$$J_1 = \int \frac{V_{-1,k} V_{k1}}{1-k^2} dk. \qquad (A.2)$$

The integral along the real axis of k, by-passing the singularities $k = \pm 1$, is broken up in the following way:

$$J_1 = \int_{-\infty}^{-1-\alpha} + \int_{-1+\alpha}^{1-\alpha} + \int_{1+\alpha}^{\infty} + \int_{\Gamma_1} + \int_{\Gamma_2}, \qquad (A.3)$$

where $\Gamma_1$ and $\Gamma_2$ are semicircles of radius $\alpha$ about the singular points. Making an error of order $\alpha$ in comparison with the leading term, we replace the exact values $V_{k,k'}$ by their asymptotic expressions (2.6) in the integrals along the straight lines, and extend the second integral to an interval from −1 to 1. The integrals over the semicircles are estimated with the help of (A.1). They give a contribution of order $\alpha$ as compared to the leading term.

In an analogous manner one can justify the application of formulae (2.9), (2.10) for the general term in the iteration series.

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# *RADIATION COOLING OF AIR. I.*
# *GENERAL DESCRIPTION OF THE PHENOMENON AND THE WEAK COOLING WAVE*

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We consider non-stationary radiation cooling of a large volume of air heated to a high temperature (on the order of tens and hundreds of thoudands degrees) by a strong explosion. It is shown that, owing to the strong temperature dependence of light absorption in the air, the cooling involves the propagation of a sharp temperature jump, i.e., of a cooling wave. Cooling from the initial high temperature to that at which the air becomes almost transparent and ceases to radiate occurs in a narrow wave front. A system of equations is derived, which permits an investigation of the internal structure of the cooling wave and leads to a connection between its parameters and the propagation velocity. A weak wave with a small temperature difference is considered.

## 1. QUALITATIVE DESCRIPTION OF THE PROCESS OF COOLING HEATED AIR

THE problem of a strong explosion in air was considered by Sedov[1] (see also Ref. 2). A strong shock wave heats the air irreversibly to a very high temperature, so that a large mass of very hot air is produced after the explosion, when the pressure returns to atmospheric.

Imagine a large mass of air with linear dimensions on the order of several hundreds of meters, heated to a high temperature — above 100,000° at the center; the temperature towards the periphery drops to below 1,000°. How is such a mass cooled? Obviously, the molecular heat conduction does not play any role at all: with a heat-diffusion coefficient (temperature conductivity) on the order of 1 $cm^2$/sec and with dimensions on the order of $10^4$ cm, it would take a year for the air to cool. The convective rise due to the difference between the densities of the hot and cold air and the mixing of the hot air with the surrounding masses of cold air, caused by the rise, are more substantial. However, the rise is small during the first 2 to 3 seconds. Obviously, the convective rise cannot exceed $gt^2/2$, which amounts to 5 m after one second, 20 m after two seconds, and 45 m after three seconds. Therefore, if we consider the first few seconds, convection can also be disregarded. The fundamental factor is the radiation of light from the air, to which this article is devoted.

A characteristic feature of this problem is that the transparency of the air depends strongly on the temperature. Cold air, as is known, is transparent to visible light, which indeed makes possible radiant cooling of a heated volume.

The continuous spectrum of light absorption in heated air is principally due to photoionization of the excited atoms. The ionization energy of a nitrogen or oxygen atom in the ground state ($I \approx 14$ ev) at temperatures on the order of 10,000° is considerably higher than the energies of the quanta, which play the principal role in a flux of energy $h\nu$ on the order of several kT. These quanta can be absorbed only by atoms excited to energies $I - h\nu$, the equilibrium number of which is proportional to the Boltzmann factor $\exp\{-(I + H\nu/kT)\}$. Therefore the free path of the light, which equals the reciprocal of the coefficient of absorption, depends very strongly on the temperature. The free path varies from kilometers at $T \approx 6{,}000°$ to meters at $T \approx 10{,}000°$ and centimeters at $T \approx 13{,}000°$.

