ENERGY DIFFUSION OF FAST IONS IN AN EQUILIBRIUM PLASMA*

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The problem of the energy diffusion of fast ions injected into an equilibrium plasma is considered. The energy distribution of the injected ions is initially monochromatic, with the energy exceeding the mean thermal energy in the plasma. It is assumed in this case that the distribution of velocity directions is isotropic. The extent to which the distribution approaches a Maxwellian one is determined. For an arbitrary initial distribution, the result can be obtained by the principle of superposition, since the equations are linear.

1. DERIVATION OF THE EQUATION

THE kinetic equation for a fully ionized plasma in the absence of external fields has the form[1]

$$\partial f/\partial t = -\operatorname{div}_{\mathbf{v}}\mathbf{j},$$

$$j_i = \frac{\pi e^2 L}{m}\sum e'^2 \int\left(f\frac{\partial f'}{m'\partial v_k'} - f'\frac{\partial f}{m\partial v_k}\right)\frac{u^2\delta_{ik} - u_i u_k}{u^3}\,d\mathbf{v}'. \quad (1)$$

Here $f = f(t, \mathbf{v})$ is the distribution function, $\mathbf{j}$ is the flux density of particles in velocity space, $L = \ln[n^{-1}(\bar{\epsilon}/e^2)^3]$ is the Coulomb logarithm, m, e are the mass and charge of particles, $u_i = v_i - v_i'$ is the component of relative velocity.

Integration over $d\mathbf{v}'$ is taken over the whole velocity space. The primed variables are summed over all types of particles that collide with the given particles.

In our case the distribution function for each kind of particle is isotropic with respect to the velocity direction. We can therefore integrate in (1) over the directions of the vector $\mathbf{v}'$. Taking into account the equalities

$$\frac{\partial f}{\partial v_k} = \frac{v_k}{v}\frac{\partial f}{\partial v}, \qquad u_k(u^2\delta_{ik} - u_i u_k) \equiv 0,$$

we obtain

$$j_i = \frac{\pi e^2 L}{m}\sum e'^2$$

$$\times \iint\left(f\frac{\partial f'}{m'v'\partial v'} - f'\frac{\partial f}{mv\partial v}\right)v_k\frac{u^2\delta_{ik} - u_i u_k}{u^3}v'^2\,dv'\,d\Omega_{\mathbf{v}'}. \quad (2)$$

The integral over the angles of $\mathbf{v}'$ has the form:

$$\int\frac{u^2\delta_{ik} - u_i u_k}{u^3}\,d\Omega_{\mathbf{v}'} = \frac{\partial^2}{\partial v_i\partial v_k}\int u\,d\Omega_{\mathbf{v}'}.$$

Evaluation of the last integral gives

$$\int u\,d\Omega_{\mathbf{v}'} = \frac{2\pi}{3vv'}\{(v+v')^3 - |v-v'|^3\}.$$

After differentiating twice and substituting into (2) we obtain for the flux in velocity space

$$j_i = \frac{\pi e^2 L}{m}\sum e'^2 \quad (3)$$

$$\times\int\left(f\frac{\partial f'}{m'v'\partial v'} - f'\frac{\partial f}{mv\partial v}\right)\frac{8\pi}{3}v_i v'^3\frac{8}{(v+v'+|v-v'|)^3}v'\,dv'.$$

This flux is directed along the vector $\mathbf{v}$, and its divergence is

$$\operatorname{div}_{\mathbf{v}}\mathbf{j} = \frac{1}{v^2}\frac{\partial}{\partial v}(v^2 j), \quad (4)$$

where j is the absolute value of the flux.

We assume that a small number of ions of definite energy, with a spherically symmetric distribution of velocity directions, is injected into an equilibrium fully-ionized plasma. Then the role played by the collisions of these particles with each other will not be great, and the change in time of the distribution function f of these particles will be determined by the collisions with the ions and the electrons of the equilibrium plasma having Maxwellian distributions:

$$f_i' = n(M/2\pi T)^{3/2}e^{-mv'^2/2T}, \quad (5)$$

$$f_e' = n(m/2\pi T)^{3/2}e^{-mv'^2/2T}, \quad (6)$$

where n is the ion or electron density, M, m are the ion and electron masses, T is the ion and electron temperature in energy units.

