -p_z'}\, dP_x; \qquad (4.3)$$
$$\partial\varepsilon_{np_z}/\partial n = \hbar\Omega = \mu H.$$

From (4.2)

$$\mathbf{j} = -\int_{-\infty}^{\infty} dp_z \int_{-\infty}^{\infty} dp_z' \sum_{n=0}^{\infty} \sum_{l=-\infty}^{\infty} \times \frac{f^0(\varepsilon_{np_z} + l\hbar\Omega + \bar{v}_z p_z') - f^0(\varepsilon_{np_z})}{l\hbar\Omega + \bar{v}_z p_z'} \mathbf{A}_{lp_z'}(n, p_z); \qquad (4.4)$$

$$\frac{\partial \varepsilon_{np_z}}{\partial p_z} = \frac{\partial}{\partial p_z}\int_0^1 \varepsilon(n, p_z, \tau)\, d\tau = \int_0^1 \frac{\partial \varepsilon}{\partial p_z} d\tau = \overline{v_z(\varepsilon_{np_z}, p_z, t_1)};$$

$$\tau = \frac{t_1}{T},\ T = \frac{2\pi m^* c}{eH};\ m^* = \frac{1}{2\pi}\frac{\partial S}{\partial \varepsilon};\ \Omega = \frac{2\pi}{T}. \quad (4.5)$$

Transforming the sum over n by Poisson's formula (see Ref. 12, for example), and keeping only the oscillating quantum-mechanical terms (since they are the only ones of interest to us; the classical part of the current density has been found in Refs. 13, 10, and 4), we have

$$\Delta j^{KB} = \sum_{s=1}^{\infty}\sum_{l=-\infty}^{\infty} G_{sl};$$

$$G_{sl} = -2\int_{-\infty}^{\infty} dp_z \int_{-\infty}^{\infty} dp_z' \int_0^{\infty} e^{2\pi i s n} \times \frac{f^0(\varepsilon_{np_z} + l\hbar\Omega + \bar{v}_z p_z') - f^0(\varepsilon_{np_z})}{l\hbar\Omega + \bar{v}_z p_z'} A_{lp_z'}\, dn. \quad (4.6)$$

Making a shift in n,*

$$n \to n - l - \bar{v}_z p_z'/\hbar\Omega,$$

and assuming that only $\epsilon \sim \epsilon_0$ and $|\bar{v}_z p_z'| \ll \epsilon_0$ play an essential role, we find

$$G_{sl} = -2\int_{-\infty}^{\infty} dp_z \int_0^{\infty} dn \cdot e^{2\pi i s n} f^0(\varepsilon_{np_z}) \int_{-\infty}^{\infty} J_{sl}\, dp_z', \quad (4.7)$$

$$J_{sl} = \frac{1 - \exp(-2\pi i s \bar{v}_z p_z'/\hbar\Omega)}{l\hbar\Omega + \bar{v}_z p_z'} A_{lp_z'} + \exp\left(-\frac{2\pi i s \bar{v}_z p_z'}{\hbar\Omega}\right)\partial A_{lp_z'}/\partial \varepsilon_{np_z}. \quad (4.8)$$

Integrals of the type (4.7) have been calculated by I. Lifshitz and Kosevich,[3] who have also shown that for $\epsilon_0/\hbar\Omega \gg 1$ only those electrons which are near an extreme cross-section $p_z = p_z^{ext}$ of the limiting Fermi surface $\epsilon(\mathbf{p}) = \epsilon_0$ make a significant contribution; this situation corresponds to the second condition for the electrons which are "necessary" for us. A region Δp_z of order

$$|\Delta p_z| \sim [(\hbar\Omega/\varepsilon_0)(2m^*\varepsilon_0)]^{1/2}. \quad (4.7a)$$

near the extremum takes part in this contribution.

We now look into which l, p_z^{ext}, and p_z' make a substantial contribution to G_{sl}, that is, for what values of these quantities J_{sl} is not small.

