MULTIMAGNON PROCESSES IN THE SCATTERING OF SLOW NEUTRONS IN FERROMAGNETS

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We calculate the cross section for the inelastic scattering of neutrons in ferromagnets, accompanied by the absorption of one magnon and the emission of another. It is shown that this cross section is small compared to that for scattering with absorption or emission of a single magnon. It is also shown that the role of scattering processes in which more than two magnons participate is negligible.

In a preceding paper of the author[1] it was shown that there are two types of multimagnon processes of scattering of slow neutrons in ferromagnets: two-magnon scattering, in which one magnon is absorbed and another emitted, and three-magnon scattering, in which two magnons are absorbed and one is emitted.

This result was based on spin wave theory in the form given by Dyson.[2,3] In Dyson's papers all types of interactions between atomic spins were neglected except for exchange interaction (which can be done only for ferromagnets with a high Curie point), and it was assumed that the magnetic atoms (or ions) form a simple Bravais lattice. Therefore the conclusion in reference 1 concerning possible multimagnon scattering processes is also valid only with respect to such ferromagnets.

We shall calculate the cross section for two-magnon scattering, limiting ourselves for simplicity to a scatterer with a cubic lattice. Using Eqs. (I.5d) and (I.6), we find the following expression for the two-magnon scattering cross section*

$$d\sigma^{\tau}_{+1,-1}/d\Omega = r_0^2\gamma^2F^2(q)\,\frac{v_0}{(2\pi)^3}\,\frac{p'}{p}\,e^{-2W_q} \times\left[1-\left(\frac{\mathbf{q}\cdot\mathbf{m}}{q}\right)^2\right]n(\mu)\,[n(\rho)+1]\,\delta(\mathbf{q}+\boldsymbol{\mu}-\boldsymbol{\rho}+\boldsymbol{\tau})\,d\boldsymbol{\rho}\,d\boldsymbol{\mu}, \qquad (1)$$

where, from conservation of energy,

$$p'^2 = p^2+\alpha(\mu^2-\rho^2), \text{ and } \alpha\gg 1.$$

From here on, as in reference 1, we replace $\mathbf{q}$ by $\boldsymbol{\tau}$ everywhere except in the argument of the δ-function, and then integrate with respect to $\boldsymbol{\rho}$. The result is:

$$d\sigma^{\tau}_{+1,-1}/d\Omega = r_0^2\gamma^2F^2(\tau)\,e_{3}^{-2W_\tau}\,\frac{v_0}{(2\pi)^3}\left[1-\left(\frac{\mathbf{m}\cdot\boldsymbol{\tau}}{\tau}\right)^2\right] \times\frac{(p'_+)^2[n(\rho_+)+1]+(p'_-)^2[n(\rho_-)+1]}{\alpha Kp\,[\cos^2\zeta-\alpha^{-1}(\alpha+1)(1-Q^2\alpha^{-1}K^{-2})]^{1/2}}, \qquad (2)$$

where

$$\mathbf{K}=\mathbf{P}+\boldsymbol{\mu},$$

$$\rho_\pm^2=\alpha^{-1}(Q^2-p'^2_\pm),$$

$$p'_\pm=\alpha K(\alpha+1)^{-1} \times\{\cos^2\zeta\pm[\cos^2\zeta-\alpha^{-1}(\alpha+1)(1-Q^2\alpha^{-1}K^{-2})]^{1/2}\}, \qquad (3)$$

and ζ is the angle between $\mathbf{K}$ and $\mathbf{p}'$.

We can now easily integrate (2) over $d\Omega$, choosing the z axis along the vector $\mathbf{K}$. (The fact that this integration can be done relatively simply is the reason why the total cross section is easier to calculate than the differential cross section, since integrating (2) with respect to $\boldsymbol{\mu}$ is very complicated.) We have to treat two cases:

$$1)\ \alpha K^2 > Q^2, \qquad 2)\ \alpha K^2 < Q^2. \qquad (4)$$

In the first case

$$1\geqslant\cos\zeta\geqslant\cos\zeta_0=\{(\alpha+1)\alpha^{-1}[1-Q^2(\alpha K^2)^{-1}]\}^{1/2}$$

while in the second,* $\cos\zeta$ varies from 1 to -1; also, in the second case $p'_- < 0$ and we need keep only terms containing p'_+ in (2). In both cases the integration over $d\Omega$ reduces to an integration over $d\rho^2_\pm$. The integral is easily done and the final result is the same for both cases:

*We use the notation of reference 1, which we cite as I.