Obviously, the radiation that cools the air is determined essentially by the layer in which the radiation free path is on the order of the dimensions of the system, i.e., by a layer of temperature on the order of 10,000°, which can be called the transparency temperature $T_2$. The colder air is transparent and does not radiate, the hotter air is opaque and radiates intensely, but its radiation is absorbed on the spot. These concepts, defining the effective radiating layer, are by no means new, and are universally used in the study of stars. However, unlike in the stars, the energy radiated by the air is not compensated for by an energy influx from an inner hotter region, since the temperature distribution is determined in our case principally by the past history of the phenomenon and is not stationary. It is therefore to be expected that if, as shown in Fig. 1, a certain smooth temperature distribution exists at the initial instant of time, the

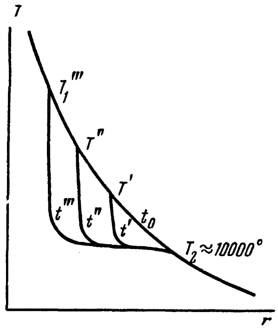


FIG. 1

first to begin cooling by radiation would be the layer with a temperature on the order of $T_2 \sim 10{,}000°$; in the subsequent instants the temperature distribution will change under the influence of radiation as shown in Fig. 1. One layer of air after another will be cooled to the transparency temperature. Propagating over the gas hotter then $T_2$ will be a temperature jump, a cooling wave (CW), in which the temperature drops sharply from an initial value $T_1$ to the transparency temperature $T_2$.

By representing the successive changes in temperature distribution as shown in Fig. 1, we disregard the changes in distribution due to purely hydrodynamic motion. Actually, the jump is formed even before the air pressure drops to atmospheric, and the hydrodynamic scattering stops at approximately that instant, when the radiant cooling of the layer of temperature $\sim 10{,}000°$ becomes comparable with the adiabatic cooling of the expanding air. Later on, when the adiabatic cooling diminishes rapidly with falling pressure, radiant cooling begins to play the principal role. To the contrary, prior to the formation of the jump, the principal role is played by the adiabatic cooling and the radiation losses are small.

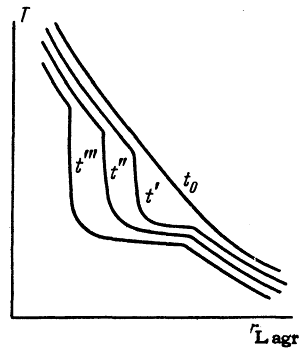


FIG. 2

Thus, taking the adiabatic cooling into account, the successive changes in the temperature distribution are shown in Fig. 2, where the abscissa represents the Lagrangian rather than the Eulerian coordinate.

It can be said that the CW propagates through air that is undisturbed by the radiation. The air temperature prior to the arrival of the wave, $T_1$, is determined only by the past history of the process and by purely hydrodynamic motion, if present. The point is that at temperatures on the order of tens and hundreds of thousands of degrees, and at temperature gradients on the order of thousands of degrees per meter, which frequently occur in the initial distribution, the radiant heat conduction is too small, owing to the strong absorption, to produce any noticeable energy flux in the region with initial temperature $T_1$. The radiant heat conduction, the coefficient of which (coefficient of proportionality between the thermal flux and the temperature gradient) is proportional to the free path of the light $l(T)$ and to the cube of the temperature, increases sharply with increasing temperature and plays a substantial role only at hundreds of thousands of degrees, limiting the temperature rise to the same order and equalizing the temperature near the center.*

Thanks to the low heat conduction on the upper edge of the CW at the temperature $T_1$, the energy flux into the wave from within is nearly zero and cannot have a significant value. All the properties of the CW, particularly its rate of propagation through the hot gas, are determined essentially by one quantity, the temperature $T_1$, of the initial gas. (The properties of the gas and its pressure are assumed specified.) The fundamental problem of the theory of the cooling wave is to find the energy flux $S_2$ radiated away from the surface of the wave. This flux lies obviously between the limits $\sigma T_1^4 > S_2 > \sigma T_2^4$ ($\sigma$ is the Stefan-Boltzmann constant). This problem is non-trivial, for the temperature changes very abruptly within the front of the CW. Once we find the flux $S_2$, the velocity of the wave is readily derived from energy-balance considerations

$$S_2 = u\rho_1 c_p (T_1 - T_2), \tag{1}$$

where $c_p$ is the specific of air at constant pressure, which we shall assume for simplicity to be constant, and $\rho_1$ is the density of the air through which the wave travels. The basis for writing such a balance is the fact that the velocity of the wave, according to estimates made, is subsonic, so that the pressure p is practically constant over the narrow front of the CW (as the air becomes cooler, it becomes compressed, so that $p \sim \rho T \approx \text{const}$).