*It might appear that a convenient shift would be $n \to n - l$, $p_z \to p_z - p_z'$. This is not so, however, because of the divergence with respect to p_z of each separate term in (4.6) for $l = 0$ the divergence occurs for $p_z = 0$).

In doing this we first of all note that from (4.2), (2.9), and (2.7)

$$|A_{lp_z'}| \sim (vE)_{lp_z'}\left/\left|\frac{1}{t_0^*} + \frac{i}{\hbar}(\varepsilon_{n+l,\, p_z+p_z'} - \varepsilon_{np_z})\right|\right.$$

$$\sim (vE)_{lp_z'}\left/\left(\frac{1}{t^l} + \frac{1}{\hbar}|\bar{v}_z p_z'|\right)\right.;$$

$$1/t^l = 1/t_0 + (\omega + l\Omega). \quad (4.9)$$

Let us designate

$$|\bar{v}_z p_z'|/(\hbar\Omega) = x.$$

Then

$$|J_{sl}| \sim \left[\frac{\hbar\Omega}{\varepsilon_0} + \frac{x}{(1+x)(l+x)}\right]\frac{1}{(\Omega t^l)^{-1} + x}(vE)_{lp_z'}. \quad (4.10)$$

The quantity x is intrinsically different for a central (x_c) and a non-central (x_{nc}) cross-section for one and the same p_z'. For a central section ($p_z = 0$), $\bar{v}_z = 0$, for a non-central one, $|\bar{v}_z| \sim |\overline{v_z}| = v_0$. This means that near a central cross-section, where (4.7a) holds, the quantity $\bar{v}_z$ is of order $|\bar{v}_z| \sim v_0(\hbar\Omega/\epsilon_0)^{1/2}$, while on a non-central section, $|\bar{v}_z| \sim v_0$. Thus, since

$$x \sim \frac{|p_z'|}{\hbar/r_z} \sim \frac{|p_z'|}{\hbar/r}\frac{|\bar{v}_z|}{v_0};\ r \sim v_0 T,\ r_z = |\bar{v}_z| T, \quad (4.11)$$

then

$$x_c \sim \frac{|p_z'|}{\hbar/r}\left(\frac{\hbar\Omega}{\varepsilon_0}\right)^{1/2},\ x_{nc} \sim \frac{|p_z'|}{\hbar/r} \gg x_c. \quad (4.12)$$

The coefficient of $(\mathbf{v}\cdot\mathbf{E})_l p_z'$ in J_{sl} is not small if $x \lesssim 1/(1+\Omega t^l)$, that is, if

$$|p_z'| \lesssim \frac{\hbar}{r_z(1+\Omega t^l)} \sim \frac{\hbar}{r(1+\Omega t^l)}\frac{v_0}{|\bar{v}_z|},$$

$$r_l = r(1 + \Omega t^l). \quad (4.13)$$

However the matrix elements $(\mathbf{v}\cdot\mathbf{E})_l p_z'$ differ significantly from zero for $p_z' \sim \hbar/\delta_{eff}^z$, where $\delta_{eff}^z = \delta_{eff}/\cos\phi$ is the effective skin depth in the direction z; δ_{eff} is the effective skin depth (in the direction ζ). Consequently the values which are important are

$$|p_z'| \sim \min\left\{\frac{\hbar}{r_l}\frac{v_0}{|\bar{v}_z|},\ \frac{\hbar}{\delta_{eff}}\cos\phi\right\};$$

$$x \sim \min\left\{\frac{1}{1+\Omega t^l},\ \frac{r_z}{\delta_{eff}^z}\right\}. \quad (4.14)$$

For a central section, this gives

$$|p_z'| \sim \min\left\{\frac{\hbar}{r_l}\left(\frac{\varepsilon_0}{\hbar\Omega}\right)^{1/2};\ \frac{\hbar}{\delta_{eff}}\cos\phi\right\}; \quad (4.15)$$

and for a non-central one,

$$|p_z'| \sim \min\left\{\frac{\hbar}{r_l},\ \frac{\hbar}{\delta_{eff}}\cos\phi\right\}. \quad (4.16)$$