*One can show that the condition $q\approx\tau$ is not violated for any angle ζ.

$$d\sigma^{\tau}_{+1,-1}/d\Omega = r_0^2\gamma^2F^2(\boldsymbol{\tau})\,e^{-2W_\tau}\,\frac{v_0}{(2\pi)^3}\left[1-\left(\frac{\mathbf{m}\cdot\boldsymbol{\tau}}{\tau}\right)^2\right]\frac{\pi}{Kp}\,n(\mu)$$
$$\times\left\{\frac{2\nu T}{T_c}\ln\frac{1-\exp(-T_c\delta^2R_-^2/2\nu T)}{1-\exp(-T_c\delta^2R_+^2/2\nu T)}+R_-^2-R_+^2\right\},\quad (5)$$

where

$$R_\pm^2 = (\alpha+1)^{-2}\,[\sqrt{(\alpha+1)\,Q^2-\alpha K^2}\mp K]^2 = \left(\frac{\alpha}{\alpha+1}\right)^2 \quad (6)$$
$$\times\left\{\sqrt{(\boldsymbol{\mu}-\alpha^{-1}\mathbf{P})^2+\frac{\alpha+1}{\alpha^2}(p^2-P^2)}\mp\alpha^{-1}|\mathbf{P}+\boldsymbol{\mu}|\right\}^2.$$

We still have to integrate (5) with respect to $\boldsymbol{\mu}$. The integration over the direction of $\boldsymbol{\mu}$ can be changed to an integration over the length of the vector $\mathbf{K} = \mathbf{P} + \boldsymbol{\mu}$, so we must find the limits of variation of K. But it is obvious, on the one hand, that

$$|P-\mu|<K<P+\mu, \quad (7)$$

and, on the other hand, we have from (6) that

$$K<(1+1/\alpha)^{1/2}Q. \quad (8)$$

The simultaneous inequalities (7) and (8) give the following results:

(1) If $p > P$, K varies between the limits (7) for all μ.

(2) If

$$P>p>\sqrt{\alpha/(\alpha+1)}P,$$

K varies between the limits (7) for all μ except those lying in the interval

$$\alpha^{-1}[P-\sqrt{(\alpha+1)(P^2-p^2)}]$$
$$\leqslant\mu\leqslant\alpha^{-1}[P+\sqrt{(\alpha+1)(P^2-p^2)}], \quad (9)$$

while for these values of μ, K varies in the interval

$$|P-\mu|\leqslant K\leqslant\sqrt{1+1/\alpha}\,Q. \quad (10)$$

(3) If

$$p<\sqrt{\alpha/(\alpha+1)}\,P,$$

then for

$$\mu<\mu_1\leqslant\alpha^{-1}[-P+\sqrt{(\alpha+1)(P^2-p^2)}] \quad (11)$$

no scattering can occur; for values of μ in the range

$$\mu_1<\mu<\alpha^{-1}[P+\sqrt{(\alpha+1)(P^2-p^2)}], \quad (12)$$

the limits for K are given by (10), while for larger μ they are given by (7).

The shape of the region of integration over K and μ changes when $P = p$ and $P = p\sqrt{(\alpha+1)/\alpha}$. Considered as a function of P, the total cross section $\sigma^{\tau}_{+1,-1}(p, P)$ therefore has a kink at these values of P. One can show, moreover, that for $P = p \pm 0$

$$\partial\sigma^{\tau}_{+1,-1}(p, P)/\partial P = \mp\infty,$$

and consequently the cross section is a maximum for $P = p$. This is related to the fact that, for $P = p$ and $\mu = 0$, both $n(\mu)$ and the logarithm in Eq. (5) become infinite (the latter because $R_+^2 = 0$ when $P = p$ and $\mu = 0$).