The distributions are normalized so that $\int f' 4\pi v'^2\,dv' = n$. Thus the flux of injected particles in velocity space is equal to:

$$j = \frac{\pi e^4 L}{M}\int\left(f\frac{\partial f_i'}{Mv'\partial v'} - f_i'\frac{\partial f}{Mv\partial v}\right)\frac{8\pi}{3}vv'^3\frac{8}{(v+v'+|v-v'|)^3}v'\,dv'$$

$$+\frac{\pi e^4 L}{m}\int\left(f\frac{\partial f_e'}{mv'\partial v'} - f_e'\frac{\partial f}{Mv\partial v}\right)\frac{8\pi}{3}vv'^3\frac{8}{(v+v'+|v-v'|)^3}v'\,dv'. \quad (7)$$

*Work carried out in 1952.

Substituting for f'_i and f'_e their values from (5) and (6), and integrating over dv', we shall obtain the final expression for the flux of particles in velocity space:

$$j = -\frac{2^{5/2} e^4 LnT^{1/2}}{3\pi^{1/2} M^{3/2} v}\left\{\left[\frac{3}{4v}\sqrt{\frac{2\pi T}{M}}\,\Phi\left(\sqrt{\frac{M}{2T}}v\right) - \frac{3}{2}e^{-Mv^2/2T}\right]\right. \tag{8}$$
$$\left. + \sqrt{\frac{M}{m}}\left[\frac{3}{4v}\sqrt{\frac{2\pi T}{m}}\,\Phi\left(\sqrt{\frac{m}{2T}}v\right) - \frac{3}{2}e^{-mv^2/2T}\right]\right\}\left(\frac{f}{T} + \frac{\partial f}{Mv\partial v}\right),$$

where $\Phi(z) = \frac{2}{\sqrt{\pi}}\int_0^z e^{-t^2}\,dt$ is Kramp's function.

The kinetic equation for the injected ions, in accordance with (1) and (4) has the form

$$\frac{\partial f}{\partial t} = -\frac{1}{v^2}\frac{\partial}{\partial v}(v^2 j). \tag{9}$$

Substituting into this the expression for j from (8) we obtain:

$$\frac{\partial f}{\partial \tau} = \frac{1}{\sqrt{x}}\frac{\partial}{\partial x}\left[\sqrt{x}\,\Gamma(x)\left(f + \frac{\partial f}{\partial x}\right)\right], \tag{10}$$

with

$$x = Mv^2/2T, \quad \tau = 2^{5/2} e^4 Lnt / 3\pi^{1/2} M^{1/2} T^{3/2},$$
$$\Gamma(x) = \frac{3}{4}\sqrt{\pi}\frac{\Phi(\sqrt{x})}{\sqrt{x}} - \frac{3}{2}e^{-x} \tag{11}$$
$$+ \sqrt{\frac{M}{m}}\left[\frac{{}^3/_4\sqrt{\pi}\,\Phi(\sqrt{mx/M})}{\sqrt{mx/M}} - \frac{3}{2}e^{-mx/M}\right].$$

We shall normalize the function f by the condition

$$\int_0^\infty f\sqrt{x}\,dx = 1. \tag{12}$$

In going over from the distribution function f in velocity space to the distribution density with respect to energy $\rho = f\sqrt{x}$, we obtain from (10) the following equation for ρ:

$$\frac{\partial \rho}{\partial \tau} = \frac{\partial}{\partial x}\left[\Gamma(x)\left(\rho - \frac{\rho}{2x} + \frac{\partial \rho}{\partial x}\right)\right], \tag{13}$$

with

$$\int_0^\infty \rho\,dx = 1. \tag{14}$$

2. SOLUTION OF THE EQUATION

The initial condition for the distribution density with respect to energy of the injected ions ρ is a monochromatic distribution with the energy equal to x_0. In future we shall assume that x_0 exceeds unity. Thus

$$\rho(x, 0) = \delta(x - x_0). \tag{15}$$

With time this distribution will be smeared out by diffusion and will approach the stationary distribution which is obtained from (13) when $d\rho/d\tau = 0$ and which is, as it ought to be, Maxwellian

$$\rho(x, \infty) = 2e^{-x}\sqrt{x/\pi}. \tag{16}$$

Our problem is to find the time variation of the distribution, particularly for large times and for energies exceeding x_0.