The lower temperature of the CW or the transparency temperature is not a strictly defined quantity. This is that temperature, below which the absorption and radiation of light become very small. More accurately, it is the temperature at which the free path of the light becomes comparable with the characteristic dimension R, over which the temperature drops from $T_2$ to a sufficiently low value, say 1,000°,

$$l(T_2) \approx R. \tag{2}$$

When the wave propagates through expanding air, this dimension is determined by the hydrodynamics of the entire motion as a whole. The faster the adiabatic cooling, the smaller this dimension. Thanks to the exceedingly sharp exponential dependence of the free path on the temperature, the transparency temperature has a rather narrow range, in spite of the arbitrariness in its definition, and depends logarithmically on the dimension R and on the air density $\rho_1$.

If the free path, suitably averaged over the spectrum, is

$$l = a(T)(\rho_0/\rho)\, e^{I/kT}, \tag{3}$$

where a is a slowly-varying function of T (we assume for air $a = 2.8 \times 10^{-12} \times T^2$ cm), $\rho_0$ is the normal air density (see below), than the transparency temperature, according to (2) is

$$T_2 = \frac{I}{k}\left(\ln\frac{R\rho}{a\rho_0}\right)^{-1}. \tag{4}$$

It will be shown below that the radiation from the surface of the CW is always generated at the lower edge of the step, regardless how high the step, i.e., at initial gas temperatures $T_1$ as high as convenient. The flux $S_2$ radiated by the CW is determined principally by the transparency temperature and equals approximately $2\sigma T_2^4$.

The speed of propagation of the CW, which is proportional to

$$u \sim 2\sigma T_2^4 / p(1 - T_2/T_1), \tag{5}$$

in the case of a sufficiently strong wave, when $T_2 \ll T_1$, thus depends principally only on the pressure.* The table gives several calculated

*The coefficient of radiant heat conduction again becomes large at low temperatures (below ~ 10,000°), owing to the sharp increase in the free path $l$, which passes through a minimum at $T \sim 50{,}000°$ [$l(T)T^3$ has a minimum at $T \sim 10{,}000°$]. However, at larger free paths, comparable with the dimensions of the system, the radiation transfer no longer has the nature of heat conduction.

*If it is taken into account that at high temperatures ionization causes the specific heat to increase with temperature, then the relations (1) and (5) become somewhat more complicated. In this case it becomes necessary to write the specific enthalpies $W(T_1 p)$ and $W(T_2 p)$ in lieu of $c_p T_1$ and $C_p T_2$.

**u, km/sec at p = 1 atmos**

| R, m | 10 | 50 | 100 |
|---|---|---|---|
| $T_2^0$ | 10 700 | 9 700 | 9 300 |
| $T_1^\circ$ | | | |
| 20 000 | 2.7 | 2.1 | 1.7 |
| 50 000 | 1.8 | 1.4 | 1.1 |
| 100 000 | 1.6 | 1.2 | 1.0 |

values of the velocity u in kilometers per second in air at atmospheric pressure, at various values of $T_1$ and $T_2$. The same table indicates also the values of R, from which the temperature $T_2$ was obtained with formula (4).

It is shown in the theory of heat conduction that the time t required to cool a hot body is proportional to $R_0^2 c_p \rho/\kappa$, where $\kappa$ is the coefficient of heat conduction, and $R_0$ is the dimension of the body. The relation $t \sim R_0^2$ is based on the assumption of a gradual similar reduction in temperature of the entire mass of the body. If a hot body is cooled by radiation, with a cooling wave traveling from the periphery toward the center, the cooling time is quite different, namely $t \sim R_0/u$. Thus, a mass of air approximately 100 m in radius, heated at atmospheric pressure to temperatures on the order of tens and hundreds of thousands of degrees, cools down by radiation to about 10,000° within approximately 0.1 seconds. The radiation cooling that follows is considerably slower and is of an entirely different, three-dimensional character (since the free path of the light becomes comparable with the dimensions of the body). The mechanism of absorption and radiation of light now becomes different.