From now on we shall assume that

$$P/\alpha\ll\beta,\quad(|p^2-P^2|/\alpha)^{1/2}\ll\beta,$$
$$\beta=\delta^{-1}\sqrt{2\nu T/T_c}. \quad (13)$$

Since $\alpha \gg 1$, these inequalities are valid over a very wide range of values of $\mathbf{p}$ and $\boldsymbol{\tau}$.

When (13) is satisfied, expression (5) simplifies considerably. In fact, when $\mu \gtrsim \beta$, $R_\pm \approx \mu \mp K/\alpha$, and the logarithm in (5) becomes

$$(4\mu/\alpha\beta^2)\,Kn(\mu)+O(\alpha^{-2}), \quad (14)$$

while, if $\mu \ll \beta$, the expression in the logarithm in (5) can be replaced by $R_-^2R_+^2$, since in this case $R_\pm^2 \ll \beta^2$.

After such a simplification, it is easy to determine the dependence of the total cross section on α. First, it is clear from (14) that the contribution to the cross section from large values of $\mu(\mu \gtrsim \beta)$ is proportional to α^{-1}.

In the region of small μ, so long as $\alpha R_+ > K$,

$$\ln(R_-^2R_+^{-2})\approx 4K(\alpha R_+)^{-1},$$

i.e., the contribution to the total cross section is also proportional to α^{-1} in this case. The condition $\alpha R_+ > K$ is violated if

$$\mu^2<\alpha^{-1}(2P\mu+P^2-p^2)+O(\alpha^{-2}),$$

but the equation $R_+^2(\mu) = 0$ has roots for just these values of μ, so that in this case (5) has a logarithmic singularity and the contribution to the cross section from this region of values of μ is in order of magnitude no greater than $\Delta(a \ln \Delta + b)$, where Δ gives the dimensions of the region and a and b do not depend on Δ. Obviously, if

$$P/\alpha>\sqrt{\alpha^{-1}|p^2-P^2|},$$

then $\Delta \sim \alpha^{-1}$, and the contribution to the cross section is of order

$$\alpha^{-1}(a'\ln\alpha+b'), \quad (15)$$

where a' and b' do not depend on α. But if $\alpha^{-1}P < \sqrt{|p^2-P^2|/\alpha}$, then from (11),

$$\Delta=\sqrt{\alpha^{-1}|P^2-p^2|}+O(\alpha^{-1})-\mu_1=O(\alpha^{-1})$$

and the contribution to the cross section has the same form (15), but with different values of a' and b'. Finally, the contribution to the total cross section from the difference $R_-^2 - R_+^2$ which appears in (5) is also proportional to $1/\alpha$, so that the total cross section has the form $(A \ln \alpha + B)/\alpha$.

As already mentioned, for given values of $\mathbf{p}$ and τ the cross section is a maximum when $P = p$. The cross section in this case is calculated in the Appendix, where the following result is obtained:

$$\sigma^{\tau}_{+1,-1}(p, p) = 16\pi^2 r_0^2 \gamma^2 F^2(\tau) \frac{v_0}{(2\pi^3)} e^{-2W_\tau} \times \left[1 - \left(\frac{\overline{\tau} \cdot \mathbf{m}}{\tau}\right)^2\right] \frac{1}{\delta^4 \alpha p} \left(\frac{2\nu T}{T_c}\right)^2 \left[\ln\left(\frac{\alpha}{p\delta}\sqrt{\frac{2\nu T}{T_c}}\right) + 2\right] + O(\alpha^{-2}). \tag{16}$$

One can also show that for the case of long-wave neutrons ($p \ll \tau$) and for temperatures which are not very low ($\tau\alpha^{-1/2} \ll \beta^{1/2}$), the total cross section for two-magnon scattering is given by

$$\sigma^{\tau}_{+1,-1}(p, \tau) = 16\pi^2 r_0^2 \gamma^2 F^2(\tau) \frac{v_0}{(2\pi)^3} e^{-2W_\tau} \left[1 - \left(\frac{\tau \cdot \mathbf{m}}{\tau}\right)^2\right] \times \frac{1}{\delta^4 \alpha p} \left(\frac{2\nu T}{T_c}\right)^2 \left[\ln\left(\frac{1}{\tau\delta}\sqrt{\frac{2\nu T\alpha}{T_c}}\right) + C\right] + O(\alpha^{-3/2}), \tag{17}$$

where $C = \frac{1}{2} + \ln 2 - \pi^2/12 \approx 0.37$.

