$$\frac{3}{8}\frac{dn}{dx}\bar{v}\int_0^{\pi} d\alpha \int_0^{\pi} \cos^2\psi \sin\psi \left(-2R\sin\alpha\sin\psi - 2\lambda\sin^2\psi\right)\int_0^{2R\sin\alpha} s e^{-s/\lambda\sin\psi} ds.$$

Integrating over ds, we reduce the problem to the evaluation of a double integral, which can only be done numerically.

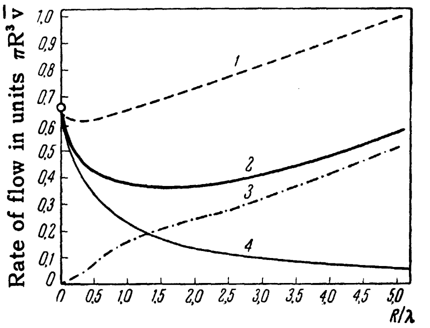

FIG. 4. 1 – observed rate of flow as function of total pressure gradient (Knudsen); 2 – our results; 3 – the anisotropic part of the rate of flow; 4 – rate of flow obtained by Pollard and Present.

At densities such that $\lambda/2 \geq R$, a good approximation to the integral is obtained by expanding the exponential function in power series and integrating numerically over the range $\pi/10 \leq \psi \leq \pi - \pi/10$. In this case we get for the "anisotropic part" of the rate of flow the values shown by the dotted curve. If we add to it the values given by the isotropic term, we get a rate of flow in good qualitative agreement with Knudsen's data,[1] which are shown by the dashed curve.

Thus we have obtained a law which automatically leads to the presence of slip (smaller than the Maxwellian slip) and of a minimum rate of flow, and which is in good agreement with experimental data.

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PROPAGATION OF AN ELECTROMAGNETIC FIELD IN A MEDIUM WITH SPATIAL DISPERSION

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General formulas are obtained for the propagation of an electromagnetic field in a semi-infinite, homogeneous, anisotropic medium with spatial dispersion. The propagation of a transverse wave along a magnetic field in a plasma is investigated, taking account of the thermal motion of the electrons. Strong absorption of the field is found in the region for which Cerenkov radiation is possible in the plasma.

IN the present paper we consider the penetration of an electromagnetic field into a semi-infinite, homogeneous, anisotropic medium with spatial dispersion. This problem is an extension of the second part of the well-known paper by Landau[1] in which the penetration of a longitudinal electric field into an isotropic plasma was treated.

In Sec. 1 we obtain general formulas which, in conjunction with appropriate boundary conditions, can be used to determine the penetration of the longitudinal and the transverse fields.* These formulas are suitable, for example, for analysis of the anomalous skin effect. In Sec. 2 we solve the specific problem involving a plasma, located in a homogeneous magnetic field, in which a transverse wave propagates along the direction of the field.

1. GENERAL FORMULAS

1. We consider a monochromatic field with time dependence of the form $e^{-i\omega t}$. The field propagates from vacuum into a medium which fills the semi-space $z > 0$. Because of spatial dispersion there is a functional relation between the electric displacement vector $\mathbf{D}$ and the electric field $\mathbf{E}$:

$$D_\alpha(\mathbf{r}) = \int K_{\alpha\beta}(\mathbf{r}, \mathbf{r}') E_\beta(\mathbf{r}') d\mathbf{r}'. \tag{1}$$

If the spatial dispersion is neglected (in a plasma this procedure corresponds to neglecting the thermal motion of the electrons[2]), we have $K_{\alpha\beta}(\mathbf{r}, \mathbf{r}') = \epsilon_{\alpha\beta}\delta(\mathbf{r}-\mathbf{r}')$ and the local relation between $\mathbf{D}$ and $\mathbf{E}$ is obtained.

In a uniform field the function $K_{\alpha\beta}(\mathbf{r}, \mathbf{r}')$, which is determined by the law of motion of the charges, depends on the vectors $\mathbf{r}$ and $\mathbf{r}'$ only through their difference $\mathbf{R} = \mathbf{r} - \mathbf{r}'$. In a semi-infinite medium the dependence on $\mathbf{r}$ and $\mathbf{r}'$ is affected by the boundary. It will be assumed that the reflection of the charges at the boundary is specular. Under these conditions, the charge distribution functions are not distorted at the boundary and the above dependence on R still applies: $K_{\alpha\beta}(\mathbf{r}, \mathbf{r}') = K_{\alpha\beta}(\mathbf{R})$.

In the general case, the electromagnetic field is determined from the integro-differential equation

$$\mathrm{curl}\,\mathrm{curl}\,\mathbf{E} - \frac{\omega^2}{c^2}\mathbf{D} = i\frac{4\pi\omega}{c^2}\mathbf{j}_{\mathrm{trans}}. \tag{2}$$

As boundary conditions we require that the tangential components of the electric and magnetic fields be continuous across the vacuum-medium interface and that the normal component of the electric induction vector be continuous:

$$[\mathbf{n}\times\mathbf{E}_e] = [\mathbf{n}\times\mathbf{E}^0],\ [\mathbf{n}\times\mathbf{H}_e] = [\mathbf{n}\times\mathbf{H}^0],\ (\mathbf{n}\cdot\mathbf{E}_e) = (\mathbf{n}\cdot\mathbf{D}). \tag{3}$$

*The terms "longitudinal" and "transverse" refer to the method of exciting the field in the anisotropic medium. Thus, if the field propagates from vacuum into the medium these terms, refer to the field in vacuum. As is well known, however, the field which penetrates into the anisotropic medium cannot, in general, be divided into a purely transverse part and a purely longitudinal part.

The subscript "e" refers to the total field, made up of the incident and reflected wave, while the superscript "0" refers to the penetrating wave; $\mathrm{n} = \{0, 0, 1\}$ is a unit vector normal to the boundary surface.

The integro-differential equation (1) is solved most conveniently by expanding all quantities in plane waves. For this purpose we transform the problem from that of finding the penetration of a field from vacuum into a semi-infinite medium $z > 0$ into the problem of finding the field in an infinite medium $-\infty < z < \infty$ excited by surface currents and charges concentrated in the plane $z = 0$. It is easy to show that the tangential component of the magnetic field, $\mathbf{H}_t = \mathbf{n}\times\mathbf{H}$, and the normal component of the electric displacement vector $\mathbf{D}_n = \mathbf{n}\cdot\mathbf{D}$ are odd functions of z and are discontinuous at the surface $z = 0$. The discontinuities in H_t and D_n correspond to surface currents and charges. The appropriate volume charge density, which yields the proper boundary conditions for H_t and D_n in the plasma, can be written

$$\mathbf{j}_{\mathrm{trans}} = \frac{c}{2\pi}\left\{[\mathbf{n}\times\mathbf{H}^0]\,\delta(z) + i\frac{\omega}{4\pi}(\mathbf{n}\cdot\mathbf{E}_e)\,\mathbf{n}\,\mathrm{Sgn}.\,z\right\}. \tag{4}$$

Expanding all quantities in plane waves

$$\mathbf{E}(z) = \int \mathbf{E}(\mathbf{k})\,e^{i\mathbf{k}\cdot\mathbf{r}}d\mathbf{k},\quad \delta(z) = \frac{1}{2\pi}\int e^{i\mathbf{k}\cdot\mathbf{r}}\delta(\mathbf{k}_\perp)\,d\mathbf{k}, \tag{5}$$

$$K_{\alpha\beta}(\mathbf{r}) = \frac{1}{(2\pi)^3}\int \varepsilon_{\alpha\beta}(\mathbf{k})\,e^{i\mathbf{k}\cdot\mathbf{r}}\,d\mathbf{k},$$

$$\mathrm{Sgn}\,z = -\int\left[\delta(k_z) - \frac{1}{i\pi k_z}\right]e^{i\mathbf{k}\cdot\mathbf{r}}\delta(\mathbf{k}_\perp)\,d\mathbf{k},$$

$$\mathbf{k}_\perp = \{k_x, k_y, 0\},$$

we find the Fourier components of Eq. (2):

$$-k_\alpha k_\beta E_\beta(\mathbf{k}) + k^2 E_\alpha(\mathbf{k}) - \frac{\omega^2}{c^2}\varepsilon_{\alpha\beta}(\mathbf{k})\,E_\beta(\mathbf{k})$$
$$= i\frac{\omega}{\pi c}[\mathbf{n}\times\mathbf{H}^0]_\alpha\delta(\mathbf{k}_\perp) + \frac{\omega^2}{c^2}(\mathbf{n}\cdot\mathbf{E}_e)\,n_\alpha\left[\delta(k_z) - \frac{1}{i\pi k_z}\right]\delta(\mathbf{k}_\perp). \tag{6}$$