Owing to the great extent of the lower edge of the CW, and also owing to absorption and radiation of light by the air cooled by the CW, the front of the CW almost always remains invisible. All these questions, including that of the possibility of experimental observation of the CW, are beyond the scope of this investigation.

We develop below an approximate theory of the CW, i.e., we examine in detail that narrow layer, in which the temperature drops sharply from $T_1$ to $T_2$.

## 2. STATEMENT OF THE COOLING-WAVE PROBLEM

Disregarding the specific dimensions and shape of the cooled air mass, we seek a solution for the non-stationary equations of radiant heat exchange in the form $T(x - ut)$, corresponding to a plane wave propagating at constant speed u in a gas of specified temperature $T_1$ and density $\rho_1$. The speed u itself should be found from equations similar to those used to determine the speed of a flame in an explosive mixture.

Actually, the equations do not have an exact solution of the form $T(x - ut)$. The point is that as the wave propagation leads to an increase in the thickness of the layer of cooled air in which the absorption of light, although small, is nevertheless different from zero, and the transparency temperature changes with time. In an unbounded medium, the layer of gas cooled to as low a temperature as desired owing to its infinite extent, turns out to be quite opaque. The flux then vanishes at infinity and no CW mode exists in the strict sense of the word.* This factor, of prime significance in the case of an unbounded medium, raises only an apparent difficulty under real conditions. In fact, the hot region is always bounded and the transparency temperature changes but little with increasing distance covered by the wave, being contained within a rather narrow interval if the system is of practical dimensions. An additional, very slow time variation of the solution, occurs only at the very lowest edge of the wave, which is quite elongated, in the almost-transparent region of the already cooled air.

If the CW propagates in expanding air, adiabatic cooling soon lowers the temperature of the radiation-cooled layers enough to make them practically transparent. An additional slow time variation of T will exist only in the region of adiabatic cooling and will hardly affect the temperature profile in the CW itself.

We shall not consider here the additional absorption of light in the region of low temperatures, on the order of several thousands of degrees, due to the nitrogen oxide and the dioxide formed in the hot air. This absorption hardly affects the wave, although it may play a substantial role in the absorption of the radiation flux from the surface of the wave in the peripheral layers of the air.†

We shall neglect, in addition, the intense molecular absorption in the low-temperature region, a substantial factor for ultraviolet radiation with $\lambda < 2{,}000$ A, since at temperatures on the order of 10,000° and below this part of the spectrum contains only a small fraction of the energy (less than

*To a certain extent, an analogous situation exists in the theory of the stationary propagation of a flame. If the speed of the chemical reaction in the uncombusted mixture is not assumed to be exactly zero, although actually it is a finite but vanishingly small quantity, the mixture will burn out before the flame front reaches it.

†The unique optical effects connected with the formation of oxides of nitrogen in a strong explosion have been considered in detail by one of the authors in Ref. 3.

4%), hardly affecting the energy balance of the CW. To formulate a cooling-wave theory it is necessary to examine, as is usually done in the theory of modes, a plane stationary process in the coordinate system in which the CW is at rest. In order to eliminate the difficulty indicated above and to make the problem stationary, i.e., to change from the true solution $T(x - ut, t)$ (with an additional slow time variation) to the idealized solution $T(x - ut)$, it is possible to employ one of two formally artificial measures. These, however are quite justified physically and, by virtue of what has been said, correspond to the real state of affairs.

It is possible, first, to introduce into the energy equation an additional constant term A, which plays the role of adiabatic cooling. This term specifies the constant dimension R which determines the transparency temperature $T_2$ and makes the absorption in the radiation-cooled region finite. The energy-balance equation becomes in the stationary case

$$u\rho_1 c_p \frac{dT}{dx} + \frac{dS}{dx} = -A, \tag{6}$$

where S is the radiation-energy flux at the point x.