For comparison, we give the expressions for one-magnon scattering in the two cases which we have treated. Using (I.23) and (I.35), we find for $P = p$:

$$\sigma^{\tau}_{1}(p, p) = \sigma^{\tau}_{-1}(p, p) + \sigma^{\tau}_{+1}(p, p) = 2\pi S\gamma^2 r_0^2 \times F^2(\tau) e^{-2W_\tau} \left[1 + \left(\frac{\mathbf{m}\cdot\tau}{\tau}\right)^2\right] \frac{1}{(p\delta)^2} \frac{2\nu T}{T_c} \left[\ln \frac{p\delta N^{1/3}}{\alpha\pi} + O(\alpha^{-2})\right], \tag{18}$$

while for $p \ll \tau$ and $\tau\alpha^{-1/2} \ll \beta^{1/2}$, we find, using (I.24):

$$\sigma^{\tau}_{1}(p, \tau) = \sigma^{\tau}_{-1}(p, \tau) = 2\pi S r_0^2 \gamma^2 F^2(\tau) e^{-2W_\tau} \left[1 + \left(\frac{\mathbf{m}\cdot\tau}{\tau}\right)^2\right] \frac{1}{\tau p \delta^2} \frac{2\nu T}{T_c} [\alpha^{-1/2} + O(\alpha^{-3/2})]. \tag{19}$$

Comparing (16) with (18) and (17) with (19), we see that

$$\sigma^{\tau}_{+1,-1}(p, p) \ll \sigma^{\tau}_{1}(p, p); \quad \sigma^{\tau}_{+1,-1}(p, \tau) \ll \sigma^{\tau}_{-1}(p, \tau). \tag{20}$$

It is obvious that in general two-magnon scattering is small compared to one-magnon scattering. When condition (13) is satisfied, this follows from the fact that in this case $\sigma^{T}_{+1,-1}$ is proportional to α^{-1}.

Let us now estimate the magnitude of the three-magnon scattering. Using (I.5d) and (I.5c), it is easy to show that*

$$\sigma^{\tau}_{+1,-2}(p, P) = \frac{v_0}{2S(2\pi)^3}\int d\boldsymbol{\mu}\, n(\mu) \frac{Q}{p} \sigma^{\tau}_{+1,-1}(Q, |\mathbf{P} + \boldsymbol{\mu}|), \tag{21}$$

where, in general, the region of integration over $\boldsymbol{\mu}$ may be smaller than an elementary cell of the reciprocal lattice. Furthermore, it is obvious that

$$\sigma^{\tau}_{+1,-1}(Q \mid P + \mu|) \leqslant \sigma^{\tau}_{+1,-1}(Q, Q),$$

while it follows from (16) that

$$\sigma^{\tau}_{+1,-1}(Q, Q) < \frac{p}{Q}\sigma^{\tau}_{+1,-1}(p, p),$$

since $p < Q$, and therefore

$$\sigma^{\tau}_{+1,-2}(p, P) < \sigma^{\tau}_{+1,-1}(p, p) \frac{v_0}{2S(2\pi)^3}\int d\boldsymbol{\mu}\, n(\mu).$$

Extending the $\boldsymbol{\mu}$ integration over the whole space, we obtain finally

$$\sigma^{\tau}_{+1,-2}(p, P) < \varkappa\sigma^{\tau}_{+1,-1}(p, p)/2S, \tag{22}$$

where

$$\varkappa = \frac{1}{8}\zeta(3/2)(v_0/\delta^3\pi^{3/2})(2\nu T/T_c)^{3/2} \ll 1. \tag{23}$$

Thus we see that the three-magnon scattering is small compared to one-magnon scattering.