Two cases are possible. If $\delta^z_{eff} \gtrsim r_0$ (that is $\cos\phi \lesssim \delta_{eff}/r_l$), then relative to the quantity **a**,

$$a = \frac{\delta^z_{eff}}{r_0}\left(\frac{\varepsilon_0}{\hbar\Omega}\right)^{1/2} = \frac{\delta_{eff}}{r_0\cos\phi}\left(\frac{\varepsilon_0}{\hbar\Omega}\right)^{1/2}, \qquad (4.17)$$

either central sections alone are important (if $a \gg 1$), so that

$$\bar{v}_\zeta = \bar{v}_z\cos\phi \ll \left(\frac{\hbar\Omega}{\varepsilon_0}\right)^{1/2}\frac{\delta_{eff}}{r_0}v_0 \ll v_0,$$

or else all extreme sections (if $a \lesssim 1$), so that

$$\bar{v}_\zeta \sim \bar{v}_z\cos\phi \ll v_0\delta_{eff}/r_0 \ll v_0.$$

If $\delta^z_{eff} \ll r_0$ (so that $\cos\phi \gg \delta_{eff}/r_0$), then only central sections make a substantial contribution to G_{sl} (the contribution of the non-central sections being smaller by a factor of $[\delta^z_{eff}/r_0 + (\hbar\Omega/\epsilon_0)^{1/2}]^{-1}$, or

$$\bar{v}_\zeta = \bar{v}_z\cos\phi \sim v_0\left(\frac{\hbar\Omega}{\varepsilon_0}\right)^{1/2}\cos\phi \ll v_0.$$

(It is obvious that the skin effect was taken to be anomalous, that is $\delta_{eff} \ll r_0$, at every point of the demonstration).

In this way the first condition is also proved: for all cases $|\bar{v}_\zeta| \ll v_0$.

We note further that, as follows from (4.10) and (4.14), only $x \ll 1$ is important both for $\Omega t^0 \gg 1$ and for $a \gg 1$, which means $l = 0$. Clearly for $a \gg 1$, central sections (or $l = 0$) play the principal role.

The first condition is easily understood physically. It is automatically satisfied in a parallel field ($\phi = 90°$). Let us study an inclined field ($\phi \sim 1$), in which, due to the fact that the electric field **E** depends on z, only $p'_z \sim \hbar/\delta_{eff}$ is important. However, because of the dependence of the electron energy on p_z, the magnitude of $|p'_z|$ is limited to $|\hbar/\bar{v}_z T|$. Thus for the anomalous skin effect, when $\delta_{eff} \ll r$, $p'_z \sim \hbar/r$ on a non-central cross-section, and

$$|p'_z| \sim \min\left\{\frac{\hbar}{\delta_{eff}}, \frac{\hbar}{r}\left(\frac{\varepsilon_0}{\hbar\Omega}\right)^{1/2}\right\},$$

on a central one, so that the non-central sections make a contribution in the next approximation with respect to the anomalousness (δ_{eff}/r) or to $(\hbar\Omega/\epsilon_0)^{1/2}$. For the normal skin effect ($\delta_{eff} \gg r$) in the bulk metal, however, all extreme cross-sections play the same role, as is to be expected.

We now prove the third assertion in the case when the contribution of electrons colliding with the surface is maximum, as is easily visualized: the constant magnetic field is aligned parallel to the surface ($\phi = \pi/2$), and the reflection of electrons from the surface is specular. The energy spectrum $E^0 = \epsilon(n, P_x, p_z)$ in this case has been found earlier;[5] **A** is non-diagonal only with respect to n; the quasi-classical matrix elements coincide with the Fourier components with respect to the time;[9] the electrons have $\bar{v}_y \sim \overline{dp_y/dt} = 0$ so that they can be separated into those which do and those which do not collide with the surface; and all extreme cross-sections make a substantial contribution to the current density.