It is possible to disregard completely the weakly-absorbing region of the gas, cooled below the transparency temperature, by determining the transparency temperature $T_2$ at the very outset from formula (4), and by assuming that the medium is absolutely transparent at $T < T_2$ ($l = \infty$).

To determine the radiation flux we employ the diffusion approximation of the rigorous kinetic equation. This approximation takes the angular distribution of the radiation into account in an approximate manner. In the diffusion approximation we add to the rigorous equation for the radiation balance

$$dS / dx = c\,(U_{\text{eq}} - U) / l \tag{7}$$

the approximate connection between the flux S and the radiation energy density* U

$$S = -{}^1/_3\, lc dU / dx. \tag{8}$$

Here

$$U_{\text{eq}} = 4\sigma T^4 / c \tag{9}$$

is the equilibrium radiation density, and c is the velocity of light. We disregard the spectral composition of the radiation, characterizing the radiation transfer in a suitable manner by means of a free path $l$ averaged over the spectrum.

It will be shown below that in a considerable portion of the CW the true radiation density U is quite close to the equilibrium density $U_{\text{eq}}$. In this case, as is known,[4] the free path is averaged as done by Rosseland. In the region of the cooled air, U differs greatly from $U_{\text{eq}}$ and the free path should be averaged, quite differently. For simplicity we shall use everywhere the Rosseland average, taking advantage of the fact that the Boltzmann exponential factor remains equal to $l$ for any averaging method, and that all the important effects in the CW depend only logarithmically on the multiplier in front of the exponential, which naturally depends on the method of averaging. The Rosseland averaging of the Kramers formula for the photoelectric absorption of quanta by excited atoms[4] yields, after substitution of known constants, the multiplier $a(T)$ in front of the exponential of formula (3) for the free path.

In Eqs. (7) and (8) it is convenient to change from the geometrical coordinate x to the optical thickness $\tau$, using the formula

$$d\tau = -dx / l, \quad \tau = -\int dx / l, \tag{10}$$

and reckoning $\tau$ from the place where $l = \infty$, in the direction of increasing absorption, i.e., of higher temperature:

$$dS / d\tau = -c\,(U_{\text{eq}} - U); \tag{11}$$

$$S = {}^1/_3 c dU/d\tau \tag{12}$$

Dispensing with exact calculation of the angular distribution of the radiation, it is also possible to write approximate integral expressions for the flux and for the density by assuming that all the quanta move "forward" and "backward" parallel to the x axis

$$S = \frac{c}{4}\left[\int_\tau^\infty U_{\text{eq}} e^{\tau-\tau'} d\tau' - \int_0^\tau U_{\text{eq}} e^{\tau'-\tau} d\tau'\right]; \tag{13}$$

$$U = \frac{1}{2}\left[\int_\tau^\infty U_{\text{eq}} e^{\tau-\tau'} d\tau' + \int_0^\tau U_{\text{eq}} e^{\tau'-\tau} d\tau'\right]. \tag{14}$$

The coefficients in front of the square brackets are chosen such as to make formulas (13) and (14) give the correct values of the flux from the surface of an absolutely black body and the correct density inside the black body, away from the boundary. To take effective account of the angular distribution it is necessary to employ in these formulas not the true free path, but half its value. It is easy to check that in this case (13) and (14) satisfy Eqs. (11) and (12), which are of the diffusion type, with a coefficient of diffusion proportional to $\frac{1}{4}$ instead

*One must not confuse the diffusion approximation with the approximation of radiant heat conduction, which is one particular case in which the true density U in Eq. (8) is replaced by the equilibrium value $U_{\text{eq}}$.

of to $1/3$.* On the upper edge of the CW, as already mentioned above, the flux is nearly zero, and therefore one of the boundary conditions for Eqs. (11) and (12) is

$$\tau = \infty, \quad S = 0. \tag{15}$$

The second boundary condition should be specified on the boundary between the absorbing and absolutely transparent media, i.e., at $\tau = 0$. This is the well known diffusion condition, by which the diffusion flux on the boundary with the "vacuum" equals half the kinetic flux

$$\tau = 0, \quad S = cU/2. \tag{16}$$

Integral expressions (13) and (14) satisfy this condition automatically.

