ON THE THEORY OF SUPERFLUIDITY

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Taking quantum effects into account, we have obtained equations describing the behavior of superfluid helium near the λ-point in the stationary case. We have considered the properties of thin films of helium and the vortex line in helium near the λ-point.

1. THE FUNDAMENTAL EQUATION

THE present paper is devoted to the properties of helium near the λ-point. It is well known that the usual quasi-microscopic approach to a superfluid liquid, where the normal part is considered as an assembly of weakly interacting elementary excitations,[1] cannot be used in the immediate neighborhood of the transition point. In that temperature region there is a natural, different approach which is applied in the phenomenological theory of phase transitions of the second order.[2] In this theory one expands the thermodynamic potential of the system in a series of powers of some "order parameter" (and its derivatives) which in equilibrium is equal to zero on one side of the transition point. The equilibrium value of this parameter on the other side of the phase transition point is determined by requiring the thermodynamic potential to be a minimum with respect to this parameter. It is clear that in our case the expansion parameter must be connected with the density ρ_s of the superfluid part of the liquid which is different from zero in He II and equal to zero in He I.

Bearing in mind the quantum nature of the phenomena in liquid helium, it is natural to take for this parameter a complex function $\psi(x, y, z) = \eta e^{i\varphi}$, which plays the role of "the effective wave function" of the superfluid part of the liquid, so that the density ρ_s and the velocity v_s of the superfluid part can be expressed as follows in terms of ψ,

$$\rho_s = m|\psi|^2, \quad v_s = \frac{\hbar}{\sqrt{mm^*}}\nabla\varphi \tag{1}$$

where m is the mass of a helium atom and m^* some effective mass. There are grounds for assuming that the ψ-function introduced by us is closely connected with the true wave function of liquid helium; it can, for instance, be expressed in terms of the single particle density matrix.

In the present paper we restrict ourselves to those stationary problems where we can assume the normal part to be at rest, that is, where we can put $v_n = 0$. In that case we can write the thermodynamic potential F per unit volume of the liquid in the following form (if we take only the first term of the expansion in the gradient of ψ),

$$F = \frac{\hbar^2}{2m^*}|\nabla\psi|^2 + F_0(p, T, |\psi|^2). \tag{2}$$

The total thermodynamic potential can be written as $\int F dV$. If we take the variation with respect to ψ^* and ψ (with the boundary condition $\psi = 0$, vide infra) we get the equation

$$-\frac{\hbar^2}{2}\nabla\left(\frac{\nabla\psi}{m^*}\right) + \frac{\partial F_0}{\partial|\psi|^2}\psi = 0, \tag{3}$$

and also its complex conjugate for ψ^*.

We note that Eq. (3) is completely analogous to the one used in the phenomenological theory of superconductivity.[3] In the stationary case with $v_n = 0$ one can also use the equation of continuity in the form

$$\operatorname{div}(\rho_s v_s) = 0. \tag{4}$$

One can, however, easily satisfy oneself that (4) follows from (3) and its complex conjugate only if m^* does not depend on the coordinates and hence neither on the temperature or pressure, since the latter may depend on the coordinates. This shows that m^* must coincide with the true mass m of the helium atom [any "effective mass" would depend on temperature and pressure; see also the footnote to equation (18)]. We replace therefore m^* by m in (3). We finally have

$$-\frac{\hbar^2}{2m}\Delta\psi + \frac{\partial F_0}{\partial|\psi|^2}\psi = 0. \tag{5}$$

We could, of course, also use the free energy $F'(\rho, T)$ instead of the potential $F(p, T)$.

For applications, the boundary conditions for Eq. (5) are also important. Bearing in mind the assumed connection between the function ψ and the true wave function of helium which tends to zero at the boundary, we must assume that $\psi = 0$ on the boundary. The following argument leads to this condition. We consider the flow of a superfluid liquid along a solid wall. If $\rho_s \neq 0$ at the wall, the current of the superfluid part must experience a jump at the wall: liquid helium wets a solid body, that is, sticks to it, and v_s can not gradually decrease coming to the wall since curl $v_s = 0$. The occurrence of a break in the current when a body moves relative to the liquid must lead to an effect of the "dry friction" kind, since the jump will be connected with a surface energy.[4] However, specially arranged experiments showed that no such effect is present,[5] leading to the conclusion that at the wall $\rho_s = 0$.

The boundary conditions at a free surface of helium are not so obvious. We shall, however, also in that case assume that $\psi = 0$ at the surface.