We solve (13) by using Laplace transforms. To do this we multiply both sides of equation (13) by $e^{-\lambda\tau}$ and integrate over $d\tau$ between the limits 0 and ∞. Integrating by parts on the left-hand side and denoting

$$\rho_\lambda(x) = \int_0^\infty \rho(x, \tau)\,e^{-\lambda\tau}\,d\tau, \tag{17}$$

we obtain from (13), after taking (15) into account,

$$\lambda\rho_\lambda - \delta(x - x_0) = \frac{d}{dx}\left[\Gamma(x)\left(\rho_\lambda - \frac{\rho_\lambda}{2x} + \frac{d\rho_\lambda}{dx}\right)\right]. \tag{18}$$

The transition from $\rho_\lambda(x)$ to $\rho(x, \tau)$ is accomplished with the aid of the formula for the inverse transformation

$$\rho(x, \tau) = \frac{1}{2\pi i}\int_{\sigma - i\infty}^{\sigma + i\infty} \rho_\lambda(x)\,e^{\lambda\tau}\,d\lambda. \tag{19}$$

Here the integration over $d\lambda$ is taken in the complex λ plane along a line lying in the right-hand half-plane to the right of the singularities of $\rho_\lambda(x)$.

Let us put Eq. (18) into a more convenient form. For this we first eliminate the first derivative by the substitution

$$\rho_\lambda = x^{1/4} e^{-x/2}\,\Gamma^{-1/2} g_\lambda. \tag{20}$$

We then replace the function g_λ and the independent variable[2] x

$$F_\lambda = \Gamma^{-1/4} g_\lambda, \tag{21}$$

$$\xi = \int_0^x [\Gamma(x)]^{-1/2}\,dx, \quad \xi_0 = \int_0^{x_0} [\Gamma(x)]^{-1/2}\,dx. \tag{22}$$

As a result we obtain

$$\frac{d^2 F_\lambda}{d\xi^2} - (\lambda - u(\xi))\,F_\lambda \tag{23}$$
$$= -x_0^{-1/4} e^{x_0/2}\,[\Gamma(x_0)]^{-1/4}\,\delta(\xi - \xi_0),$$

where

$$u(\xi) = \frac{d^2\left(x^{1/4} e^{-x/2}[\Gamma(x)]^{1/4}\right)/d\xi^2}{x^{1/4} e^{-x/2}[\Gamma(x)]^{1/4}}. \tag{24}$$

Equation (23) with $\Gamma(x)$ as given by formula (11) does not have a solution in finite analytic form (except for the case $\lambda = 0$). However, one can obtain, with sufficient degree of accuracy, an approximate solution by making use of the specific form of the function $u(\xi)$. A graph of the function $u(\xi)$ for the value $m/M = 1/3600$ (deuterium plasma) is given in Fig. 1.

In the range of values of ξ from 5 to 30 (while x varies from 4 to 25) the function $u(\xi)$ maintains an approximately constant value equal to 0.14, as can be seen from the graph. We denote this value by a. (The remaining values of the function $u(\xi)$ in the region $\xi > 30$, which are of no practical interest, increase monotonically and go through a maximum (on the order of M/m) at ξ, and then asymptotically approach zero for still greater values of ξ.

For an an approximate solution of (23), we shall choose such an expression for $\Gamma(x)$ [which we shall denote by $\gamma(x)$], as to make $u(\zeta)$ a constant equal to a, i.e., in accordance with (24)

$$\frac{d^2\left(x^{1/4}e^{-x/2}[\gamma(x)]^{1/4}\right)/d\zeta^2}{x^{1/4}e^{-x/2}[\gamma(x)]^{1/4}} = a. \quad (25)$$

With such a value of γ, Eq. (23) can be solved in finite analytic form. We have denoted by ζ the quantity ξ corresponding to the new function $\gamma(x)$. Such an approximation allows us to obtain, with a sufficient degree of accuracy, the distribution function in the range of values of ξ from 5 to 30. On solving Eq. (25) for the function $x^{1/4}e^{-x/2}[\gamma(x)]^{1/4}$ we find

$$x^{1/4}e^{-x/2}[\gamma(x)]^{1/4} = C\exp(-\sqrt{a}\zeta) + C_1\exp(\sqrt{a}\zeta), \quad (26)$$

where C and C_1 are constants, which for the time being are arbitrary and will be determined later. The dependence of ζ on x is now given instead of (22) by the formula:

$$\zeta = \int_0^x [\gamma(x)]^{-1/2}dx, \quad \zeta_0 = \int_0^{x_0} [\gamma(x)]^{-1/2}dx. \quad (27)$$