## 3. WEAK COOLING WAVE

Let us consider the limiting case of a weak CW, in which the upper temperature $T_1$ barely exceeds the lower one $T_2$. The free path, however will be assumed here quite strongly dependent on the temperature, so that the following conditions become compatible: the condition $l(T_1) \ll l(T_2)$, which is necessary for the very existence of the CW, and the condition $T_1 \approx T_2$, $T_1^4 \approx T_2^4$, which is necessary if the wave is to be considered a weak one.

The examination of the weak wave is of interest essentially as far as method goes. With this example, by simplification of the initial equations, we can obtain an exact analytical solution of the equations. Let us use the first of the artifices indicated in the preceding section and assume that constant adiabatic cooling A exists along with radiant heat exchange, so that the energy equation is written in form (6). The integral of the energy equation (6) contains an integration constant C, determined by the choice of the origin for the coordinate x, i.e., arbitrary (the equation has a translation group):

$$u\rho_1 c_p T + S = -Ax + C. \tag{17}$$

On the lower and upper edges of the CW, where the flux S tends to $S_2$ (the flux that goes to infinity) and to zero, the quantity $u\rho_1 c_p T$ tends asymptotically to two straight lines

$$u\rho_1 c_p T = -Ax - S_2 + C_2, \quad x \to \infty; \tag{18}$$

$$u\rho_1 c_p T = -Ax + C, \quad x \to -\infty, \tag{19}$$

whose ordinates are apart by the amount of flux $S_2$ that goes to infinity.

The step in the CW is contained between these two lines: our problem consists of finding the position of this step. Let us now use the condition that the wave is weak. Since the phenomenon plays itself out in a narrow temperature range, it is possible to assume approximately in the equations for the radiation transfer that the factor $U_{eq}$, to which the radiating ability is proportional, is a constant. Obviously this factor, whose limits in the wave are

$$4\sigma T_2^4/c < U_{eq} < 4\sigma T_1^4/c,$$

can be set equal to any of these limits, since the two limits are nearly equal. We shall assume specifically that $U_{eq} = 4\sigma T_2^4/c$. Here obviously the flux going to infinity is

$$S_2 = \sigma T_2^4, \tag{20}$$

If $U_{eq}$ is constant, the equations for the radiation transfer become much simpler. We start with the integral expression (13) and obtain

$$S = S_2 e^{-\tau}. \tag{21}$$

Inserting (21) into (17) we get

$$u\rho_1 c_p T = -Ax - S_2 e^{-\tau} + C. \tag{22}$$

When the temperature obtained from this formula is inserted into (10) we get a first-order differential equation for the function $x(\tau)$ and, returning to (22), we get $T(\tau, S_2)$ and $T(x, S_2)$. To solve this equation we note that, in the narrow temperature range of interest to us, the actual Boltzmann dependence of the free path on the temperature, which is given by formula (3) (we neglect the weak temperature dependence of the multiplier ahead of the exponent), can be approximated by an exponential one

$$l = (a\rho_0/\rho) \exp\left\{\frac{I}{kT_0} - (T - T_0)\frac{I}{kT_0^2}\right\}$$

$$= l(T_0) \exp\left\{-\frac{(T - T_0) I}{kT_0^2}\right\}, \tag{23}$$

where $T_0$ is a certain temperature about which the exponent is expanded.

