$$\rho_\alpha = 0.5\cdot 10^{-8},\ \rho_\beta = 0.45\cdot 10^{-8},\ \rho_\gamma = 0.34\cdot 10^{-8}\ \text{sec.}$$

In weak fields ($H_0^2 \ll H_i^2$) we get $\rho = 0.2 \cdot 10^{-8}$ sec. For a polycrystalline specimen

$$\rho = {}^1/_3 (\rho_\alpha + \rho_\beta + \rho_\gamma).$$

The resulting dependence of the relaxation time on the orientation of the $CuSO_4 \cdot 5H_2O$ crystal in an external magnetic field H_0 agrees very well with the experimental results obtained by Volokhova.[9] With a certain approximation the dependence of ρ on the absolute value of H_0 also agrees with the results obtained by Volokhova. The temperature dependence of ρ is determined by the integral I_8.

If the constant λ is isotropic, then the probability of the relaxation transition A_{+-} (7) will also be isotropic. The anisotropy of ρ will be determined in this case by the anisotropy of the g-factor only. With $g_\parallel = 2.4$ and $g_\perp = 2.1$ the computed values of ρ for H_0 directed along the three magnetic axes of the crystal differ by no more than 4%, i.e., it is difficult to attribute to the anisotropy of the g-factor even a comparatively small (10 to 20%) anisotropy of ρ as observed in the crystals of Tutton's copper salts.[9] It may be presumed that the constant λ of the spin-orbit coupling is also anisotropic here, the more so since these salts are similar to $CuSO_4 \cdot 5H_2O$ in their crystalline structure.

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SOVIET PHYSICS JETP VOLUME 34(7), NUMBER 6 DECEMBER, 1958

KINETIC THEORY OF THE FLOW OF A GAS THROUGH A CYLINDRICAL TUBE

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By introducing a certain "anisotropy function" a good description of the flow of a gas can be obtained, even at pressures at which only empirical formulas have been used hitherto. Furthermore, the term corresponding to the slipping of the gas relative to the walls is obtained automatically, without any additional hypotheses; the same is true of the minimum rate of flow at intermediate pressures. Our final formula is qualitatively correct at all pressures, including intermediate ones.

IT is well known that hydrodynamics cannot provide the solution of the problem of the flow of gases at low pressures. Knudsen[1] succeeded in establishing the correct laws of flow at such pressures by using kinetic theory. In an intermediate range of pressures, however, neither hydrodynamics nor the kinetic theory as developed by Knudsen can give correct results.

To obtain the laws of flow at arbitrary pressures one must resort to interpolation, as was done by Knudsen. None of these formulas, however, gives an explanation of the laws of flow at intermediate pressures.

Pollard and Present[2] developed a theory for small pressures, in which they took account not only of collisions of the molecules with the walls but also of collisions between molecules. These authors assume that out of the $(n\bar{v}/\lambda)\,d$ molecules that collide during unit time in the volume $d\tau$, the number that leave the volume element in the solid angle $d\omega$ is independent of its orientation relative to the axis of the tube and is given by

$$(n\bar{v}/\lambda)\,d\tau\,d\omega/4\pi. \qquad (1)$$

Consequently the method of Pollard and Present amounts essentially to a kinetic theory of diffusion in tubes. Their results for the diffusion of gases at low pressures are in better agreement with experimental data than the results of the ordinary kinetic theory of gases.

Hiby and Pahl[3-5] have shown that the collision process in a flowing gas gives rise to an anisotropy in the distribution of the molecules that have had collisions. The result is an additional current due to this anisotropy. They have given a purely kinetic formulation of the problem of gas flow. In this formulation, however, the calculations are extraordinarily complicated, so that these authors have in fact had to confine themselves to obtaining a second-order correction to the calculations of Pollard and Present. The predictions of the theory of Hiby and Pahl are in good agreement with the results of experiments at low pressures.

We attempt below to give a unified phenomenological approach to the problem of gaseous flow, which leads to good results at both low and ordinary pressures. The basis of our arguments is the method of Pollard and Present, to which we add certain plausible physical assumptions.

Let us consider a very long cylindrical tube, between the ends of which there is a pressure drop that is extremely small in comparison with the average pressure in the tube. Our hypothesis is that the number of molecules leaving the volume element $d\tau$ in the tube in the solid angle $d\omega$ per unit time is given by

$$\frac{n(x)+f_a}{\lambda}\bar{v}\,d\tau\frac{d\omega}{4\pi},$$

where $\lambda^{-1}n(x)\bar{v}d\tau d\omega/4\pi$ is the isotropic contribution and f_a is a function that characterizes the

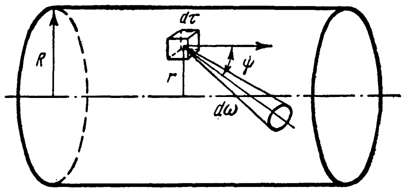

FIG. 1

anisotropy and depends on: (a) the density $n(x)$ of the gas in $d\tau$, (b) the gradient of the density along the axis of the tube, dn/dx, (c) the distance r between $d\tau$ and the axis of the tube, (d) the angle ψ between the axis of the solid angle $d\omega$ and the axis of the tube, and (e) a specific molecular quantity s_p (see Fig. 1).