Taking account of the δ-function in $\mathbf{k}_\perp$ on the right-hand side of this equation, we have on the left-hand side $k_x = k_y = 0$. We also introduce the notation $k_z = N\omega/c$. In component form, the equations become

$$N^2E_x - \varepsilon_{x\alpha}(N)\,E_\alpha = -i\frac{c}{\pi\omega}H_y^0\delta(\mathbf{k}_\perp),$$
$$N^2E_y - \varepsilon_{y\alpha}(N)\,E_\alpha = i\frac{c}{\pi\omega}H_x^0\delta(\mathbf{k}_\perp), \tag{6'}$$
$$-\varepsilon_{z\alpha}(N)\,E_\alpha = -\frac{c}{\omega}E_{ez}\left[\delta(N) - \frac{1}{i\pi N}\right]\delta(\mathbf{k}_\perp).$$

2. We now consider the longitudinal field. Setting $H_x^0 = H_y^0 = 0$, we introduce the quantities

$$\eta_{xx} = \varepsilon_{xx} - \varepsilon_{xz}\varepsilon_{zx}/\varepsilon_{zz};\ \eta_{yy} = \varepsilon_{yy} - \varepsilon_{yz}\varepsilon_{zy}/\varepsilon_{zz},$$
$$\eta_{xy} = \varepsilon_{xy} - \varepsilon_{xz}\varepsilon_{zy}/\varepsilon_{zz};\ \eta_{yx} = \varepsilon_{yx} - \varepsilon_{yz}\varepsilon_{zx}/\varepsilon_{zz}. \tag{7}$$

Using the first two equations in (6) to eliminate E_x and E_y we have

$$E_z = -\frac{c}{\omega}\frac{E_{ez}}{\varepsilon_{zz}(N)}\frac{(N^2-\varepsilon_1^0)(N^2-\varepsilon_2^0)}{(N^2-\varepsilon_1)(N^2-\varepsilon_2)}\left[\delta(N) - \frac{1}{i\pi N}\right]\delta(\mathbf{k}_\perp). \tag{8}$$

Here we have introduced the notation

$$N^4 - N^2(\eta_{xx}+\eta_{yy}) + \eta_{xx}\eta_{yy} - \eta_{xy}\eta_{yx} \equiv (N^2-\varepsilon_1)(N^2-\varepsilon_2),$$
$$N^4 - N^2(\varepsilon_{xx}+\varepsilon_{yy}) + \varepsilon_{xx}\varepsilon_{yy} - \varepsilon_{xy}\varepsilon_{yx} \equiv (N^2-\varepsilon_1^0)(N^2-\varepsilon_2^0). \tag{9}$$

The expression for the field in the medium assumes the form

$$E_z(z) = -E_{ez}\int_{-\infty}^{+\infty}\frac{(N^2-\varepsilon_1^0)(N^2-\varepsilon_2^0)}{(N^2-\varepsilon_1)(N^2-\varepsilon_2)}\frac{e^{i\omega Nz/c}}{\varepsilon_{zz}(N)}\left[\delta(N) - \frac{1}{i\pi N}\right]dN. \tag{10}$$

In an isotropic plasma $\epsilon_1^0 = \epsilon_1$, $\epsilon_2^0 = \epsilon_2$ and Eq. (10) coincides with Landau's expression for the longitudinal field. The function K_k which appears in reference 1 is related to $\epsilon_{zz}(N)$ by the expression $\epsilon_{zz}(N) = 1 - K_k$ so that $K_0 = 1 - \epsilon_{zz}(0)$.

In the absence of spatial dispersion (i.e., when $\epsilon_{\alpha\beta}$ is independent of N) the integral in Eq. (10) is computed easily:

$$E_z(z) = \frac{E_{ez}}{\varepsilon_{zz}}\left\{\frac{\varepsilon_1^0\varepsilon_2^0}{\varepsilon_1\varepsilon_2} + \frac{(\varepsilon_1-\varepsilon_1^0)(\varepsilon_1-\varepsilon_2^0)}{\varepsilon_1(\varepsilon_1-\varepsilon_2)}\exp\left\{i\frac{\omega}{c}\sqrt{\varepsilon_1}z\right\} + \frac{(\varepsilon_2-\varepsilon_2^0)(\varepsilon_2-\varepsilon_1^0)}{\varepsilon_2(\varepsilon_2-\varepsilon_1)}\exp\left\{i\frac{\omega}{c}\sqrt{\varepsilon_2}z\right\}\right\}. \tag{11}$$

3. We next consider the penetration of the transverse field. Let $E_{ez} = 0$. From the last equation in (6′) we find

$$E_z(z) = -(\varepsilon_{zx}E_x + \varepsilon_{zy}E_y)/\varepsilon_{zz}. \tag{12}$$

Substituting this expression in the first two equations we obtain

$$(N^2-\eta_{xx})E_x - \eta_{xy}E_y = -i\frac{c}{\omega\pi}H_y^0\delta(\mathbf{k}_\perp),$$
$$-\eta_{yx}E_x + (N^2-\eta_{yy})E_y = i\frac{c}{\omega\pi}H_x^0\delta(\mathbf{k}_\perp). \tag{13}$$

We introduce the notation

$$E = E_x + \lambda E_y,$$
$$H^0 = H_y^0 - \lambda H_x^0, \tag{14}$$

and determine the values $\lambda = \lambda_1$ and $\lambda = \lambda_2$ and $\epsilon = \epsilon_1$ and $\epsilon = \epsilon_2$ from the relation

$$\varepsilon = \eta_{xx} + \lambda\eta_{yx} = \eta_{yy} + \eta_{xy}/\lambda. \tag{15}$$

Then we obtain in place of Eq. (13) two equations of the form

$$(N^2-\varepsilon)E = -i\frac{c}{\omega\pi}H^0\delta(\mathbf{k}_\perp). \tag{16}$$

The roots $\lambda_{1,2}$ of Eq. (15) determine the ratio of the x-th and y-th field components of the normal ordinary and extraordinary waves.

The corresponding functions $\epsilon(N) = \epsilon_1(N)$ and $\epsilon(N) = \epsilon_2(N)$ are determined through the tensor $\epsilon_{\alpha\beta}(\mathbf{k})$ in the same way as the squares of the refractive indices for the ordinary and extraorinary waves in the absence of spatial dispersion. It is easily shown that

$$\lambda_1\lambda_2 = -\frac{\eta_{xy}}{\eta_{yx}};\quad \lambda_1+\lambda_2 = \frac{\eta_{yy}-\eta_{xx}}{\eta_{yx}} \tag{17}$$

and

$$\varepsilon_1+\varepsilon_2 = \eta_{xx}+\eta_{yy};\ \varepsilon_1\varepsilon_2 = \eta_{xx}\eta_{yy} - \eta_{xy}\eta_{yx}. \tag{18}$$

It is apparent from Eqs. (18) and (9) that both definitions of $\epsilon_{1,2}$ (9) and (15) are the same. In accordance with Eqs. (5) and (16), the expression for the electric field can be written in the form

$$E_{1,2}(z) = -\frac{i}{\pi}\int_{-\infty}^{+\infty}\frac{H_{1,2}^0 e^{i\omega Nz/c}}{N^2-\varepsilon_{1,2}(N)}dN. \tag{19}$$

If there is no spatial dispersion, we obtain by the method of residues

$$E_{1,2}(z) = \frac{H_{1,2}^0}{\sqrt{\varepsilon_{1,2}}}\exp\{i\sqrt{\varepsilon_{1,2}}z\}. \tag{20}$$

Using the boundary conditions given in (3), it is easy to determine from Eq. (19) all components of an electromagnetic field that propagates in an anisotropic medium with spatial dispersion.