Such an approximation was made by Frank-Kamenetskii in the theory of thermal explosions.[5] This formula automatically insures that the flux tends to zero at $x \to -\infty$, where the temperature tends to infinity [and its gradient, according to (22), is finite], which is essential for the existence of the mode. The temperature $T_0$, about which the expansion is made and which can be specified arbitrarily, will be defined by the equation

*Using half the value of the free path means that the average cosine of the "forward" and "backward" quanta is assumed to be one-half. The differential equations equivalent to the integral expressions in (13) and (14) are known in astrophysics as the Schwarzschild approximation.[4]

$$Al(T_0)\,I/u\rho_1 c_p T_0 k T_0 = 1. \tag{24}$$

Let us change to dimensionless quantities, using the formulas

$$\Theta = (T - T_0)\,I/T_0 k T_0, \tag{25}$$

$$d\xi = dx/l(T_0); \tag{26}$$

$$s = \frac{S_2}{u\rho_1 c_p T_0}\frac{I}{kT_0} = \frac{S_2}{Al(T_0)} = \frac{I}{kT_0}\frac{(T_1 - T_2)}{T_0}. \tag{27}$$

Equations (22) and (10) assume the following form in dimensionless quantities

$$\Theta = -\xi - se^{-\tau} + C, \tag{28}$$

$$d\xi = -e^{-\Theta}\,d\tau. \tag{29}$$

Their solutions, with the boundary conditions $\tau = 0$, $\xi = \infty$ $(x = \infty)$

$$\xi = -\ln\left[\int_0^\tau e^{se^{-\tau}} d\tau\right] + C, \tag{30}$$

$$\Theta = \ln\left[\int_0^\tau e^{se^{-\tau}} d\tau\right] - se^{-\tau} \tag{31}$$

yields the parametric relation for $\Theta(\xi)$ i.e., the temperature profile in a weak CW. Using the substitution $z = e^{-\tau}$, the integral in (30) or (31) is expressed in terms of the tabulated $\overline{\mathrm{Ei}}(x)$ functions (Ref. 6)

$$\overline{\mathrm{Ei}}(x) = \int_{-\infty}^{x} e^{y}\frac{dy}{y}, \tag{32}$$

namely:

$$J = \int_0^t e^{se^{-\tau}} d\tau = \overline{\mathrm{Ei}}(s) - \overline{\mathrm{Ei}}(se^{-\tau}). \tag{33}$$

On the lower edge of the CW, the temperature approaches the lower straight line asymptotically (at $\tau \ll 1$, $s\tau \ll 1$, $\Theta \to -\infty$), in accordance with

$$\xi = -\Theta - s + s(1 - \exp(-e^{\Theta}) + C. \tag{34}$$

On the higher-temperature side the profile $\Theta(\xi)$ has the character of a step, whose slope increases all the time with increasing $\Theta$. Only when $\Theta$ almost reaches the upper straight line does the curve $\Theta(\xi)$ pass through the point of inflection and begins to approach the upper straight line asymptotically, again in accordance with (34), but for

$$\tau \gg 1,\quad se^{-\tau} \ll 1,\quad \Theta \to +\infty,$$

These laws are illustrated in Fig. 3, which shows the plot of $\Theta(\xi)$ at $s = 5$. $\xi$ is measured from the point at which $\Theta = 0$. It is natural to assume the front of the CW to be the point of inflection of $\Theta(\xi)$, a point at which the slope of the step has a maximum, and to assume the upper and lower

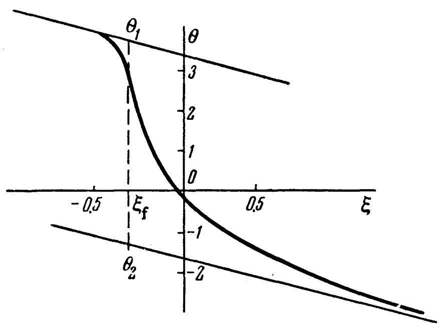


FIG. 3

temperatures of the CW to be the values of $\Theta$ on the asymptotic lines at the coordinate of the point of inflection (see Fig. 3).