Equation (23), without its right-hand side, has for $\lambda = 0$ a solution which falls off as $\xi \to \infty$, or, what is the same, for $x \to \infty$. This solution has, according to (24), the form

$$F_\lambda = \mathrm{const}\, x^{1/4}e^{-x/2}[\Gamma(x)]^{1/4}.$$

A solution of (23) (for $\lambda = 0$) which falls off at infinity, must have for $u(\xi) = a$ the form

$$F_\lambda = \mathrm{const}\, x^{1/4}e^{-x/2}[\gamma(x)]^{1/4}. \quad (28)$$

Comparing (28) with (26) we find that $C_1 = 0$, and thus

$$x^{1/4}e^{-x/2}[\gamma(x)]^{1/4} = C\exp(-\sqrt{a}\zeta). \quad (29)$$

The constant C is determined from the normalization condition and turns out to be equal to

$$C = (\pi a)^{1/4}. \quad (30)$$

Now we can find from (27), (29) and (30) the dependence of γ and ζ on x. We obtain

$$\zeta = -\frac{1}{2\sqrt{a}}\ln\left[1 - \Phi(\sqrt{x}) + 2\sqrt{\frac{x}{\pi}}e^{-x}\right]; \quad (31)$$

$$\gamma = \pi a\left[\frac{1-\Phi(\sqrt{x})}{\sqrt{x}e^{-x}} + \frac{2}{\sqrt{\pi}}\right]^2. \quad (32)$$

Equation (23) assumes the form

$$\frac{d^2F_\lambda}{d\zeta^2} + (a+\lambda)F_\lambda = x_0^{1/4}e^{x_0/2}[\gamma(x_0)]^{-1/4}\delta(\zeta-\zeta_0), \quad (33)$$

while ρ_λ and F_λ are related by the expression

$$\rho_\lambda = x^{1/4}e^{-x/2}[\gamma(x)]^{-1/4}F_\lambda. \quad (34)$$

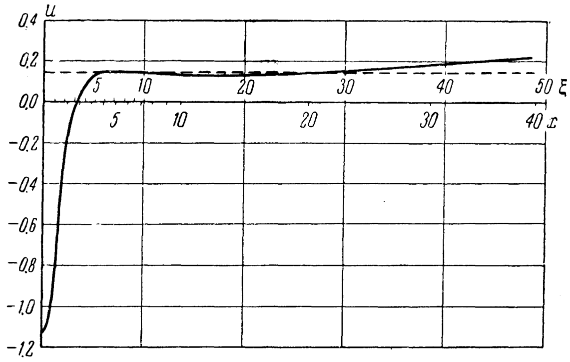

FIG. 1

Let us determine the boundary conditions for Eq. (33). Integrating (13) with respect to x between the limits 0 and ∞, and noting that it follows from (14) that $\frac{\partial}{\partial\tau}\int_0^\infty \rho dx = 0$, we obtain

$$\left\{\gamma(x)\left(\rho - \frac{\rho}{2x} + \frac{\partial\rho}{\partial x}\right)\right\}\Big|_0^\infty = 0. \quad (35)$$

As $x \to \infty$ both $\rho(x, \tau)$ and $\partial\rho/\partial x$ should tend to zero, and since at the same time $\gamma(x)$ remains finite, it must follow that

$$(\rho - \rho/2x + \partial\rho/\partial x)_{x\to\infty} = 0, \quad (36)$$

$$\{\gamma(x)(\rho - \rho/2x + \partial\rho/\partial x)\}_{x\to 0} = 0. \quad (37)$$

Upon applying the Laplace transformation to the last two equations, we shall have the same conditions also for $\rho_\lambda(x)$; obviously $(\rho_\lambda)_{x\to\infty} = 0$. From the condition $(\rho_\lambda)_{x\to\infty} = 0$, from the specific forms of the functions $\zeta(x)$ and $\gamma(x)$ given by (31) and (32) for $x \to \infty$, and from the relation (34), we obtain the boundary condition for F_λ as $x \to \infty$ ($\zeta \to \infty$)

$$(F_\lambda)_{\substack{x\to\infty\\ \lambda>0}} = 0. \quad (38)$$

Similarly we obtain the boundary condition

$$\left(\frac{1}{F_\lambda}\frac{dF_\lambda}{d\zeta}\right)_{\zeta\to 0} = -\sqrt{a}. \quad (39)$$

In addition to satisfying conditions (38) and (39), the function $F_\lambda(\zeta)$ must be continuous in the whole region from zero to ∞, while at the point $\zeta = \zeta_0$ the derivative $dF_\lambda/d\zeta$ must have a discontinuity whose magnitude is found by integrating (23) over a small interval which contains the point $\xi = \xi_0$:

$$\left.\frac{dF_\lambda}{d\zeta}\right|_{\zeta_0-0}^{\zeta_0+0} = x_0^{1/4} e^{x_0/2} [\gamma(x_0)]^{-1/4} = (\pi a)^{1/4} e^{-\sqrt{a}\zeta_0}. \quad (40)$$