Formulas (10) and (19) are of interest in connection with the penetration of a field into a plasma located in a magnetic field. It is important to note that the functions $\epsilon_{\alpha\beta}(\mathbf{k})$ and, thus, $\epsilon_{1,2}(N)$ are not analytic when thermal motion of the electrons in the plasma is taken into account. Thus, for example, the integral in (19) cannot be computed by residues and the field in the plasma cannot be expressed by a simple formula such as (20).

2. TRANSVERSE ELECTROMAGNETIC FIELD PROPAGATING IN A PLASMA ALONG THE MAGNETIC FIELD

1. In a plasma which is located in a homogeneous magnetic field H_0, the tensor $\epsilon_{\alpha\beta}(\mathbf{k})$ is given by the equation

$$\varepsilon_{\alpha\beta}(\mathbf{k}) = \delta_{\alpha\beta} + i\frac{4\pi e^2 n_0}{\omega T}\left\langle \int_0^\infty v_\alpha(t)\, v_\beta(0)\exp\left\{i\left(\omega t - \mathbf{k}\int_0^t \mathbf{v}(\xi)\,d\xi\right) - \nu t\right\}dt\right\rangle. \quad (21)$$

Here n_0 is the electron density in the plasma and $\mathbf{v}(t)$ describes the motion of an electron in the homogeneous magnetic field; if the magnetic field is along the z axis,

$$v_x = v_\perp(0)\cos(\omega_H t + \varphi);\quad v_y = v_\perp(0)\sin(\omega_H t + \varphi),$$

$$v_z = v_z(0); \tan\varphi = v_{y0}/v_{x0};\quad \omega_H = |e|H_0/mc\sqrt{1-(v/c)^2}. \quad (22)$$

The triangular brackets denote averages taken over a Maxwellian velocity distribution, T is the temperature of the plasma, expressed in energy units, and ν is the frequency of collisions between electrons and heavy particles in the plasma.

Equation (21) is an inversion of the correlation function for microcurrents.[3] It can be shown that this formula is the same as the expression for $\epsilon_{\alpha\beta}(\mathbf{k})$ obtained in references 4 to 6.

If the wave is propagated in the z direction (along the magnetic field) and $T \ll mc^2$, the tensor $\epsilon_{\alpha\beta}$ assumes the form

$$\varepsilon_{\alpha\beta} = \delta_{\alpha\beta} - \frac{\omega_0^2}{\omega^2}\frac{1}{\beta\sqrt{\pi}N}\left\{\frac{1}{2}W\left(\frac{\omega - \omega_H + i\nu}{\omega\beta N}\right)\begin{pmatrix}1 & -i & 0\\ i & 1 & 0\\ 0 & 0 & 0\end{pmatrix} + \frac{1}{2}W\left(\frac{\omega + \omega_H + i\nu}{\omega\beta N}\right)\begin{pmatrix}1 & i & 0\\ -i & 1 & 0\\ 0 & 0 & 0\end{pmatrix} + W_1\left(\frac{\omega + i\nu}{\omega\beta N}\right)\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 1\end{pmatrix}\right\}. \quad (23)$$

Here $\omega_0^2 = 4\pi e^2 n_0/m$; $\beta^2 = 2T/mc^2$ and

$$W(u) = \int_{-\infty}^{+\infty}\frac{e^{-t^2}}{u-t}\,dt = -i\pi e^{-u^2}\left\{\operatorname{Sgn}\operatorname{Im}u + \frac{2i}{\sqrt{\pi}}\int_0^u e^{t^2}dt\right\},$$

$$W_1(u) = \int_{-\infty}^{+\infty}\frac{2t^2e^{-t^2}}{u-t}\,dt = 2u^2W(u) - 2\sqrt{\pi}u. \quad (24)$$

According to Eqs. (7), (15), and (23), $\lambda^2 = \epsilon_{xy}/\epsilon_{yx} = -1$. The values $\lambda = \pm i$ correspond to a field $E = E_x \pm iE_y$ which is circularly polarized. The function $\epsilon(N) = \epsilon_{1,2}(N)$, which appears in Eq. (19), is found to be $\epsilon = \epsilon_{xx} \pm i\epsilon_{yx}$; in view of Eqs. (23) and (24),

$$\varepsilon = 1 + i\frac{\sqrt{\pi}}{\beta}\frac{\omega_0^2}{\omega^2N}e^{-(x/N)^2}\left(\operatorname{Sgn}N + \frac{2i}{\sqrt{\pi}}\int_0^{x/N}e^{t^2}dt\right),$$

$$x = (\omega \pm \omega_H + i\nu)/\omega\beta. \quad (25)$$

2. The electric field in the plasma is expressed by Eq. (19), where $H_{1,2}^0 = H_x^0 \mp iH_y^0$. For the ordinary wave $x = (\omega + \omega_H + i\nu)/\omega\beta \gg 1$ and the spatial dispersion is unimportant since, when $x/N \gg 1$, the expression for ϵ assumes the form

$$\varepsilon = 1 - \omega_0^2/\beta\omega^2 x = 1 - \omega_0^2/\omega(\omega + \omega_H + i\nu).$$

We next consider the penetration of the extraordinary wave, setting $\nu = 0$. The situation here is similar to that which obtains in the oscillations of a plasma in an external electric field. The integral in (19) can be computed approximately by transforming to the complex plane. In order to carry out this procedure, we introduce in place of $\epsilon(N)$ (25) the analytic functions $\epsilon^{(1)}(N)$ and $\epsilon^{(2)}(N)$ which coincide with $\epsilon(N)$ for $N < 0$ and $N > 0$ respectively:

$$\varepsilon^{(1)}(N) = 1 - i\frac{\sqrt{\pi}}{\beta}\frac{\omega_0^2}{\omega^2N}e^{-(X/N)^2}\left(1 - \frac{2i}{\sqrt{\pi}}\int_0^{X/N}e^{t^2}dt\right);$$

$$\varepsilon^{(2)}(N) = 1 + i\frac{\sqrt{\pi}}{\beta}\frac{\omega_0^2}{\omega^2N}e^{-(X/N)^2}\left(1 + \frac{2i}{\sqrt{\pi}}\int_0^{X/N}e^{t^2}dt\right). \quad (26)$$

Hereinafter $X \equiv (\omega - \omega_H)/\omega\beta$.

As N approaches zero, both functions tend to the limiting value ϵ_0, which equals the square of the refractive index for the extraordinary wave if thermal motion is disregarded

$$\varepsilon_0 = 1 - \omega_0^2/\omega(\omega - \omega_H);\quad (\omega \neq \omega_H). \quad (27)$$

In the upper half of the complex variable N, the same limiting value is obtained for $\epsilon^{(1)}(N)$ as N approaches zero by any path which does not pass through the sector $\pi/4 < \arg N < 3\pi/4$ when $\omega < \omega_H$. The same holds for $\epsilon^{(2)}(N)$ if the path which does not pass through this sector when $\omega > \omega_H$.