The optical thickness $\tau_f$ corresponding to the front of the CW can be found from the equation $d^2\Theta/d\xi^2 = 0$. Differentiation of (30) and (31) gives a transcendental equation for $\tau_f$ as a function of the parameter s.

$$J_f = \overline{\mathrm{Ei}}(s) - \overline{\mathrm{Ei}}(\beta) = e^{\beta}/(1-\beta),\quad \beta = se^{-\tau_f}. \tag{35}$$

The temperature at the point of inflection, $\Theta_f$, and also the upper and lower temperatures $\Theta_1$ and $\Theta_2$ of the CW, are

$$\Theta_f = -\ln(1 - \beta), \tag{36}$$

$$\Theta_1 = \Theta_f + \beta, \tag{37}$$

$$\Theta_2 = \Theta_f + \beta - s. \tag{38}$$

The problem of finding the lower temperature $T_2$ of the CW, and consequently the velocity u of the CW for specified upper temperature $T_1$ and for adiabatic cooling A, is readily solved by successive approximations. We assume some value of the parameter s and use Eqs. (34) to (38) to calculate $\Theta_1$ and $\Theta_2$. Then, changing to real temperatures in accordance with (25), we determine $T_0$ and $T_2$. Inserting these values into (27) we find the parameter s in the next approximation, etc. The successive approximations converge rapidly, since $T_0$ depends logarithmically on s.

It is more convenient to proceed in the reverse manner: specify the values of the parameter s and any one of the two quantities characterizing the CW, either $T_1$ or $T_2$, and then determine the second quantity and the adiabatic cooling A necessary to insure the existence of a stationary mode. Thus, for the case $s = 5$ illustrated in Fig. 3, we obtain from Eqs. (35) to (38): $\beta = 0.93$, $\tau_f = 1.69$, $\Theta_f = 2.7$, $\Theta_1 = 3.7$, and $\Theta_2 = -1.3$. For example, at an upper temperature $T_1 = 12{,}250°$, the lower temperature turns out to be $T_2 = 9200°$, with $T_0 = 10{,}000°$ (I is assumed to be 14 ev for air).

The only really interesting values of the parameter s are those much greater than unity. In fact, it follows from (23) that

$$s = \Theta_1 - \Theta_2 = l(T_1)/l(T_2), \tag{39}$$

and the ratio of the free paths must be greater than unity for the very existence of the CW, as already mentioned at the beginning of this section. (On the other hand, s is bounded from above by the condition that the wave must be weak.) In the case when $s \gg 1$, all the formulas become substantially simplified and an approximate relation can be established in explicit form between the lower temperature of the CW and the value of the adiabatic cooling. In this case the temperature $T_0$ about which the range is expanded drops out entirely from the equation.

Using the asymptotic expression $\overline{\mathrm{Ei}}(s) \approx e^s/s$ for $s \gg 1$, and noting that when $s \gg 1$ the root of (35) is $\beta \approx 1$, ($\tau_f \approx \ln s$), we obtain from (31) and (35)

$$\Theta_f \approx \ln \overline{\mathrm{Ei}}(s) - 1 \approx s - \ln s - 1. \tag{40}$$

From this we obtain from (38)

$$\Theta_2 \approx -\ln s. \tag{41}$$

Returning to the true temperature with the aid of (25) and taking (27) and (23) into account, we obtain the desired relation

$$Al(T_2) = S_2 = \sigma T_2^4, \tag{42}$$

It must be noted that according to (37) $\Theta_1 > \Theta_f > 0$, and according to (41) $\Theta_2 < 0$, i.e., the free path is expanded in accordance with (23) about the intermediate temperature in the CW:

$$T_2 < T_0 < T_f < T_1.$$

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# *EXCITATION OF VIBRATIONAL AND ROTATIONAL STATES OF NUCLEI DUE TO SCATTERING OF NUCLEONS*

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Scattering of fast nucleons by black nuclei possessing vibrational or rotational levels is considered in the adiabatic approximation. It is shown that, in the diffraction region of scattering angles, the shape of the angular distributions of nucleons of definite energy, scattered with excitation of a given collective level of an even-even nucleus, does not depend on whether the level is a rotational or vibrational one.

WE consider the scattering of fast neutrons or protons from nuclei possessing vibrational or rotational excited states.[1] We shall assume that the wavelength of the incident particle $k^{-1}$ is much smaller than the nuclear dimension R ($kR \ll 1$), that the energy of the proton significantly exceeds the value of the Coulomb barrier ($Ze^2/RE \ll 1$), and that the nucleus absorbs all particles incident upon it (black nucleus). These assumptions correspond to neutron energies $E \gtrsim 10$ Mev and pro-