As was stated at the beginning of the paper, all the possible multi-magnon scattering processes are included in the two we have treated if we can neglect all types of interaction of the atomic spins except the exchange interaction, and if the magnetic atoms form a simple Bravais lattice.

In the general case one can assert[4] only that the principal multi-magnon processes will be those in which the total number of spin waves changes by no more than unity. This follows from the fact that the projection of the total spin of the ferromagnet on the direction of magnetization is at least approximately an integral of the motion. In fact, the scattering amplitude depends linearly on the operators S^{+}_{l}, S^{-}_{l} and S^{z}_{l}. But it was shown in reference 1 that if the projection of the total spin of the system is conserved, the matrix elements of the operator S^{z}_{l} are different from zero only for transitions involving no change in the projection of the total spin, while the operators $S^{\pm}_{l}$ give non-zero matrix elements only for transitions in which the projection of the total spin changes by ± 1. It follows that the operators S^{z}_{l} are responsible for the scattering without change in the total number of magnons, and the operators $S^{\mp}_{l}$ for the scattering in which this number changes by ± 1.

If, however, we take account of the fact that the projection of the total spin of the ferromagnet is not an exact integral of the motion, then obviously scattering processes are possible in which the change in the total number of magnons is greater than unity. The simplest processes of this type are scattering with absorption of two magnons and scattering with emission of two magnons. Let us assume, as before, that the energy of a magnon is a quadratic function of its momentum and that α is a large number. Then, just as in the case of $\sigma^{T}_{+1,-1}$, we can show that $\sigma^{T}_{2,0}$ and $\sigma^{T}_{0,-2}$ are pro-

*For simplicity we assume the scatterer to be unmagnetized.

portional to α^{-1} (provided, of course, that condition (13) is satisfied), and consequently $\sigma^{T}_{2,0}$, $\sigma^{T}_{0,-2} \ll \sigma^{T}_{+1,-1}$.

We now show that if there is scattering with emission of two magnons and absorption of one magnon, this cross section is small compared to $\sigma^{T}_{+1,-1}$. In fact, as in the derivation of (22), we can write

$$\sigma^{\tau}_{+2,-1}(p, P) \lesssim \tilde{\sigma}^{\tau}_{+2,0}(p, p)\frac{v_0}{(2\pi)^3}\int d\nu n(\nu) = \varkappa\tilde{\sigma}^{\tau}_{+2,0}(p, p), \tag{24}$$

where $\tilde{\sigma}^{T}_{+2,0}$ is the total cross section which would be gotten from (1) by replacing $n(\mu)$ with $n(\mu)+1$ in (1), and changing the sign of μ^2 in the expression for p'.* The actual cross section for scattering with emission of two magnons, if it occurs, differs from $\tilde{\sigma}^{T}_{+2,0}$ by a factor which is small compared to unity. As we stated above, $\tilde{\sigma}^{T}_{+2,0}$ is proportional to $1/\alpha$, and consequently $\tilde{\sigma}^{T}_{+2,0} \lesssim \tilde{\sigma}^{T}_{+1,-1}$, so that $\sigma^{T}_{+2,-1} \ll \sigma^{T}_{+1,-1}$. In this same way we can show that the cross section for any other multimagnon scattering process is small compared to $\sigma^{T}_{+1,-1}(p, P)$.

We have thus found the following result.

If the energy of a magnon is a quadratic function of its momentum, the cross sections for all possible multi-magnon scattering processes are small compared with $\sigma^{T}_{+1,-1}(p, P)$. On the other hand, $\sigma^{T}_{+1,-1}(p, P)$ in turn is small compared with the cross section for one-magnon scattering (at least, for large α). Therefore multi-magnon scattering processes play a minor role in the inelastic scattering of slow neutrons in ferromagnets, and in particular they cannot be used to explain the observed[5,6] large value of the total cross section for inelastic magnetic scattering in ferromagnets at high temperatures.