In accordance with Eqs. (19) and (26), the electric field can be written in the form*

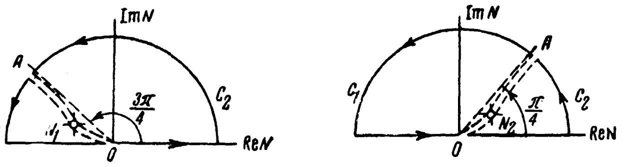
*The first expression for the field of a transverse wave propagating along a magnetic field was derived by Silin.[7] In his paper, however, no account was taken of the difference in the sign of Im K ± (k) for k > 0 and k < 0. In our case this difference leads to the difference in the functions $\varepsilon^{(1)}(N)$ and $\varepsilon^{(2)}(N)$ [Sgn N in Eq. (25)!]. The function $\varepsilon^{(1)}(N)$, which differs from $\varepsilon^{(2)}(N)$, appears because of the necessity for introducing advanced potentials as well as retarded potentials in the boundary value problem.

$$E(z) = -\frac{1}{\pi} H^0 \left\{ \int_{-\infty}^{0} \frac{e^{i\omega Nz/c}}{N^2 - \varepsilon^{(1)}(N)} dN + \int_{0}^{\infty} \frac{e^{i\omega Nz/c}}{N^2 - \varepsilon^{(2)}(N)} dN \right\} \tag{28}$$

or, going over to integration in the complex plane

$$E(z) = -\frac{i}{\pi} H^0 \left\{ \oint_{C_1} \frac{e^{i\omega Nz/c}}{N^2 - \varepsilon^{(1)}(N)} dN + \oint_{C_2} \frac{e^{i\omega Nz/c}}{N^2 - \varepsilon^{(2)}(N)} \right.$$
$$\left. + \frac{2i\sqrt{\pi}}{\beta} \frac{\omega_0^2}{\omega^2} \int_{OA} \frac{\exp\{-X^2/N^2 + i\omega Nz/c\}}{N[N^2 - \varepsilon^{(1)}(N)][N^2 - \varepsilon^{(2)}(N)]} dN \right\}. \tag{29}$$

The contours C_1, C_2 and the path of integration OA for the frequencies $\omega < \omega_H$ and $\omega > \omega_H$, are chosen as shown in Fig. 1. The curve OA passes



a FIG. 1 b

through the saddle point of the integrand in the third term in Eq. (29). The first and second integrals in Eq. (29) are computed by the method of residues. If N_0 is a root of the equation $N^2 - \epsilon^{(\alpha)}(N) = 0$ $(\alpha = 1, 2)$, the corresponding integral makes the following contribution to the field:

$$E_1(z) = \frac{H^0}{N_0} \frac{e^{i\omega N_0 z/c}}{1 - (1/2N_0)\, d\varepsilon^{(\alpha)}(N_0)/dN_0}. \tag{30}$$

This part of the field propagates as a damped wave in the plasma. The asymptote for large values of z in the last integral can be obtained by the method of steepest descent. We first exclude the resonance region. Then the saddle points are

$$N_1 = (2cX^2/\omega z)^{1/3} e^{i5\pi/6} \quad \text{for } \omega < \omega_H;$$
$$N_2 = (2cX^2/\omega z)^{1/3} e^{i\pi/6} \quad \text{for } \omega > \omega_H.$$

At large values of z, $N_{1,2}$ is small so that we can write in the denominator of the integrand $N^2 - \epsilon^{(1)}(N) = N^2 - \epsilon^{(2)}(N) = -\epsilon_0$. A simple calculation of the last term in the field expression (29) yields

$$E_2(z) = \frac{2H^0}{\sqrt{3}\beta} \frac{\omega_0^2}{\omega^2 \varepsilon_0^2} \left[\frac{2\beta c}{z(\omega_H - \omega)}\right]^{1/3} \exp\left\{-\frac{3}{2}\left(\frac{z(\omega_H - \omega)}{2\beta c}\right)^{2/3} + i\left[\frac{3\sqrt{3}}{2}\left(\frac{z(\omega_H - \omega)}{2\beta c}\right)^{2/3} + \frac{\pi}{6}\right]\right\}; \ (\omega < \omega_H), \tag{31}$$

$$E_2(z) = \frac{2H^0}{\sqrt{3}\beta} \frac{\omega_0^2}{\omega^2 \varepsilon_0^2} \left[\frac{2\beta c}{z(\omega - \omega_H)}\right]^{1/3} \exp\left\{-\frac{3}{2}\left(\frac{z(\omega - \omega_H)}{2\beta c}\right)^{2/3} - i\left[\frac{3\sqrt{3}}{2}\left(\frac{z(\omega - \omega_H)}{2\beta c}\right)^{2/3} - \frac{5\pi}{6}\right]\right\}; \ (\omega > \omega_H).$$

When $\beta = 0$ the field $E_2(z)$ vanishes and, in accordance with Eqs. (29) and (30), we are left with the simple wave $E(z) = (H^0/\sqrt{\epsilon_0}) \exp\{i\omega\sqrt{\epsilon_0} z/c\}$. When $\epsilon_0 > 0$ this expression is obtained from a second term in Eq. (29); when $\epsilon_0 < 0$ it is obtained from the first. If $\beta \neq 0$, the chief contribution is associated with the field $E_2(z)$.

3. We consider Eqs. (29) to (31) in three frequency regions, assuming at the outset that $\omega_0^2 \gg \omega_H^2 \beta$ (high electron density).

(a) $\omega < \omega_H$. In this region the equation $N^2 - \epsilon^{(1)}(N) = 0$ has no roots in the contour C_1 and the first integral in Eq. (29) vanishes. The equation $N^2 - \epsilon^{(2)}(N) = 0$ has a root $N_0 = n + iq$, which is determined at resonance by the expressions

$$q^3 + q/4 = \sqrt{\pi}\omega_0^2/8\beta\omega_H^2; \quad n^2 = 1 + 3q^2 \tag{32}$$

and far from resonance, $n \gg q$, by the approximation formulas:

$$n^2 = 1 + \frac{2}{3}\frac{\omega_0^2}{\omega^2 n} \exp\{-(X/n)^2\} \int_0^{-X/n} e^{t^2} dt \approx 1 + \frac{\omega_0^2}{\omega(\omega_H - \omega)},$$

$$q = \frac{\sqrt{\pi}}{2\beta} \frac{\omega_0^2}{\omega^2 n^2} \exp\{-(X/n)^2\}. \tag{33}$$

These formulas define the absorption line shape for a linear oscillator (in this case the electron in the magnetic field) when the Doppler effect is taken into account. It is easy to show that when $\beta = 0$ in the plasma the square of the refractive index has the following form [in place of (27)]:

$$\varepsilon = 1 - \frac{\omega_0^2}{\omega(\omega - \omega_H)} + i\pi \frac{\omega_0^2}{\omega} \delta(\omega - \omega_H);$$

the δ-function in the imaginary part of ϵ means that the electrons absorb energy at the resonance frequency. If there is a spread in the electron velocities v_z the absorption line is broadened as a consequence of the Doppler effect and Eq. (33) is obtained.

Equation (33) differs from the well-known expression for the Doppler-broadened absorption of an oscillator in that the phase velocity is c/n rather than the free-space velocity. As a result E_1 [Eq. (30)] falls off rapidly (is absorbed) over a relatively wide range of frequencies (greater than $\omega_H \beta$) when the electron density is high. Absorption is especially important when $\omega_0^2/\omega_H^2 \geq 1/\beta^2$ or $H_0^2/8\pi \leq n_0 T$. Thus, for example, when $\omega_0^2/\omega_H^2 = 1/\beta^2$, the absorption factor is $q = 0.27\beta$ at a frequency which is one-fifth of the resonance frequency $(\omega = \frac{1}{5}\omega_H)$.

At high values of z the electric field is given by Eq. (31); this expression also shows strong attenutation of the field amplitude.

(b) $\omega_H \leq \omega < \omega_1$, $\omega_1 = \omega_H/2 + \sqrt{\omega_H^2/4 + \omega_0^2}$. In this region $\epsilon_0 < 0$, and propagation does not take place if thermal motion is not taken into account. When $\omega \geq \omega_H$, the equation $N^2 - \epsilon^{(1)}(N) = 0$ has one pure imaginary root in the upper half of the plane, $N_0 = iq$ where q is determined from the equation

$$q^2 + 1 = \frac{\sqrt{\pi}}{\beta}\frac{\omega_0^2}{\omega^2 q}\exp\{(X/q)^2\}\left(1 - \frac{2}{\sqrt{\pi}}\int_0^{X/q} e^{-t^2}dt\right). \quad (34)$$

The quantity q falls off monotonically from $\omega = \omega_H$ to a value $\omega = \omega_1$ at which it vanishes. As $q \to 0$, using the asymptotic value of the error integral which appears in (34) we have

$$q^2 = \omega_0^2/\omega^2\beta X_1 - 1 = \omega_0^2/\omega(\omega - \omega_H) - 1 = -\varepsilon_0$$

and the limiting frequency, at which $q = 0$, is ω_1. The corresponding field E_2 is attenuated, just as in the case in which the thermal motion is not considered.