We expand the thermodynamic potential F_0, as in the usual theory of phase transitions, in powers of $|\psi|^2$,

$$F_0 = F_I(p, T) - \alpha|\psi|^2 + \frac{\beta}{2}|\psi|^4 \tag{6}$$

(expansion (6) was already applied to helium in Ref. 6; F_I is the potential of helium I). At the λ-point $\alpha(p, t_\lambda) = 0$ and expanding in powers of $T - T_\lambda$ we get

$$\alpha = \left(\frac{d\alpha}{dT}\right)_{T_\lambda}(T - T_\lambda) = \alpha'_\lambda(T_\lambda - T).$$

As far as expansion (6) is concerned, we must note that it is not completely well founded, since the true character of the singularity at the transition point of a second-order transition is not known at the present time. Apart from that, the large value of the anomalous heat capacity in helium near the λ-point and the strong dependence of the jump on the pressure possibly indicate the proximity of a critical Curie point where $\beta = 0$. In that case we must also take into account in the expansion the term of the form $\gamma|\psi|^6$. However, bearing in mind the insufficiency of the experimental data, we shall for the sake of simplicity restrict ourselves to the expansion (6). The transition to another form of the function $F_0(|\psi|^2)$ does not introduce any fundamental difficulties.

If the state is uniform in space, and in equilibrium, we have

$$\frac{\partial F_0}{\partial|\psi|^2} = 0,\ |\psi|_e^2 = \frac{\rho_s}{m} = \frac{\alpha}{\beta} = \frac{\alpha'_\lambda(T_\lambda - T)}{\beta},$$

$$\Delta c_p = c_{pII} - c_{pI} = T_\lambda(\alpha'_\lambda)^2/\beta_\lambda.$$

Hence

$$\alpha'_\lambda = \Delta c_p m/T_\lambda\left|\frac{\partial\rho_s}{\partial T}\right|_\lambda, \qquad \beta_\lambda = \alpha'_\lambda m\Big/\left|\frac{\partial\rho_s}{\partial T}\right|_\lambda.$$

If we put

$$\Delta c_p = 1.0\cdot 10^7\ \text{erg-degree}^{-1}\,\text{cm}^{-3},$$

$$\left|\frac{\partial\rho_s}{\partial T}\right|_\lambda = 0.7\ \text{g-degree}^{-1}\,\text{cm}^{-3},$$

we get

$$\alpha'_\lambda \approx 4.5\cdot 10^{-17}\ \text{erg/degree and}\ \beta_\lambda \approx 4\cdot 10^{-40}\ \text{erg-cm}^3.$$

We note the tentative nature of these estimates.

We turn now to Eq. (5) and introduce new variables (the x_i are Cartesian coordinates)

$$\psi_0 = \psi/|\psi|_e = \psi/\sqrt{\alpha/\beta};\ \xi_k = x_k/l,\ l = \hbar/\sqrt{2m\alpha}.$$

The expression for the thermodynamic potential then becomes

$$F = F_I + \frac{\alpha^2}{2\beta}\{-2|\psi_0|^2 + |\psi_0|^4 + 2|\nabla_\xi\psi_0|^2\}, \tag{7}$$

and instead of (5) we have

$$\Delta_\xi\psi_0 = (|\psi_0|^2 - 1)\psi_0. \tag{8}$$

Using the calculated value of α we get

$$l \approx 4\cdot 10^{-8}/\sqrt{T_\lambda - T}.$$

In order that this macroscopic theory be applicable, it is necessary that l be much larger than the interatomic distances, i.e., that the condition $l \gg a \sim 3\times 10^{-8}$ cm be satisfied. If $l \sim a$, there are no special reasons to restrict ourselves in (2) to only the squares of the first derivatives. In that sense (2) gives us the first terms in an expansion in $(a/l)^2$. It is thus clear that the theory under consideration can only be applied in the immediate neighborhood of the λ-point.

2. SOLUTIONS OF SOME PROBLEMS

We consider first of all helium near a solid wall which we shall take as the xy plane. We shall assume that the helium is not moving; the presence of some velocity v_s leads only to a change in the coefficient of ψ_0 in Eq. (8), completely insignificant for all real velocities. If $v_s = 0$, the function ψ can be considered to be real and to depend on the z coordinate only. Equation (8) is then of the form

$$d^2\psi_0/d\xi^2 = (\psi_0^2 - 1)\psi_0 \tag{9}$$

with the boundary conditions

$$\xi = 0,\ \psi_0 = 0;\ \xi \to \infty,\ \psi_0 \to 1. \tag{10}$$

Equation (9) has a first integral

$$\left(\frac{d\psi_0}{d\xi}\right)^2 + \psi_0^2 - \frac{\psi_0^4}{2} = C = \text{const}.$$