The expression given above can be written in the form

$$\bar{v}d\tau\frac{d\omega}{4\pi}\left[n(x)+f_a\left(r, R, \frac{dn}{dx}, \psi, n, s_p\right)\right]\Big/\lambda. \qquad (2)$$

In order to obtain the explicit form of the function f_a we make use of the characteristic data of our problem. Since there is a small difference of pressure between the ends of the tube, f_a can be expanded in a power series in dn/dx, in which we keep only the first two terms. If there is no density gradient, the anisotropy function is zero and the first term in the expansion is zero. Thus we have

$$f_a=\frac{dn}{dx}f(r,\ \psi,\ n,\ s_p).$$

If we assume that the molecules are diffusely reflected by the walls, then we can suppose that the anisotropy function is zero or close to zero at $r = R$. The maximum anisotropy is attained at the axis of the tube, so that if we expand $f(r, \psi, n, s_p)$ in powers of r we get the following form:

$$f={}^1/_2(r^2-R^2)\,f_1(\psi,\ n,\ s_p).$$

We shall assume that the dependence of f on ψ is such that a larger number of molecules leaves $d\tau$ in the direction of flow of the gas than in the opposite direction. This means that f is positive for values of ψ which correspond to the direction of the flow and is negative for the opposite direction. In relation to the molecules that emerge after colliding, the flowing gas acts like a semitransparent mirror which makes the majority of these molecules leave the volume element in the direction of the flow. We shall assume that in the perpendicular direction the distribution remains unchanged as compared with the isotropic case. From the multitude of angular functions that satisfy the requirements given above, we choose the simplest:

$$f_1(\psi,\ n,\ s_p)=f_2(n,\ s_p)\cos\psi.$$

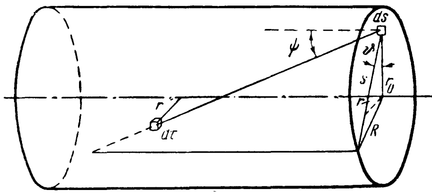

FIG. 2

(It is obvious that in a rigorous treatment one would have to use Fourier series, but for our purpose the approximation just made is satisfactory.)

With a correct choice of $f_2(n, s_p)$, the anisotropy function increases with increasing pressure. (It is obvious that at very small pressures the anisotropy vanishes, and that it increases as the pressure is raised.) We set

$$f_2(n,\ s_p) = k/\lambda$$

and shall show that this choice leads to results in agreement with the experimental data. Let us consider the tube shown in Fig. 2. The number of molecules that traverse the element of area dA of a cross-section in unit time is given by the expression of Pollard and Present with an added anisotropic term:

$$dN = dN_{is} + kdA\frac{\bar v}{4\pi}\int_\tau \frac{(r^2 - R^2)\cos^2\psi e^{-\rho/\lambda}}{2\lambda^2\rho^2}\, d\tau\, \frac{dn}{dx}. \quad (3)$$

Choosing spherical coordinates with the center at the point dA and remembering that for very long tubes dn/dx is practically constant, we get

$$dN = dN_{is}\cdot$$
$$+ kdA\frac{dn}{dx}\frac{\bar v}{4\pi}\int_0^{2\pi} d\varphi \int_0^{\pi} d\psi \cos^2\psi \sin\psi \int_0^{s/\sin\psi} \frac{r^2 - R^2}{\lambda^2} e^{-\rho/\lambda} d\rho. \quad (4)$$

Using the notations of Fig. 2 we have

$$r^2 = r_0^2 + \rho^2\sin^2\psi - 2r_0\rho\sin\psi\cos\vartheta,$$
$$R^2 = r_0^2 + s^2 - 2r_0 s\cos\vartheta.$$

Substituting this into Eq. (4) and integrating with respect to φ, we get

$$dN = dN_{is} + kdA\frac{dn}{dx}\frac{\bar v}{4}\int_0^{\pi} d\psi \cos^2\psi\sin\psi \times$$
$$\times \int_0^{s/\sin\psi} (r_0^2 - R^2 + \rho^2\sin^2\psi - 2r_0\rho\sin\psi\cos\vartheta)\lambda^{-2}e^{-\rho/\lambda}d\rho. \quad (5)$$

In the case of a small pressure drop along the tube we can regard the mean free path as approximately constant, so that in Eq. (5) we may carry out the integration with respect to ρ:

$$N = N_{is}$$
$$+ \frac{k\bar v}{4}\iint dA \int_0^{\pi} \cos^2\psi \sin\psi \left\{\left(\frac{r_0^2 - R^2}{\lambda} - 2r_0\sin\psi\cos\vartheta + 2\lambda\sin^2\psi\right)\right.$$
$$\left. + (2r_0\sin\psi\cos\vartheta - 2s\sin\psi - 2\lambda\sin^2\psi)e^{-s/\lambda\sin\psi}\right\} d\psi. \quad (6)$$