The formula for the current density $\mathbf{j}(Y)$ in this case is written in the form

$$\mathbf{j}(Y) = -\int_{-\infty}^{\infty} dp_z \sum_{n=0}^{\infty}\sum_{l=-\infty}^{\infty}\int_{-\infty}^{\infty} dP_x \times \frac{f^0(\varepsilon_{n+l,\,p_z,\,P_x}) - f^0(\varepsilon_{np_zP_x})}{\varepsilon_{n+l,\,p_z,\,P_x} - \varepsilon_{np_z,\,P_x}}\mathbf{D}_l(n,\,p_z,\,P_x), \qquad (4.18)$$

$$\mathbf{D}_l = K\psi_l[\mathbf{v}\,\delta(Y - \hat{y})]_{-l}$$

or, considering that for $\Delta\mathbf{j}^{qu}$ for $\phi = \pi/2$, $a = \infty$, only $l = 0$ is important,

$$\Delta\mathbf{j}^{KB}(Y) = \sum_{s=1}^{\infty}\mathbf{G}_{s0};\ \mathbf{G}_{s0} \qquad (4.19)$$

$$= -2\int_{-\infty}^{\infty} dp_z \int_{-\infty}^{\infty} dP_x \int_0^{\infty} e^{2\pi isn}\frac{\partial f^0}{\partial\varepsilon_{np_zP_x}}\mathbf{A}_0(n,\,p_z,\,P_x)\,dn.$$

We now separate out from G_{s0} the contribution of electrons which do not collide with the surface $y = 0$. For simplicity, we consider the case of a half-space. In classical mechanics, $y(t) = (P_x - p_x(t)/|eHc|$. An electron does not collide with the surface $y = 0$ if at all times during the motion $y(t) > 0$, that is, if

$$y_{min}(t) = (P_x - p_x^{max})\,|c/eH| > 0; \qquad (4.20)$$
$$P_x > P_0 = p_x^{max},\ p_x = p_x(\varepsilon,\,p_z,\,t_1),\ P_0 = P_0(\varepsilon_{np_z},\,p_z).$$

(The maximum is taken with respect to t_1). The corresponding inequality in the quasi-classical case is

$$P_x > P_0(\varepsilon_{np_z},\,p_z). \qquad (4.21)$$

Here we have considered that the electrons which do not collide with the surface have the same spectrum as in the bulk metal:

$$\varepsilon = \varepsilon(n,\,p_z), \qquad (4.22)$$

that is, the energy is infinitely degenerate with respect to P_x.

The degeneracy with respect to P_x is removed for electrons colliding with the surface, or $\epsilon = \epsilon(n, p_z, P_x)$.

A calculation analogous to that carried out by Kosevich and Lifshitz[5] shows that, because of the

degeneracy with respect to P_x and the resulting absence of an integration of $\partial f^0/\partial\epsilon_{np_z}$ over P_x, the integral over dP_x from P_0 to ∞ proves, as in Ref. 5, to be $(\epsilon_0/\hbar\Omega)^{1/2} \gg 1$ times larger than the integral from $-\infty$ to P_0. This fact indicates that it is sufficient to give attention only to those electrons which do not collide with the surface.

A case which is more convenient for calculations is naturally one in which

$$a = (\delta_{\text{eff}}/r_0)\,(\varepsilon_0/\hbar\Omega)^{1/2}/\cos\phi \gg 1, \qquad (4.23)$$

so that the spectrum of the electrons which give a substantial contribution to Δj^{qu} can be regarded as coinciding with the spectrum of the bulk metal, and the relation between the periods of the quantum oscillations and the areas of the extreme cross-sections is the same as in Ref. 1; in the calculation of Δj^{qu} only $l = 0$ is important.