Outside the sector $\pi/4 < \arg N < 3\pi/4$, the equation $N^2 - \epsilon^{(2)}(N) = 0$ has a root with an imaginary part which approaches infinity as $\beta \to 0$. The corresponding field is not of interest because it is so small.

At large values of z, the most important contribution is due to $E_2(z)$ which propagates in the plasma and is absorbed in exponential fashion (the exponent is $z^{2/3}$). The existence of an absorbed wave means that if $\beta \neq 0$ the electromagnetic wave is not completely reflected in the region where $\epsilon_0 < 0$.

(c) $\omega > \omega_1$. In this region the second and third terms in Eq. (29) assume major importance. The equation $N^2 - \epsilon^{(2)}(N) = 0$ has the root $N_0 = n + iq$ where $q \ll n$, thus Eqs. (33) hold

$$n^2 = \varepsilon_0 = 1 - \omega_0^2/\omega(\omega - \omega_H),$$
$$q = \frac{\sqrt{\pi}}{2}\frac{\omega_0^2}{\omega^2\beta\varepsilon_0}\exp\{-(X/\varepsilon_0)^2\}. \quad (35)$$

In contrast to Eq. (33), here $n^2 = \epsilon_0$ is less than unity and absorption is not important, It should be kept in mind that in obtaining the tensor $\epsilon_{\alpha\beta}(\mathbf{k})$ use has been made of a nonrelativistic Maxwellian electron distribution ($mc^2 \gg T$) so that the terms which contain the factor $e^{-\beta^{-2}}$ in the formulas should be considered precisely zero. Thus, when $\omega \gg \omega_H$, q vanishes and an unattenuated wave $E = (H^0/\sqrt{\epsilon_0})\exp\{i\omega\sqrt{\epsilon_0}\, z/c\}$ propagates in the plasma.

4. We now consider the case in which ϵ is slightly different from unity $\omega_0^2/\omega_H^2\beta < 1$ (low electron density $n_0 < 0.9 \times 10^5 \beta H_0^2$).

In this case the equation $N^2 - \epsilon^{(2)}(N) = 0$ has the solution

$$N_0^2 = 1 - \frac{2\omega_0^2}{\omega^2\beta}\exp\{-X^2\}\int_0^X e^{t^2}dt + i\frac{\sqrt{\pi}}{\beta}\frac{\omega_0^2}{\omega^2}\exp\{-X^2\} \quad (36)$$

where $\operatorname{Re} N_0 \gg \operatorname{Im} N_0 > 0$ for all frequencies. The solution of the equation $N^2 - \epsilon^{(1)}(N) = 0$ differs in the sign of the imaginary part. Thus when it is slightly different from unity, the refractive index is calculated from usual dispersion equations (5) and (6), where the integrals of the type in (24) are taken along a path which goes around the pole $t = u$ from below (corresponding to the condition $\operatorname{Sgn} \operatorname{Im} u = 1$). It is apparent from Eq. (36) that, with the exception of the narrow resonance region, $\operatorname{Im} N_0 \ll 1$ everywhere. The amplitude of the propagating part $E_1(z)$ thus turns out to be considerably larger than that of the nonpropagating part $E_2(z)$. The analysis of the penetration of the electromagnetic wave is thus reduced to the usual problem involving the dispersion equation in (36).

5. If the magnetic field is at some arbitrary angle $\theta \neq 0$ with respect to the direction of propagation of the wave (z axis), all the formulas become much more complicated. In this case the tensor $\epsilon_{\alpha\beta}(\mathbf{k})$ contains all nine components. Resonances appear at frequencies which are multiples of ω_H.[5] The additional terms in $\epsilon_{\alpha\beta}(\mathbf{k})$ however, are proportional to different powers of β and when $\beta \ll 1$ these terms are unimportant. Hence, a first approximation to $\epsilon_{\alpha\beta}(\mathbf{k})$, with thermal motion taken into account, is given by Eq. (23). From this formula it is apparent that the tensor $\epsilon_{\alpha\beta}(\mathbf{k})$ is the same as the tensor $\epsilon_{\alpha\beta}$ computed for the case in which $\beta = 0$ (ref. 9) if

$$\text{(a)}\ (\omega - \omega_H)/\omega\beta N\cos\theta \gg 1 \quad \text{(b)}\ 1/\beta N \gg 1.$$

The first condition means that the imaginary parts of the components of the dielectric permittivity tensor ϵ_{xx}, ϵ_{yy}, $i\epsilon_{xy}$ must be small. The second means that the imaginary part of ϵ_{zz} must be small. These conditions are not satisfied in two regions: (a) close to the resonance frequency, and (b) at frequencies for which the index of refraction calculated with the usual formulas, with $\beta = 0$, is large. The violation of the first condition is associated with magnetic radiation at a frequency ω_H and a corresponding resonance absorption. The violation of the second condition is due to coherent Cerenkov radiation and the radiation absorption associated with this effect (this question is considered in detail in the Appendix). The width of the absorption band, as can be seen from the example involving longitudinal propagation, may be very large.

CONCLUSIONS

1. At a given frequency ω a transverse electromagnetic wave in a medium with spatial dispersion cannot, in general, be represented in the form of a wave $e^{i(\omega N_0 z/c - \omega t)}$ as is the case in the absence of dispersion. If a solution is sought in this form in the case of a wave which propagates along a magnetic field in a plasma, the following dispersion equation is obtained:

$$N^2 = 1 + i\frac{\sqrt{\pi}}{\beta}\frac{\omega_0^2}{\omega^2 N}\exp\left\{-\left(\frac{\omega-\omega_H}{\omega\beta N}\right)^2\right\}$$

$$\times\left\{\operatorname{Sgn}\operatorname{Im}\left(\frac{\omega-\omega_H}{\omega\beta N}\right) + \frac{2i}{\sqrt{\pi}}\int\limits_0^{(\omega-\omega_H)/\omega\beta N} e^{t^2}dt\right\}.$$

This expression contains the nonanalytic function $\operatorname{Sgn}\operatorname{Im}(\omega - \omega_H)\,\omega\beta N$ and leads to an incorrect solution when $\omega > \omega_H$, where this equation has no roots at all.

The electromagnetic field must be determined in each individual case, taking account of the boundary conditions. This is the procedure used in analyzing the field in oscillations of a plasma in an external longitudinal electric field.[1]

2. Analysis of the propagation of an electromagnetic field in a plasma in a magnetic field indicates that this field consists of two parts. One part of the field is characterized by a phase velocity which depends on the z coordinate ($v_{\text{phase}} \sim z^{1/3}$) and decays exponentially (exponent $z^{2/3}$). The other part of the field is the usual wave.

At high values of electron density, $n_0 \gg 0.9 \times 10^5\,\beta H_0^2$, the refractive index for this wave is determined from the equation $N^2 - \epsilon^{(2)}(N) = 0$ in the region $\epsilon_0 > 0$ and from the equation $N^2 - \epsilon^{(1)}(N) = 0$ in the region $\epsilon_0 < 0$ [cf. Eq. (26)].

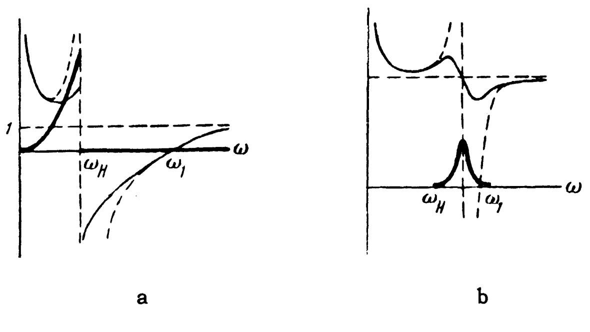


FIG. 2. The square of the refractive index N_0^2 for an extraordinary wave which propagates along the magnetic field: a – for $\omega_0^2 \gg \omega_H^2\beta$, b – for $\omega_0^2 < \omega_H^2\beta$. The thin lines are $\operatorname{Re} N_0^2 \equiv n^2 - q^2$, the heavy lines are $\operatorname{Im} N_0^2 \equiv 2nq$, and the dashed lines $N_0^2 = \varepsilon_0 \equiv 1 - \omega_0^2/\omega(\omega - \omega_H)$ (thermal motion is neglected); $\omega_H = |e|H_0/mc$; $\omega_0^2 = 4\pi e^2 n_0/m$; $\omega_1 = \omega_H/2 + \sqrt{\omega_H^2/4 + \omega_0^2}$.