Conditions (38), (39), (40) and the requirement of continuity completely determine $F_\lambda(x)$ for $x > 0$. The function F_λ which satisfies all the above requirements has the form:

$$F_\lambda = \begin{cases} \dfrac{\exp\{\sqrt{a}\zeta_0 - \sqrt{a+\lambda}\zeta_0\}}{2(\pi a)^{1/4}\sqrt{a+\lambda}(\sqrt{a+\lambda}-\sqrt{a})} [(\sqrt{a+\lambda}-\sqrt{a})\exp(\sqrt{a+\lambda}\zeta) \\ \qquad + (\sqrt{a+\lambda}+\sqrt{a})\exp(-\sqrt{a+\lambda}\zeta)], \quad \zeta < \zeta_0 \\ \dfrac{\exp\{\sqrt{a}\zeta_0 - \sqrt{a+\lambda}\zeta\}}{2(\pi a)^{1/4}\sqrt{a+\lambda}(\sqrt{a+\lambda}-\sqrt{a})} [(\sqrt{a+\lambda}-\sqrt{a})\exp(\sqrt{a+\lambda}\zeta_0) \\ \qquad + (\sqrt{a+\lambda}+\sqrt{a})\exp(-\sqrt{a+\lambda}\zeta_0)], \quad \zeta > \zeta_0. \end{cases} \quad (41)$$

The function $F_\lambda(\xi)$ is defined for complex values of λ, including also negative values of λ, by means of analytic continuation with respect to λ, starting with positive values of λ.

According to (17) and (34) the desired distribution $\rho(x, \tau)$ is equal to

$$\rho = x^{1/4} e^{-x/2} \gamma^{-1/4} \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} F_\lambda e^{\lambda\tau} d\lambda$$
$$= \frac{(\pi a)^{1/4} e^{-\sqrt{a}\zeta}}{\gamma^{1/2}} \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} F_\lambda e^{\lambda\tau} d\lambda. \quad (42)$$

The path of integration in the complex λ plane may be deformed in the following manner. The integrand $F_\lambda e^{\lambda\tau}$ has singularities at $\lambda = 0$ and $\lambda = -a$; in particular, the point $\lambda = 0$ is a pole with a residue equal to $2(a/\pi)^{1/4} e^{-\sqrt{a}\,\xi}$, while the point $\lambda = -a$ is a branch point of the first order. After making a cut along the negative part of the real axis from $\lambda = -a$ to $\lambda = -\infty$, we take the path of integration along the edges of the cut in the form of a loop surrounding the point $\lambda = -a$. Then the integral in formula (42) will reduce to an integral over this loop and an integral over a circle surrounding the point $\lambda = 0$, as shown in Fig. 2. Both functions F_λ in formula (41) give, on integration, the same result for $\rho(x, \tau)$. On carrying out the integration we obtain:

$$\rho = \frac{2\sqrt{a}}{\gamma^{1/2}} \exp(-2\sqrt{a}\zeta) + \frac{e^{-a\tau}}{2\pi\gamma^{1/2}} \exp\{-\sqrt{a}(\zeta-\zeta_0)\}$$
$$\times \int_0^\infty \frac{d\mu e^{-\mu\tau}}{\sqrt{\mu}} \left[\cos\sqrt{\mu}(\zeta-\zeta_0) + \frac{\mu-a}{\mu+a}\cos\sqrt{\mu}(\zeta_0+\zeta) - \frac{2\sqrt{\mu a}}{\mu+a}\sin\sqrt{\mu}(\zeta_0+\zeta)\right]. \quad (43)$$

The first term in (43) represents the integral over the circle, while the second term represents the integral over the loop. On carrying out the necessary integrations in the second term, we obtain the final expression for the distribution $\rho(x, \tau)$:

$$\rho = \gamma^{-1/2}\sqrt{a}\exp\{-2\sqrt{a}\zeta\}\left[1 + \Phi\left(\sqrt{a\tau} + \frac{\zeta+\zeta_0}{2\sqrt{\tau}}\right)\right]$$
$$+ \gamma^{-1/2}\exp\{-\sqrt{a}(\zeta-\zeta_0)\} \frac{e^{-(\zeta-\zeta_0)^2/4\tau} + e^{-(\zeta+\zeta_0)^2/4\tau}}{2\sqrt{\pi\tau}} \cdot e^{-a\tau}, \quad (44)$$