If (4.23) is satisfied, then for the value $l = 0$ which we require the expression for ψ is considerably simplified. From (2.7) for $l = 0$ and according to (4.23), $|\bar{v}_z p_z' t_0^*/\hbar| \ll 1$,

$$\psi = \left(\frac{1}{t_0^*}\right)^{-1} \mathbf{vE}. \qquad (4.24)$$

As already pointed out, because of the quasi-classicalness only those electrons are important which are near an extreme cross-section $\epsilon(\mathbf{p}) = \epsilon_0$, $p_z = p_z^{\text{ext}}$ (and, for $\pi/2 - \phi \gg \delta_{\text{eff}}/r$, near the central cross-section, $p_z = 0$, $\epsilon(\mathbf{p}) = \epsilon_0$).

Because the skin effect is anomalous, only electrons with $|v_\zeta| \ll v_0$, $v_\zeta'(t_1) < 0$ are important (this will subsequently be shown to be analogous to other results[13,14,4]; physically this assertion is related to the fact that only those electrons which spend a maximum amount of time in the skin layer, that is, those having $v_\zeta \approx 0$, play a significant role).

Consequently in the current density in the symbol for a matrix element, it would be possible everywhere to take out $\mathbf{v}$ at the point $\epsilon(\mathbf{p}) = \epsilon_0$, $p_z = p_z^{\text{ext}}$ (or $p_z = 0$), $v_\zeta = 0$, $v_\zeta'(t_1) > 0$, and to sum over all extreme cross-sections (or over all non-central ones, in the case of a non-parallel field).

5. CALCULATION OF THE QUANTUM CORRECTION TO THE CURRENT DENSITY

In the preceding paragraph it was shown how one might determine the quantum correction to the classical formula for the current density.

The simplest case is one in which the magnetic field is aligned parallel to the surface. Here the physical quantities depend only on y and consequently are non-diagonal only with respect to one quantum number n; the quantum-mechanical correction to the current density is determined by (4.19), (4.21), and (4.22), where

$$K = 2e^2/h^2, \qquad (5.1)$$

in which it is easy to demonstrate the transition to the classical case.

Evaluating the integrals, we obtain

$$\Delta j_i^{\text{KB}}(Y) = \sum \frac{h^3}{2\pi} H^2 \chi_i(\varepsilon_0, p_z^{\text{ext}}) \left(\frac{d\ln S_{\text{ext}}}{d\varepsilon}\right)^2 \frac{\partial \Delta M_z^{\text{ext}}}{\partial H}, \qquad (5.2)$$

where ΔM_z^{ext} has been found elsewhere;[1] the sum is taken over all extreme sections, and for the film thickness D

$$\chi_i(Y) = \frac{4\pi e^2}{h^3 \overline{t_0^{*-1}}} \left\{ v_i(t)\, S\left(Y - \frac{p_x^{\max} - p_x(t)}{|eH/c|}\right) \times S\left(D - Y - \frac{p_x(t) - p_x^{\min}}{|eH/c|}\right) \times \overline{v_j(t')\, E_j\left(Y + \frac{p_x(t) - p_x(t')}{|eH/c|}\right)} \right\}_{\text{cp}}; \qquad (5.3)$$

$$S(w) = \begin{cases} 1\ (w > 0) \\ 0\ (w < 0) \end{cases}; \quad D > \left|\frac{c}{eH}\right| (p_x^{\max} - p_x^{\min});$$

$$\bar{\varphi} = \varphi_{\text{cp}} = \frac{1}{T}\int_0^T \varphi\, dt,$$

which corresponds to (4.24). The averages are taken over the period t' (designated by a bar) and over t (designated by "av").

Inasmuch as the calculations were actually carried out in terms of the variables ζ_{chem}, H (where ζ_{chem} is the chemical potential), and physically it is not ζ_{chem} but the total number of electrons which is conserved, it would be necessary to allow for oscillations of ζ_{chem}. However, since $\Delta\zeta_{\text{chem}}^{qu} \sim (\hbar\Omega/\epsilon_0)^{3/2}$ and $\Delta j_i^{qu} \sim \hbar\Omega/\epsilon_0)^{1/2}$,[1,3] the oscillations of ζ_{chem} can be ignored. (As will be seen later, this is true for any inclination of the magnetic field). Finding the quantum contribution to the current density for an inclined magnetic field is quite complicated.