In the region in which ϵ_0 (the square of the refractive index) is much greater than unity when the thermal motion is not taken into account, there is strong absorption of the wave. The approximate behavior of the real and imaginary parts of the square of the refractive index for the extraordinary wave in longitudinal propagation is shown in Fig. 2a.

At low values of the electron density, $n_0 < 0.9 \times 10^5\,\beta H_0^2$, the refractive index is determined from the equation $N^2 - \epsilon^{(2)}(N) = 0$ and has the same form as that for a simple resonator in which the Doppler effect is taken into account[8] (Fig. 2b).

In conclusion we wish to express our gratitude to Academician M. A. Leontovich for suggesting this work and for help in its execution.

APPENDIX

Cerenkov Radiation

1. If the medium contains electrons with thermal velocities greater than the phase velocity of an electromagnetic wave at some given frequency, the electrons radiate at this frequency. In accordance with Kirchhoff's Law, these electrons will also absorb electromagnetic waves at the same frequency (Cerenkov absorption). This can be seen directly in the case of a weakly-absorbing medium in which the radiancy of the medium η_ω is related to the absorption α_ω by the expression

$$\eta_\omega = \alpha_\omega I_\omega, \tag{1}$$

where I_ω is the equilibrium radiation intensity.

As an example we may consider the following model. A gas of free electrons, with a Maxwellian velocity distribution, is located in a transparent dielectric, the refractive index of which satisfies the condition $n > 1$ in the absence of the electron gas. In this example it is most fruitful to consider methods of calculating the Cerenkov absorption and the mechanism responsible for this absorption.

The absorption coefficient is determined by calculating the dielectric permittivity of the medium, taking account of the thermal motion of the electrons. The value of the dielectric permittivity, ϵ, which determines the propagation of a plane wave, can be obtained from Eq. (21) of Sec. 2. For nonrelativistic electron velocities

$$\varepsilon = n^2 - \frac{4\pi e^2 n_0}{m\omega}\int\left\{\frac{1}{\omega - \mathbf{kv}} - i\pi\delta(\omega - \mathbf{kv})\right\} f_0(\mathbf{v})\,d\mathbf{v}, \tag{2}$$

$$f_0(\mathbf{v}) = (m/2\pi T)^{3/2}\exp(-mv^2/2T),$$

where m and n_0 are the electron mass and electron density respectively, $\mathbf{k}$ is the wave vector, and T is the temperature expressed in energy units. We assume that the electron density is small, $4\pi e^2 n_0/m\omega^2 \ll 1$, so that the real part of the integral can be neglected as compared with n^2 and we can write $|k| = n\omega/c$. The square of the total index of refraction is found to be

$$N^2 = \varepsilon = n^2 + i\,\frac{4\pi e^2 n_0}{m\omega^2}\,\frac{c\pi}{n}\left(\frac{m}{2\pi T}\right)^{1/2} e^{-mc^2/2Tn^2} \tag{3}$$

while the absorption coefficient is

$$\alpha = \frac{\omega}{c}\,\frac{\operatorname{Im} N^2}{n} = \frac{\omega_0^2}{\omega c}\,\frac{\pi^{1/2}}{\beta n^2}\,e^{-1/\beta^2 n^2}, \tag{4}$$

$$\omega_0^2 = 4\pi e^2 n_0/m;\ \beta^2 = 2T/mc^2.$$

As is apparent from Eq. (2), the imaginary part of ϵ, which is responsible for absorption, is nonvanishing when the electron velocities satisfy the Cerenkov condition $\omega = \mathbf{k}\cdot\mathbf{v}$ or

$$v\cos\theta = c/n, \tag{5}$$

where θ is the angle between the vectors $\mathbf{k}$ and $\mathbf{v}$.

Inasmuch as the electron velocity cannot exceed the velocity of light $v < c$, this condition means in general that there is a sharp boundary (in terms of ω), for the absorption band. If the relativistic electron velocity distribution is used, this boundary is taken into account automatically. In this case the absorption coefficient assumes the form [in place of Eq. (4)]:

$$\alpha = \frac{2\pi^2\omega_0^2}{\omega c n^2}\,C\,\frac{e^{-b}}{b}\left(1+\frac{1}{b}\right),\ b = \frac{mc^2}{T(1-1/n^2)^{1/2}},\ n > 1; \tag{6}$$

$$\alpha = 0,\ n \leqslant 1.$$

Here

$$C = (4\pi)^{-1}\left[\frac{T}{mc^2}\,K_0\left(\frac{mc^2}{T}\right) + 2\left(\frac{T}{mc^2}\right)^2 K_1\left(\frac{mc^2}{T}\right)\right]$$

is the normalization constant in the Maxwell distribution.[10] When $mc^2 T \gg 1$

$$C \approx (mc)^3 (2\pi mT)^{-3/2} e^{mc^2/T}.$$

As follows from Eq. (6), the approximate formula (4) applies if $n^2 \gg 1$ and $(1 - 1/n^2)^{1/2} = 1 - 1/2n^2$.

To demonstrate that the absorption in question is actually related to Cernekov radiation we compute the absorption coefficient using Kirchhoff's law (1). In order to carry out this procedure, it is first necessary to compute the radiancy η_ω. An electron moving with a velocity v in a medium of refractive index n radiates the following energy per unit time and frequency interval ω:[3]

$$\mathscr{E}_\omega = \frac{e^2\omega}{c^2}\left(1 - \frac{c^2}{n^2v^2}\right)v. \tag{7}$$

Averaging this expression over a Maxwellian velocity distribution, we find in the nonrelativistic approximation

$$\eta_\omega = \int n_0\,\mathscr{E}_\omega\,f_0(\mathrm{v})\,d\mathrm{v} = \frac{2e^2\omega n_0}{\sqrt{\pi}c}\,\beta e^{-1/\beta^2 n^2}. \tag{8}$$

Substituting the equilibrium intensity of the radiation obtained by the Rayleigh-Jeans Law for a medium of refractive index n, $I_\omega = (\omega^2 T/\pi^2 c^2)\,n^2$, and the value of η_ω of (8) into Eq. (1), we obtain, in complete agreement with (4)

$$\alpha_\omega = \frac{\omega_0^2}{\omega c}\,\pi^{1/2}\,\frac{1}{\beta n^2}\,e^{-1/\beta^2 n^2}.$$

To explain the mechanism of Cerenkov absorption, we introduce a coordinate system which moves with a velocity $\mathbf{v}$ at an angle θ to the wave vector $\mathbf{k}$. If the velocity v and the angle θ are chosen in accordance with Eq. (5) the field in this coordinate system is independent of time [$\omega' = \omega(1 - vn\cos\theta/c)/\sqrt{1-\beta^2} = 0$]. As can be shown easily by the usual transformation formulas, in this coordinate system the nonvanishing component of the electric field E' lies in the plane defined by the vectors $\mathbf{k}$ and $\mathbf{v}$. The amplitude of this field is related to the amplitude of the laboratory field component E, which lies in the plane of the vectors $\mathbf{k}$ and $\mathbf{v}$ (cf. Fig. 3), by the relation

$$E' = E\,\frac{v\sqrt{n^2-1}}{c\sqrt{1-v^2/c^2}}\,\sin\theta. \tag{9}$$

An electron which moves in the laboratory coordinate system with a velocity $\mathbf{v}$ which satisfies (5) is at rest in the moving system. Consequently, this electron is acted on by a force $f = eE'$ which is constant in time. The continuous acquisition of energy by electrons under the influence of this force results in a reduction in the energy flux of the electromagnetic field, i.e. absorption.