The solution of equation (9) with the boundary conditions (10) is of the form

$$\psi_0(\xi) = \tanh(\xi/\sqrt{2}).$$

Here

$$\rho_s = \rho_{s_e} \tanh^2 (z/\sqrt{2}l).$$

The presence of a boundary leads to the appearance of an additional surface energy

$$\sigma = \int (F - F_{0e})\, dz = \frac{\alpha^2}{2\beta} l \int_0^\infty \left(\psi_0^4 - 2\psi_0^2 + 2\left(\frac{d\psi_0}{d\xi}\right)^2 + 1\right) d\xi$$

$$= \frac{4\sqrt{2}}{3}\left(\frac{\alpha^2}{2\beta}\right) l. \qquad (11)$$

We note that $\sigma \sim (T_\lambda - T)^{3/2}$ while

$$F_I - F_{0e} = \alpha^2/2\beta \sim (T_\lambda - T)^2.$$

We consider now a helium film, that is, a layer of helium of thickness α. The boundary conditions of Eq. (9) are now of the form*

$$\xi = 0,\ \psi_0 = 0;\ \xi = d/2,\ d\psi_0/d\xi = 0.$$

The corresponding solution of Eq. (9) is as follows

$$\xi = \int_0^{\psi_0} \frac{d\psi_0}{\sqrt{\psi_0^4/2 - \psi_0^2 + C}} = \frac{\sqrt{2}}{b} \int_0^{\psi_0/a} \frac{du}{\sqrt{(1-u^2)(1-k^2u^2)}}$$

$$\equiv \frac{\sqrt{2}}{b} F(\varphi, k), \quad \varphi = \arcsin(\psi_0/a),\ k = a/b,$$

$$a^2 = (1 - \sqrt{1-2C}),\ b^2 = (1 + \sqrt{1-2C}), \qquad (12)$$

where C follows from the equation

$$\frac{d}{2l} = \frac{\sqrt{2}}{b} F\left(\frac{\pi}{2}, \frac{a}{b}\right) \equiv \frac{\sqrt{2}}{b} K\left(\frac{a}{b}\right). \qquad (13)$$

A plot of the function $\psi_0(\xi)$ has the form of a dome, which is symmetric with respect to the middle of the film and the height of which decreases with decreasing film thickness d. Also, starting from some thickness d_k there is no solution with $\psi_0 \neq 0$, that is, the film ceases to be superfluid.

To find d_k it is sufficient to note that the complete elliptic integral $K(a/b)$ reaches its minimum value $\frac{1}{2}\pi$ for $a/b = 0$, that is, when $b = \sqrt{2 - a^2} = \sqrt{2}$. We get thus

$$d_k = \pi l = \pi\hbar/\sqrt{2m\alpha}. \qquad (14)$$

This result means that the temperature of the λ-point in the film is lower than in the case of large volumes of helium. The corresponding change ΔT_λ can be found from Eq. (14) which we can consider to be the equation determining T_λ for a given thickness d. We get finally

$$\Delta T_\lambda = \pi^2\hbar^2 / 2m\alpha'_\lambda d^2 \approx 2 \cdot 10^{-14}/d^2. \qquad (15)$$

The fact that T_λ was lower in films was observed experimentally.[7,8] Unfortunately, the film thickness in those experiments was only known as to order of magnitude. At the same time the theory can only be applied to films of thickness $d \gg a \sim 3 \times 10^{-8}$ cm since in the opposite case it is impossible, in particular, to take for α'_λ data referring to He II in bulk, and one must consider in detail the nature of the interaction with the solid wall. We shall thus give only one estimate. In Ref. 7, $\Delta T_\lambda = 0.146°$ for a film of thickness of about 18 atomic layers. At the same time we get from Eq. (15), for $d = 18 \times 3.6 \times 10^{-8} = 6.5 \times 10^{-7}$ cm, a change $\Delta T_\lambda \approx 5 \times 10^{-2}$. The discrepancy by a factor of 3, which we have found, can completely be caused by the inaccuracy of the values of ΔT_λ, d, and l we have used, even apart from the possible inapplicability of the expansion (6).

The thermodynamic potential of the film per unit volume is equal to

$$\overline{F} = F_I + \frac{\alpha^2}{2\beta}\frac{2l}{d}\int_0^{d/l} (C - 2\psi_0^2 + \psi_0^4)\, d\xi$$

$$= F_I + \frac{4\alpha^2 l}{3\beta d}\left\{\left(\frac{C}{2} - b^2\right)\frac{d}{2l} + \sqrt{2}\, bE\left(\frac{a}{b}\right)\right\}, \qquad (16)$$

where

$$E(a/b) = \int_0^{\pi/2} \sqrt{1 - (a/b)^2 \sin^2\varphi}\, d\varphi$$

is the complete elliptical integral of the second kind, and where the quantities a, b, and C are determined in terms of the film thickness [see Eqs. (12) and (13)].