We obtain a closed solution of this problem for the case of a strong magnetic field which satisfies the conditions

$$\Omega \gg \omega, \frac{1}{t_0}, \quad \omega\cdot\omega t_0; \quad \frac{\delta_{\text{eff}}}{r\cos\phi}\left(\frac{\varepsilon_0}{\hbar\Omega}\right)^{1/2} = \frac{\delta_{\text{eff}}}{|\bar{v}_z|\,T} \ll 1 \qquad (5.4)$$

(the second condition allows us to use the spectrum for electrons in the bulk metal).

It is obvious that we would be able to solve this problem without superimpositng this restriction on the size of the magnetic field. Since, as proved in Section 4, the nature of the reflection of electrons from the surface does not influence the result, the

solution of the problem for specular reflection (Section 3) should parallel the solution for any type of reflection. However a similar solution is extremely involved. All the results can be understood in the most important case (5.4), however, when the formula for Δj^{qu} takes a very simple form.

We note that the condition (5.4) leads to an additional reduction of the relative contribution of the electrons corresponding to a central section and colliding with the surface, since these electrons are only accelerated once during the time that the electrons which do not collide with the surface are accelerated many times (since $\Omega \gg \omega$, $1/t_0$). It is this circumstance which facilitates the solution of the problem.

For simplicity let us investigate the case when the Fermi surface is a closed convex surface.

It follows from Section 4 that the quantum contribution to the current density from (5.4) in a film which is not too thin differs from that in an unbounded sample of metal, with the same electric field distribution as in the film, only in that it is necessary to consider only the electrons which do not collide with the surface. Thus we write Δj^{qu} for an unbounded volume of metal in such a form that we can separate the electrons which do intersect the surface $\zeta = 0$, D from those which do not.

In an unbounded metal, $\mathbf{j}$ is given by formulas (4.4) and (4.3).

Under the conditions which we have assumed, the oscillating quantum-mechanical part will, according to Section 4, be given just as accurately by the equation

$$\mathbf{j} = -\int_{-\infty}^{\infty} dp_z \sum_{n=0}^{\infty} \frac{\partial f^0}{\partial \varepsilon_{np_z}} \int_{-\infty}^{\infty} \mathbf{A}_{0p'_z}\, dp'_z. \tag{5.5}$$

But from equation (4.9) it is clear that, because of (5.4),

$$\mathbf{A}_{0p'_z} \gg \mathbf{A}_{lp'_z}\ (l \neq 0);\quad A_{0p'_z} \approx \sum_{l=-\infty}^{\infty} A_{lp'_z}, \tag{5.6}$$

or

$$\int_{-\infty}^{\infty} \mathbf{A}_{0p'_z}\, dp'_z \approx \int_{-\infty}^{\infty} dp'_z \sum_{l=-\infty}^{\infty} \mathbf{A}_{lp'_z}$$

$$\sim \int_{-\infty}^{\infty} dP_x \int_{-\infty}^{\infty} dp'_z \sum_{l=-\infty}^{\infty} \psi_{lp'_z} [\mathbf{v}\,\delta(\mathbf{Z} - \hat{\zeta})]_{-l,\,-p'_z}$$

$$= \int_{-\infty}^{\infty} dP_x [\psi\,\mathbf{v}\,\delta(\mathbf{Z} - \hat{\zeta})]_{0,0} \sim \overline{\psi\,\mathbf{v}\,\delta(\mathbf{Z} - \hat{\zeta})}. \tag{5.7}$$

Thus this integral is proportional to the corresponding quasi-classical quantity in a state with $\epsilon = \epsilon(n, p_Z)$ and p_Z. This is to be expected, since if, in the formula

$$\mathbf{j} = -\int_{-\infty}^{\infty} dp_z \sum_{n=0}^{\infty} \frac{\partial f^0}{\partial \varepsilon_{np_z}} \int_{-\infty}^{\infty} dp'_z \sum_{l=-\infty}^{\infty} \mathbf{A}_{lp'_z}$$

we replace the summation over n by an integration and remember that $\partial(n, p_Z)/\partial(\epsilon, p_Z) = 1/\hbar\Omega$, we obtain the classical formula for the current density.