At large pressures, for which $R \gg \lambda$, we keep only the first term in Eq. (6). Then

$$N \approx \frac{k\bar v}{4}\frac{dn}{dx}\iint dA \int_0^{\pi} \frac{r_0^2 - R^2}{\lambda}\cos^2\psi\sin\psi\, d\psi = \frac{R^4\pi k\bar v}{12\lambda}\frac{dn}{dx}.$$

If we take $k \approx \frac{3}{2}(\overline{v^2}/\bar v^2) \approx \frac{3}{2}$, then this formula reduces to Poiseuille's law; this shows that the choice of the anisotropy function was correctly made. With this value of k we have in the general case

$$N = N_{is}$$
$$+ \frac{3}{8}\frac{dn}{dx}\bar v\iint dA\int_0^{\pi}\cos^2\psi\sin\psi\, d\psi\left[\left(\frac{r_0^2 - R^2}{\lambda} - 2r_0\sin\psi\cos\vartheta\right.\right.$$
$$\left.\left. + 2\lambda\sin^2\psi\right) + (2r_0\sin\psi\cos\vartheta - 2s\sin\psi - 2\lambda\sin^2\psi)e^{-s/\lambda\sin\psi}\right].$$

To perform the integration we choose a coordinate system in the transverse section in the way shown in Fig. 3. We have here

$$A = \int_0^{\pi} d\alpha \int_0^{2R\sin\alpha} s ds;\ r\cos\vartheta = s - R\sin\alpha$$

and for N we now have

$$N = N_{is} - \left[\frac{\pi R^4\bar v}{8\lambda} + \frac{\pi R^3\bar v}{12} - \frac{\lambda\pi R^2\bar v}{5}\right]\frac{dn}{dx}$$
$$+ \frac{3}{8}\frac{dn}{dx}\bar v\iint dA\int_0^{\pi}\cos^2\psi\sin\psi\,(-2R\sin\psi\sin\alpha - 2\lambda\sin^2\psi)$$
$$\times \exp(-s/\lambda\sin\psi)\, d\psi.$$

Changing the order of the integrations, we bring the last term into the form

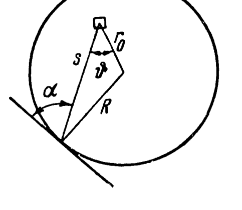

FIG. 3

$$\frac{3}{8}\frac{dn}{dx}\bar{v}\int_0^\pi d\alpha\int_0^\pi \cos^2\psi\sin\psi(-2R\sin\alpha\sin\psi-2\lambda\sin^2\psi)\int_0^{2R\sin\alpha} se^{-s/\lambda\sin\psi}ds.$$

Integrating over ds, we reduce the problem to the evaluation of a double integral, which can only be done numerically.

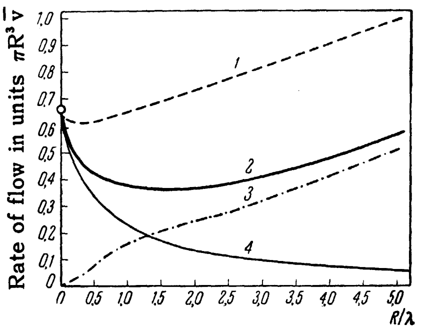

FIG. 4. 1 – observed rate of flow as function of total pressure gradient (Knudsen); 2 – our results; 3 – the anisotropic part of the rate of flow; 4 – rate of flow obtained by Pollard and Present.

At densities such that $\lambda/2 \geq R$, a good approximation to the integral is obtained by expanding the exponential function in power series and integrating numerically over the range $\pi/10 \leq \psi \leq \pi - \pi/10$. In this case we get for the "anisotropic part" of the rate of flow the values shown by the dotted curve. If we add to it the values given by the isotropic term, we get a rate of flow in good qualitative agreement with Knudsen's data,[1] which are shown by the dashed curve.

Thus we have obtained a law which automatically leads to the presence of slip (smaller than the Maxwellian slip) and of a minimum rate of flow, and which is in good agreement with experimental data.

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PROPAGATION OF AN ELECTROMAGNETIC FIELD IN A MEDIUM WITH SPATIAL DISPERSION

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General formulas are obtained for the propagation of an electromagnetic field in a semi-infinite, homogeneous, anisotropic medium with spatial dispersion. The propagation of a transverse wave along a magnetic field in a plasma is investigated, taking account of the thermal motion of the electrons. Strong absorption of the field is found in the region for which Cerenkov radiation is possible in the plasma.

In the present paper we consider the penetration of an electromagnetic field into a semi-infinite, homogeneous, anisotropic medium with spatial dispersion. This problem is an extension of the second part of the well-known paper by Landau[1] in which the penetration of a longitudinal electric field into an isotropic plasma was treated.

In Sec. 1 we obtain general formulas which, in