2. Cerenkov absorption can play an important role in a plasma in a magnetic field. The magnetic field causes an increase in the refractive index in certain frequency regions. The resulting retardation of the wave causes marked broadening of the absorption band in the region of the cyclotron resonance; moreover, the increased refractive index means that the plasma electrons can become "faster-than-light" electrons, i.e. electrons with velocities greater than the velocity of light in the "medium". Both of these effects become especially important when $nT \approx H^2/8\pi$ (i.e., when the pres-

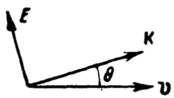


FIG. 3

sure associated with the electron plasma component becomes comparable with the magnetic pressure) because under these conditions the phase velocity in the low frequency region is smaller than the mean thermal velocity of the electrons.

Cerenkov radiation arises in a plasma as a consequence of free motion of electrons along the strong magnetic field lines. The possibility of this radiation has been indicated by Veksler.[11] The radiation due to a charge moving with uniform motion along a magnetic field has been calculated by Kolomenskii.[12] If $nT \approx H^2/8\pi$, the energy radiated per unit time by an electron which moves with velocity v along the field is approximately $d\mathscr{E}/dt \approx e^2\omega_H^2 v/c^2$ where $\omega_H = eH/mc$. This energy is approximately c/v times greater than the energy radiated by an electron which moves with the same velocity, but in a circle in a plane perpendicular to the magnetic field! The large amount of energy radiated by an isolated electron obviously does not mean that the plasma loses enormous amounts of energy by radiation (since the maximum loss, in accordance with the laws of thermal radiation, cannot depend on the nature of the radiator). The foregoing is merely an indication of the correspondingly strong absorption of this type of radiation in the plasma.

Since the refractive index is approximately unity at high frequencies, the Cerenkov absorption is important only at relatively low frequencies $\omega \leq \omega_H$. If $T \ll mc^2$, as has been shown in Sec. 2, we need not consider in the dielectric permittivity tensor ϵ_{ik} terms associated with higher resonances $\omega = 2\omega_H$, $\omega = 3\omega_H$, . . . , except in the case in which propagation takes place across the magnetic field. Under these conditions the tensor components are the same when thermal motion is neglected (magnetic field parallel to the axis)

$$\varepsilon_{ik} = \begin{pmatrix} \varepsilon & ig & 0 \\ -ig & \varepsilon & 0 \\ 0 & 0 & \eta \end{pmatrix}, \tag{10}$$

where

$$\varepsilon = 1 - \frac{\omega_0^2}{2\omega}\langle - i\pi\delta_+(\omega - \omega_H - kv\cos\theta) - i\pi\delta_+(\omega + \omega_H - kv\cos\theta)\rangle;$$

$$g = \frac{\omega_0^2}{2\omega}\langle - i\pi\delta_+(\omega - \omega_H - kv\cos\theta) + i\pi\delta_+(\omega + \omega_H - kv\cos\theta)\rangle; \tag{11}$$

$$\eta = 1 - \frac{\omega_0^2}{\omega}\langle - i\pi\delta_+(\omega - kv\cos\theta)\rangle;\quad \delta_+(x) = \delta(x) + i/\pi x.$$

The triangular brackets denote averages over the distribution function $f_0(v)\,dv = (m/2\pi T)^{1/2} \times e^{-mv^2/2T}\,dv$, θ is the angle between the direction of propagation and the magnetic field, and v is the electron velocity along the z axis.

The imaginary parts of ϵ and g, which contain δ-functions with argument $\omega \pm \omega_H - kv\cos\theta$, describe the radiation or absorption at the resonance frequency ω_H, and take account of the Doppler effect which arises as a consequence of the free motion of the charges along the magnetic field lines.

The Cerenkov radiation or absorption lies in the imaginary part of the tensor component $\epsilon_{zz} = \eta$ which contains the δ-function with argument $\omega - kv\cos\theta$. If the absorption is small, we can write as an approximation $k = n\omega/c$ where n is the real part of the refractive index, $N = n + iq$, and the condition $\omega - kv\cos\theta = 0$ reduces to (5). It is apparent that the Cerenkov radiation obtains for waves which propagate at an angle θ, for which the imaginary part of the refractive index q depends on η, while the real part meets the requirement $n > 1$. In longitudinal propagation $(\theta = 0)$ we have $N^2 = \epsilon \pm g$ and there is no Cerenkov absorption. This result is clear from an examination of Eq. (9): when $\theta = 0$, we have $E' = 0$. In transverse propagation $(\theta = \pi/2)$ we have $N^2 = (\epsilon^2 - g^2)/\epsilon$ for the extraordinary wave; the absence of Cerenkov absorption is due in this case to the fact that, in accordance with (5), the electron velocity would have to be infinite when $\theta = \pi/2$. The number of such electrons is zero. For the ordinary wave $N^2 = \eta$; since $\mathrm{Re}\,\eta = 1 - (\omega_0/\omega)^2 < 1$, there is no absorption. This case is the same as for propagation in the absence of a magnetic field. Thus, Cerenkov absorption takes place only in oblique propagation $(\theta \neq 0,\ \theta \neq \pi/2)$.

The general expression for the square of the refractive index is:

$$N^2 = A/B$$
$$A = (\varepsilon^2 - g^2 - \varepsilon\eta)\sin^2\theta + 2\varepsilon\eta \pm \sqrt{(\varepsilon^2 - g^2 - \varepsilon\eta)^2\sin^4\theta + 4\eta^2 g^2\cos^2\theta}, \tag{12}$$
$$B = 2(\varepsilon\sin^2\theta + \eta\cos^2\theta).$$

It is of interest to determine the boundary of the absorption band in which $\mathrm{Im}\,N^2 \ll \mathrm{Re}\,N^2$. In this frequency region the tensor components ϵ_{ik} can be written in the form

$$\varepsilon = \varepsilon_0 + i\varepsilon_1,\ g = g_0 + ig_1;\ \eta = \eta_0 + i\eta_1, \tag{13}$$

where ϵ_0, g_0 and η_0 are the values of ϵ, g, and η when thermal motion is not taken into account and ϵ_1, g_1 and η_1 are small corrections which take account of the absorption. In the approximation which is linear in ϵ_1, g_1 and η_1

$$N^2 = n^2 (1 + ia/A_0 - ib/B_0), \qquad (14)$$

where n^2 is the square of the refractive index when thermal motion is not taken into account, A_0 and B_0 are the values of A and B under the same conditions,

$$a = [2(\varepsilon_0\varepsilon_1 - g_0g_1) - \varepsilon_0\eta_1 - \eta_0\varepsilon_1]\sin^2\theta + 2(\eta_0\varepsilon_1 + \varepsilon_0\eta_1)$$

$$\pm \frac{(\varepsilon_0^2 - g_0^2 - \varepsilon_0\eta_0)(2\varepsilon_0\varepsilon_1 - 2g_0g_1 - \varepsilon_0\eta_1 - \varepsilon_1\eta_0)\sin^4\theta + 4(\eta_0^2 g_0g_1 + \eta_0g_0^2\eta_1)\cos^2\theta}{[(\varepsilon_0^2 - g_0^2 - \varepsilon_0\eta_0)^2\sin^4\theta + 4\eta_0^2g_0^2\cos^2\theta]^{1/2}}, \quad b = 2(\varepsilon_1\sin^2\theta + \eta_1\cos^2\theta), \qquad (15)$$

and in the expressions for $\epsilon_1(k)$, $g_1(k)$ and $\eta_1(k)$ we take $k = n\omega/c$.

The Cerenkov absorption is most important when:

$$\text{(a) } \omega_0^2/\omega_H^2 \gg 1, \quad \text{(b) } \omega \ll \omega_H. \qquad (16)$$

Let us consider this case in greater detail. If the first condition is satisfied, $n^2 \sim \omega_0^2/\omega_H^2 \gg 1$; in particular, if $nT \sim H^2/8\pi$, $n^2 \sim \beta^{-2}$ and the majority of the plasma electrons are "faster-than-light" electrons. This condition simplifies the calculation considerably because when $\omega_0^2/\omega_H^2 \gg 1$, for values of θ for which the following condition is satisfied:

$$\sin^4\theta/\cos^2\theta \ll 4\omega_0^4/\omega^2\omega_H^2, \qquad (17)$$

we have the case of so-called "quasi-longitudinal propagation,"[9] in which the square of the refractive index is, disregarding thermal motion,

$$n^2 = -\omega_0^2/\omega(\omega \pm \omega_H\cos\theta). \qquad (18)$$

The refractive index for the extraordinary wave exceeds unity (minus sign in the denominator) at frequencies ω which satisfy the condition $0 < \omega < \omega_H \cos\theta$. Near the right-hand boundary of this frequency region, as follows from the results of Sec. 2, we have resonance absorption when $\omega_0^2/\omega_H^2 \gg 1$. If the second condition in (16) is satisfied, however, resonance absorption no longer plays a role and in the expressions for ϵ and g we can omit the imaginary parts and deal only with the pure Cerenkov absorption. Thus, the Cerenkov absorption determines the shape of the total absorption band on the low-frequency side.