APPENDIX

In this appendix, we shall calculate the cross section for two-magnon scattering to terms of order α^{-2}, for the case where $P = p$.

When condition (13) is satisfied, we easily obtain from (5) the expression:

$$\sigma^{\tau}_{+1,-1}(p, p) = 2\pi^2 r_0^2\gamma^2 F^2(\tau)e^{-2W_\tau}\frac{v_0}{(2\pi)^3}\left[1-\left(\frac{\mathbf{m}\cdot\boldsymbol{\tau}}{\tau}\right)^2\right]p^{-2}(J_1+J_2+J_3); \tag{A.1}$$

$$J_1 = \frac{4}{3}\alpha^{-3}\int_0^\infty d\mu\cdot n(\mu)\left[|p+\alpha\mu|^3 - |p-\alpha\mu|^3\right]; \tag{A.2}$$

$$J_2 = \beta\int_0^{\zeta\beta^{1/2}} d\mu\cdot\mu n(\mu)\left\{(p+\mu)\ln\left(\frac{|p-\alpha\mu|+p+\mu}{|p-\alpha\mu|-p-\mu}\right)^2 - |p-\mu|\ln\left(\frac{p+\alpha\mu+|p-\mu|}{p+\alpha\mu-|p-\mu|}\right)^2 + Q\ln\left[\frac{(|p-\alpha\mu|-Q)(p+\alpha\mu+Q)}{(|p-\alpha\mu|+Q)(p+\alpha\mu-Q)}\right]^2\right\}; \tag{A.3}$$

$$J_3 = \frac{8}{\alpha}p\int_{\zeta\beta^{1/2}}^\infty d\mu\cdot\mu^3 n(\mu), \tag{A.4}$$

where ζ is a number such that $\zeta \ll 1$, $\zeta\beta^{1/2} \gg p/\alpha$.

Since $p/\alpha \ll \beta^{1/2}$, we get

$$J_1 = \frac{4p}{\alpha}\beta^2\int_0^\infty\frac{x\,dx}{e^x-1} + O(\alpha^{-3}) \approx \frac{2}{3\alpha}\pi^2\beta^2 p. \tag{A.5}$$

It is convenient to split J_2 into two parts, in the first of which we integrate from zero to p/α, and in the second from p/α to $\zeta\beta^{1/2}$. We note that if $p < \beta^{1/2}$, we can, without violating the assumptions made about ζ, choose ζ so that p is greater than $\zeta\beta^{1/2}$, and can therefore assume that in J_2, p is greater than μ. If we replace $n(\mu)$ by $\beta\mu^{-2}$ in J_2, and introduce the new integration variable $x = \mu/p$, we get:

$$J_2 = p\beta^2(I_1+I_2); \tag{A.6}$$

$$I_1 = \int_1^{1/\alpha}\frac{dx}{x}\left\{\ln\left[\frac{2-(\alpha-1)x}{2+(\alpha-1)x}\right]^2 + x\ln\left[\frac{4-(\alpha-1)^2x^2}{(\alpha+1)^2x^2}\right]^2 + \sqrt{1+\alpha x^2}\ln\left[\frac{(\alpha+1)x+2\sqrt{1+\alpha x^2}}{(\alpha+1)x-2\sqrt{1+\alpha x^2}}\right]^2\right\}; \tag{A.7}$$

$$I_2 = \int_{1/\alpha}^{\zeta\beta^{1/2}/p}\frac{dx}{x}\left\{\ln\left[\frac{(\alpha+1)^2x^2}{4-(\alpha-1)^2x^2}\right]^2 + x\ln\left[\frac{2+(\alpha-1)x}{2-(\alpha-1)x}\right]^2 + \sqrt{1+\alpha x^2}\ln\left[\frac{\alpha^2x^2-(1+\sqrt{1+\alpha x^2})^2}{\alpha^2x^2-(1-\sqrt{1+\alpha x^2})^2}\right]^2\right\}. \tag{A.8}$$