Consequently, the transition from the classical equation for $\mathbf{j}$,

$$\mathbf{j} = -\frac{2|e|^3 H}{ch^3} \int \frac{\partial f^0}{\partial \varepsilon}\, \mathbf{v}\,\psi\, d\varepsilon\, dp_z\, dt_1, \tag{5.8}$$

to the quantum-mechanical one gives

$$\mathbf{j} = -\frac{2|e|^3 H}{ch^3} \int_{-\infty}^{\infty} dp_z \sum_{n=0}^{\infty} \frac{\partial f^0}{\partial \varepsilon_{np_z}}\, \hbar\Omega$$

$$\times \int_0^T \mathbf{v}(\varepsilon_{np_z}, p_z, t_1)\, \psi(\zeta, \varepsilon_{np_z}, p_z, t_1)\, dt_1. \tag{5.9}$$

Substituting the expression (2.12) here for ψ, we write $j_i(\zeta)$ in the form

$$j_i(\zeta) = -\frac{2|e|^3 H}{ch^3} \int_{-\infty}^{\infty} dp_z \sum_{n=0}^{\infty} \frac{\partial f^0}{\partial \varepsilon_{np_z}} \int_{-\infty}^{\infty} d\zeta_0 \tag{5.10}$$

$$\times \left\{ v_i(t_1)\, \delta(\zeta - \zeta_0 - \zeta_1(t_1)) \right.$$

$$\left. \times \int_{-\infty}^{t_1} \exp\left\{ -\frac{t_1 - t'_1}{t_0^*} \right\} v_j(t'_1)\, E_j(\zeta_0 + \zeta_1(t'_1))\, dt'_1 \right\}_{\text{av}},$$

$$\zeta_1(t_1) = \int_{t'_0}^{t_1} v_\zeta dt_2. \tag{5.11}$$

We take t'_0 to be the time when

$$v_\zeta(\varepsilon_{np_z}, p_z, t'_0) = 0;\quad v'_\zeta(t'_0) > 0,\quad t'_0 = t'_0(\varepsilon_{np_z}, p_z). \tag{5.12}$$

Strictly speaking, there are an infinite number of values of t'_0 which are solutions of (5.12). However, only the quantum correction Δj_i^{qu} is of interest to us (the classical part j_i has been determined already[13,14,4]), and for this only the point $\overline{v}_\zeta(\epsilon_{np_Z}, p_Z) = 0$ is important. For electrons in these states, the various values of t'_0 differ by integral numbers of periods, thereby not affecting $\zeta_1(t_1)$; ζ_0 is in this case the minimum distance of an electron from the surface.

It is now a simple matter to separate out the electrons which do not collide with the surface:

$$j_i(\zeta) = -\frac{2|e|^3 H}{ch^2} \int_{-\infty}^{\infty} dp_z \sum_{n=0}^{\infty} \frac{\partial f^0}{\partial \varepsilon_{np_z}} \int_0^Q d\zeta_0$$

$$\times \left\{ v_i(t_1)\, \delta\left(\zeta - \zeta_0 - \int_{t'_0}^{t_1} v_\zeta dt_2\right) \right.$$

$$\times \int_{-\infty}^{t_1} \exp\left(-\frac{t_1-t_1'}{t_0^*}\right) v_j(t_1') E_j\left(\zeta_0 + \int_{t_0'}^{t_1'} v_\zeta dt_2\right) dt_1'\Big\}_{\text{av}}; \quad (5.13)$$

$$v_\zeta(t_0') = 0; \quad v_\zeta'(t_0') > 0; \quad v_\zeta(t_0'') = 0;$$

$$v_\zeta'(t_0'') < 0; \quad D > \left|\int_{t_0'}^{t_0''} v_\zeta dt_2\right|_{p_z=0}, \quad Q = D - \int_{t_0'}^{t_0''} v_\zeta dt_2. \quad (5.14)$$

$\zeta_0 \leq Q$, since the lowest point of the orbit $\zeta_0 + \int_{t_0'}^{t_0''} v_\zeta dt_2$ for electrons which do not collide with the surface lie inside the metal, that is, $\zeta_0 + \int_{t_0'}^{t_0''} v_\zeta dt_2 \leq D$. It is obvious that this formula gives accurately only Δj_i^{qu}, and not the classical value j_i, for which it is essential to make allowances also for the electrons which do collide with the surface.