Under the assumptions indicated in (16), the components of the tensor ϵ_{ik} (10) assume the form:

$$\varepsilon = \varepsilon_0 = \omega_0^2/(\omega_H^2 - \omega^2); \quad g = g_0 = \varepsilon_0\omega_H/\omega;$$

$$\eta = \eta_0 + i\eta_1 = -\frac{\omega_0^2}{\omega^2} \qquad (19)$$

$$+ i\sqrt{\pi}\,\frac{2\omega_0^2}{\omega^2(\beta n\cos\theta)^3}\exp\left\{-\frac{1}{\beta^2n^2\cos^2\theta}\right\}.$$

Substituting these values in Eqs. (14) and (15) we have

$$N^2 = \frac{\omega_0^2}{\omega(\omega_H\cos\theta - \omega)}\left\{1 + i\sqrt{\pi}\,\frac{\omega^{5/2}(\omega_H\cos\theta - \omega)^{1/2}}{\omega_0^3}\right.$$

$$\left.\times \frac{\sin^2\theta}{(\beta\cos\theta)^3}\exp\left[-\frac{\omega(\omega_H\cos\theta - \omega)}{\omega_0^2\beta^2\cos^2\theta}\right]\right\}. \qquad (20)$$

The absorption coefficient is

$$\alpha = \frac{\omega}{c}\frac{\operatorname{Im}N^2}{n} = \sqrt{\pi}\,\frac{\omega}{c}\,\frac{\omega^2}{\omega_0^2}\,\frac{\sin^2\theta}{(\beta\cos\theta)^3}\exp\left[-\frac{\omega(\omega_H\cos\theta - \omega)}{\beta^2\omega_0^2\cos^2\theta}\right]. \qquad (21)$$

These formulas apply when $\omega \ll \omega_H$ and for values of θ which satisfy the condition given in (17); from Eq. (16) $\omega_0^2 \gg \omega_H^2$ so that it follows from Eq. (17) that Eqs. (20) and (21) are valid over the entire region of variation of θ with the exception of the narrow cone close to $\theta = \pi/2$, defined by the condition

$$|\pi/2 - \theta| < \omega|\omega_H|/2\omega_0^2.$$

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*VELOCITY AND TEMPERATURE DISCONTINUITIES NEAR THE WALLS OF A BODY AROUND WHICH RAREFIED GASES FLOW WITH TRANSONIC VELOCITIES**

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New and more general formulas are derived for velocity and temperature discontinuities on a gas-wall surface for rarefied gas flows of arbitrary Mach number. The equation for the velocity discontinuity is practically the same as that for $M \ll 1$; on the other hand, the relation for the temperature discontinuity differs markedly from the well-known Maxwell formula for a gas at rest near a wall.

IN previous investigations, even in the most detailed,[1] the effects of the slipping of rarefied gases along the walls and the temperature discontinuities on the gas-wall boundary have been studied only in cases corresponding to $M \ll 1$. Here M is a dimensionless quantity, equal to the ratio of the speed of flow far from the wall to the speed of sound (the Mach number). Furthermore, there is a great need to know the laws governing these effects for flows with arbitrary values of M, since the gas dynamics of rarefied gases are of considerable interest at the present time, principally in connection with the problem of the flight of rocket missiles and apparatus at the upper levels of the atmosphere.

The work below had as its aim the solutions of these problems.

1. INITIAL ASSUMPTIONS; THE VELOCITY DISTRIBUTION FUNCTION f

As is well known,[2] the equations of gas dynamics preserve their usual form in relation to the expression for the heat flow q_μ and the stress tensor $\tau_{\mu\nu}$ if

$$Ml/L = M^2/R < 1, \tag{1.1}$$

where l is the length of the molecular mean free path, L a characteristic linear dimension of the object (or channel) in the flow, and R the Reynolds number. q_μ and $\tau_{\mu\nu}$ are in this case expansions in powers of the parameter Ml/L, with factors in the form of first, second and higher derivatives, with respect to the coordinates x_α, of the mean velocity $\bar{u}_\mu$ and of the temperature T.

We assume condition (1.1). Then

$$\rho\bar{u}_\nu = \rho v_\nu, \quad \rho\overline{u_\mu u_\nu} = p\delta_{\mu\nu} - \tau_{\mu\nu} + \rho v_\mu v_\nu,$$
$$\rho\left(\frac{1}{2}\overline{u^2 u_\nu} + \overline{\varepsilon u_\nu}\right) = p v_\nu + q_\nu - \tau_{\mu\nu} v_\mu + \rho v_\nu\left(\frac{v^2}{2} + c_v T\right), \tag{1.2}$$

where v_μ is the mean velocity of the macroscopic motion of the medium, ϵ is the internal energy of the molecule of the gas, and c_V is the specific heat at constant volume.

The laws of conservation of mass, momentum, and energy at the gas-wall boundary can be written in the following form, if we denote the unit normal vector by n_ν:

$$\rho\bar{u}_\nu n_\nu = (\rho\bar{u}_\nu)^* n_\nu, \quad \rho\overline{u_\mu u_\nu} n_\nu = \rho(\overline{u_\mu u_\nu})^* n_\nu,$$
$$\rho\left(\frac{1}{2}\overline{u_\mu u_\mu u_\nu} + \overline{\varepsilon u_\nu}\right) n_\nu = \rho\left(\frac{1}{2}\overline{u_\mu u_\mu u_\nu} + \overline{\varepsilon u}\right)^* n_\nu. \tag{1.3}$$

*This work was completed in 1950.

ERRATA TO VOLUME 7

Page	Reads	Should Read
533, title	Nuclear magnetic moments of Sr^{87} and Mg^{95}	Nuclear magnetic moments of Sr^{87}
645 Eq. (1)	$\ldots + \varkappa \sqrt{j_0 (j_0 + 1)}$	$\ldots - \varkappa \sqrt{j_0 (j_0 + 1)}$
647 Eq. (11)	$(L+1) \lvert B_L^- \rvert^2 - L \lvert B_L^+ \rvert^2$	$L(L+1) [\lvert B_L^- \rvert^2 - \lvert B_L^+ \rvert^2]$
894 Eq. (12)	$\varepsilon_{11} = 1 - \sum \frac{\cdots}{\sqrt{\pi/\mu}}$	$\varepsilon_{11} = 1 - \sum \frac{\cdots}{\sqrt{\pi}\, \mu}$
897 Eq. (45)	$\sqrt{\pi/2}$	$\sqrt{\pi/8}$
979 Table II, heading	$\lvert E_\gamma > 50$ **Mev** $\lvert E_\gamma > 50$ **Mev**	$\lvert E_\gamma < 50$ **Mev** $\lvert E_\gamma > 50$ **Mev**
1023 Figure caption		a) $\omega < \omega_H$, b) $\omega > \omega_H$
1123 Eq. (2)	$\Gamma = \mu_2/\mu_1$	$\Gamma = \mu_2/\mu_1,\ \mu_\perp = (\mu_1^2 - \mu_2^2)/\mu_1$

ERRATA TO VOLUME 8

Page	Reads	Should Read
375 Figure caption	a) positrons of energy up to 0.4 ϵ, b) positrons of energy up to 0.3 ϵ.	a) positrons of energy up to 0.3 ϵ, b) positrons of energy up to 0.4 ϵ.
816 Beginning of Eq. (8)	$I_2^5 = (4\pi)^2 \ldots$	$I_2^2 = (4\pi)^5 \ldots$