If we make the substitution $x = t/\alpha$ in I_1 and then expand in powers of $1/\alpha$, we get

$$I_1 = \frac{6}{\alpha}\left(1+\frac{3}{4}\ln 3\right) + O(\alpha^{-2}). \tag{A.9}$$

The integral of the second term in I_2 is easily done and gives

$$I_2' = \frac{8}{\alpha}\left[\left(1-\frac{3}{4}\ln 3\right) + \ln\frac{\alpha\zeta\beta^{1/2}}{p}\right] + O\left(\frac{p}{\alpha^2\zeta\beta^{1/2}}\right). \tag{A.10}$$

In deriving this expression we used the fact that $\zeta\beta^{1/2} \gg p\alpha^{-1}$.

The remaining part of integral I_2 can be split into two terms:

*$\tilde{\sigma}^{\tau}_{+2,0}(p, P)$ and $\tilde{\sigma}^{\tau}_{0,-2}(p, P)$ are maximum at $P = p$ for the same reasons that were given for $\sigma^{\tau}_{+1,-1}(p, P)$.

$$I_2'' = I_2 - I_2' = I_+ + I_-;$$

$$I_\pm = 2\int_{(1+\alpha^{-1})^{1/2}}^{(1+\alpha\zeta^2\beta p^{-2})^{1/2}} dy\cdot y\,(y^2-1)^{-1}\left\{\ln\frac{\alpha+1}{\alpha-1} \mp (y\pm 1)\ln\left|1 \pm \frac{2}{(\alpha-1)(y\pm 1)}\right|\right\} \quad \text{(A.11)}$$

We easily find for I_+:

$$I_+ = \int_{(1+\alpha^{-1})^{1/2}}^{(1+\alpha\zeta^2\beta p^{-2})^{1/2}} dy^2\,(y^2-1)^{-1}\left\{-\frac{2}{\alpha^2} + \frac{4}{\alpha^2(y+1)} + O(\alpha^{-3})\right\}.$$

But this expression is smaller in absolute value than $4\alpha^{-2}\ln(\alpha\zeta\beta^{1/2}p^{-2})$, and can therefore be neglected. After substituting $t = 2[(\alpha-1)(y-1)]^{-1}$, we get for I_-:

$$I_- = \frac{2}{\alpha-1}\int_{t_1}^{t_2}\frac{dt}{t^2}\{1+[1+(\alpha-1)t]^{-1}\}$$

$$\times\left\{\ln|1-t| + \frac{1}{2}(\alpha-1)t\ln\frac{\alpha+1}{\alpha-1}\right\}; \quad \text{(A.12)}$$

$$t_1 = 2[(\alpha-1)(\sqrt{1+\alpha\zeta^2\beta p^{-2}}-1)]^{-1} \ll 1,$$
$$t_2 = 2[(\alpha-1)(\sqrt{1+\alpha^{-1}}-1)]^{-1} \approx 4 + O(\alpha^{-1}). \quad \text{(A.13)}$$

From (A.12) and (A.13) we easily get

$$I_2'' = 2\alpha^{-1}\left(\frac{3}{4}\ln 3 - 1\right) + O\left(\alpha^{-2}\ln\alpha\right). \quad \text{(A.14)}$$

We have only J_3 left to consider. But from (A.4) we get:

$$J_3 = \frac{4}{\alpha}p\beta^2\int_{\zeta^2}^{\infty} xdx\frac{1}{(e^x-1)^2} = \frac{4}{\alpha}p\beta^2\{-\ln(1-e^{-\zeta^2})$$

$$+\int_0^\infty dx\frac{1}{e^x-1}\left(\frac{x}{e^x-1}-1\right) - \int_0^{\zeta^2}\frac{dx}{e^x-1}\left(\frac{x}{e^x-1}-1\right)\}.$$

and, since $\zeta \ll 1$, we have finally

$$J_3 = \frac{8}{\alpha}p\beta^2\left(-\ln\zeta + \frac{1}{2} - \frac{\pi^2}{12}\right). \quad \text{(A.15)}$$

Using these formulas, it is easy to get (16).

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302