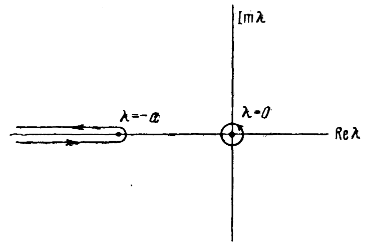

FIG. 2

where ξ and ξ_0 are determined by formula (27), while $\gamma(x)$ is determined by formula (32). As $\tau \to 0$ we obtain from (44) $\rho = \delta(x - x_0)$, while for $\tau \to \infty$, $\rho = 2e^{-x}\sqrt{x/\pi}$, as it should be.

3. RESULTS

Expression (44) obtained above for the distribution density with respect to energy $\rho(x, \tau)$ will describe, to a sufficient degree of approximation, the energy distribution of ions injected into plasma in the range of values of ξ and ξ_0 from 5 to 30. The range of the corresponding values of ζ and ζ_0 is approximately the same, while x and x_0 vary between 4 and 25. In order to describe the manner in which the distribution function approaches the stationary Maxwellian distribution

$$\rho_M = 2\gamma^{-1/2}\sqrt{a}\,e^{-2\sqrt{a}\zeta}$$

it is convenient to introduce the ratio ρ/ρ_M which is equal to

$$\frac{\rho}{\rho_M} = \frac{1}{2}\left[1 + \Phi\left(\sqrt{a\tau} - \frac{\zeta + \zeta_0}{2\sqrt{\tau}}\right)\right]$$

$$+ \frac{\exp\{-a\tau + \sqrt{a}(\zeta + \zeta_0)\}}{4\sqrt{\pi a \tau}}\left[e^{-(\zeta-\zeta_0)^2/4\tau} + e^{-(\zeta+\zeta_0)^2/4\tau}\right]. \quad (45)$$

The variables ζ and ζ_0 occur in ρ/ρ_M in a symmetric way.

For given values of ζ and ζ_0 ($\zeta \neq \zeta_0$) lying within the region in which the solution is applicable the quantity ρ/ρ_M regarded as a function of time increases from zero (at $\tau = 0$) to a maximum value greater than unity (for $\tau = \tau^*$) and then decreases asymptotically approaching unity from above. For small values of the time

$$\tau \ll (\zeta + \zeta_0)/2\sqrt{a} < \zeta\zeta_0$$

ρ/ρ_M has the form:

$$\frac{\rho}{\rho_M} = \frac{1}{4\sqrt{\pi a \tau}} \exp\left\{-\left(\frac{\zeta + \zeta_0}{2\sqrt{\tau}} - \sqrt{a\tau}\right)^2\right\}\left(1 + e^{\zeta\zeta_0/\tau}\right). \quad (46)$$

For large values of the time

$$\tau \gg \zeta\zeta_0 > (\zeta + \zeta_0)/2\sqrt{a}$$

we have:

$$\frac{\rho}{\rho_M} = 1 + \frac{1}{4\sqrt{\pi a}\,\tau^{3/2}}$$
$$\times \exp\left\{-\left(\sqrt{a\tau} - \frac{\zeta + \zeta_0}{2\sqrt{\tau}}\right)^2\right\}\left(\zeta - \frac{1}{\sqrt{a}}\right)\left(\zeta_0 - \frac{1}{\sqrt{a}}\right). \quad (47)$$

The time $\tau = \tau^*$ at which ρ/ρ_M has a maximum is found from the condition $\partial(\rho/\rho_M)/\partial\tau = 0$ which, in accordance with (45), assumes the form:

$$e^{\zeta\zeta_0/\tau^*} = \frac{[\sqrt{a\tau^*} + (\zeta + \zeta_0)/2\sqrt{\tau^*}]^2 - 1/2}{a\tau^* - (\zeta - \zeta_0)^2/4\tau^* + 1/2}. \quad (48)$$

Assuming that the quantity $\zeta\zeta_0/\tau^*$ is large compared with unity (this condition is fulfilled as may be seen from the result (49) obtained above for our range of ζ and ζ_0) we obtain τ^* by equating to zero the denominator of the right-hand side of (48)

$$\tau^* = \frac{\sqrt{1 + 4a(\zeta - \zeta_0)^2} - 1}{4a}. \quad (49)$$

In conclusion, I express my gratitude to A. B. Migdal for suggesting the problem.

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