Evaluating (5.13), with $\bar{v}_\zeta = 0$ and taking note of (5.4), we obtain

$$\Delta j_i^{qu}(\zeta) = \frac{h^3}{2\pi} H^2 \sum \chi_i \left(\frac{d \ln S}{d\varepsilon}\right)^2 \frac{\partial \Delta M_z}{\partial H}\Big|_{\varepsilon=\varepsilon_0,\ p_z=0}; \quad (5.15)$$

$$\chi_i = \frac{4\pi e^2}{h^3 t_0^{*-1}} \Big\{ v_i(t_1) S\left(\zeta - \int_{t_0'}^{t_1} v_\zeta dt_2\right) S\left(D - \zeta - \int_{t_1}^{t_0''} v_\zeta dt_2\right)$$

$$\times \int_{-\infty}^{t_1} \exp\left(-\frac{t_1-t_1'}{t_0^*}\right) v_j(t_1') E_j\left(\zeta - \int_{t_1'}^{t_1} v_\zeta dt_2\right) dt_1'\Big\}_{\text{av}}\Big|_{p_z=0}$$

$$\approx \frac{4\pi e^2}{h^3 t_0^{*-1}} \Big\{ v_i(t_1) S\left(\zeta - \int_{t_0'}^{t_1} v_\zeta dt_2\right) S\left(D - \zeta - \int_{t_1}^{t_0''} v_\zeta dt_2\right)$$

$$\times v_j(t_1') E_j\left(\zeta - \int_{t_1'}^{t_1} v_\zeta dt_2\right)\Big\}_{\text{cp}}\Big|_{p_z=0}. \quad (5.16)$$

This formula has the same form as (5.3), but it has a considerably narrower range of applicability. The fact that it appears to be exact in a parallel field is connected with the fact that the errors of the two approximations were reduced: in Eq. (5.6) and through the transformation in (5.16).

We note that in the general case in Eqs. (5.15), (5.16), $p_z = p_z^{ext}$ and the sum is carried out over all extreme sections, if $\pi/2 - \phi \ll \delta_{eff}/l$, and over central cross-sections ($p_z = 0$) if $\pi/2 - \phi \gg \delta_{eff}/r$. The form of Eq. (5.16) corresponds completely to what was said at the end of Section 4.

Thus $\Delta j_i^{qu} \sim (\hbar\Omega/\epsilon_0)^{1/2}$ always. The case when $v_i = 0$ at the point indicated in Section 4 might be an exception, also the case of a field aligned perpendicularly for an isotropic dispersion law, when $\bar{v}|_{p_z=0} = 0$ enters into the expression for Δj^{qu}, since at all times during the motion, $v_z = 0$. This leads to a decrease of Δj_i^{qu} by a factor of $\epsilon_0/\hbar\Omega$, but when Δj_i^{qu} increases essentially in conformance with the fact that the electron is at all times located in the skin layer and the normal skin effect occurs. For all cases, the oscillations of the chemical potential $\Delta\zeta_{chem} \sim (\hbar\Omega/\epsilon_0)^{3/2}$ do not have to be taken into account in any way.

Since $|\Delta j_i^{qu}| \ll j_i$, it is a simple matter to obtain the quantum-mechanical correction to the surface impedance. Its determination and also the solution of a series of other questions will be the subject of a separate paper.

I am grateful to L. D. Landau and to I. M. Lifshitz for valuable discussions.

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Translated by W. M. Whitney
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