The quantity $F_I - \overline{F}$ decreases with decreasing film thickness and for $d = d_k$ we have, of course, $\overline{F} = F_I$, that is, the transition to He I takes place. The heat capacity of the film decreases also when it gets thinner, which is in agreement with Frederikse's results.[9] Unfortunately the available data are insufficient for a quantitative comparison of the theory with experiment.

We shall now consider, on the basis of Eq. (8), a vortex filament in He II. The function ψ in that case can be written in the form

$$\psi(r, \varphi) = \Phi(r) e^{in\varphi}, \qquad n = 1, 2 \ldots. \qquad (17)$$

*The boundary conditions which we have written down are, under the assumptions we have made, strictly valid for the case of a film enclosed between solid walls. In the case of a free surface the problem of the boundary conditions is insufficiently clear.

where r, φ, and z are cylindrical coordinates; the z axis is taken along the axis of the filament. We shall see below that vortices with $n > 1$ are energetically unfavorable.

The velocity v_s has only a component $v_{s\varphi}$, and (see also Ref. 10)*

$$\oint v_s ds = v_{s\varphi} \cdot 2\pi r = \frac{2\pi n\hbar}{m}. \qquad (18)$$

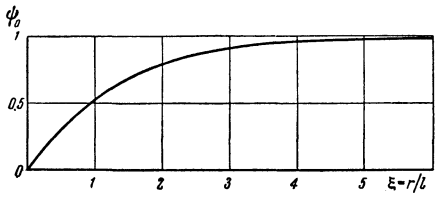
Furthermore, Eq. (8) will be of the form ($\xi = r/l$, $\psi_0 = \Phi\sqrt{\alpha/\beta}$)

$$\xi^2 \frac{d^2\psi_0}{d\xi^2} + \xi\frac{d\psi_0}{d\xi} + (\xi^2 - n^2)\psi_0 - \xi^2\psi_0^3 = 0. \qquad (19)$$

As $\xi \to 0$, the function ψ_0 is of the form $c_1 r^{\pm|n|}$. The solution with $\psi_0 \sim c_1 r^{-|n|}$ ($\psi_0 \to \infty$ as $\xi \to 0$) has no physical meaning and must be discarded. As $\xi \to \infty$ the density $\rho_s = m\psi_0^2$ must equal the unperturbed density ρ_{se}, i.e., $\psi_0^2 = 1$. One sees easily that in the region of large ξ Eq. (19) has the approximate solution

$$\psi_0^2 = 1 - n^2/\xi^2, \qquad \xi \gg 1. \qquad (20)$$

Equation (19) cannot be solved by quadrature over the whole range, and has been solved numerically. The result for $n = 1$ is given in the figure.



The thermodynamic potential per unit length of filament is equal to (the problem is here to find the difference of the potential when a solenoidal velocity is present or not)

$$f = (\alpha^2/2\beta)\,4\pi l^2 N = (\pi\hbar^2/m^2)\rho_{se}N, \qquad N = N_1 + N_2, \qquad (21)$$

$$N_1 = \frac{1}{2}\int_0^\infty \left[\psi_0^4 - 2\psi_0^2 + 2\left(\frac{d\psi_0}{d\xi}\right)^2 + 1\right]\xi\, d\xi, \quad N_2 = n^2\int_0^{R/l}\psi_0^2\frac{d\xi}{\xi}.$$

In this equation R is some maximum radius of integration (for instance, the diameter of the container, or the distance between vortex filaments) which must be introduced in connection with our consideration of an infinitely long filament. If we put $\rho = \rho_{se} = \text{const.}$, as should be the case, and if we do not take the gradient term into consideration, we find $N = N_e = n^2 \ln(R/a)$, where a is some distance of atomic dimensions. From a numerical calculation it follows that for $n = 1$, $N = \ln(1.46\, R/l)$, for $n = 2$, $N = 4\ln(0.59\, R/l)$, and for $n = 3$, $N = \ln(0.38\, R/l)$. If R is sufficiently large, the difference between N and N_e is of small importance; this is also natural since the main contribution to f comes from the distant, classical region. For the same reason we find that for large values of R the energy is roughly proportional to n^2 and states with $n > 1$ need not be taken into account (it is more favorable to form two vortex lines with $n = 1$ than one with $n > 1$). We note that solutions related to the one considered for the vortex line were earlier investigated by Abrikosov[11] on the basis of a theory of superconductivity.[3]

In conclusion the authors use this opportunity to thank Academician L. D. Landau for discussing the problems considered in the present paper.

If we had in Eq. (5) instead of the helium atom mass m some effective mass m, the right hand side of Eq. (18) would be $2\pi n\hbar/\sqrt{mm^*}$. This circulation can, however, not change with temperature. If $T = 0$, Eq. (18) is valid as follows from the arguments given in Ref. 10. It follows thus that $m^* = m$, as followed also from the considerations at the beginning of the present paper.

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248