PHENOMENOLOGICAL THEORY OF KINETIC PROCESSES IN FERROMAGNETIC DIELECTRICS

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The relaxation time resulting from the interaction of spin waves with each other, in ferromagnetic dielectrics, is calculated.

THE present paper is concerned with those relaxation processes, in ferromagnets, that result from the interaction of spin waves with each other. In contrast to the work of Akhiezer,[1] the treatment is carried through without any assumption about the nominal magnetization of the ferromagnetic in its ground state.

Relaxation processes in a ferromagnetic are not limited to interactions within the spin system; spin waves also interact with lattice vibrations. However, as will be shown below, there are a number of cases in which interactions between spin waves play the fundamental role in the establishment of equilibrium.

1. THE ENERGY SPECTRUM OF A FERROMAGNET

As was shown by Herring and Kittel,[2] the energy spectrum of a ferromagnet in the neighborhood of the ground state can be obtained without assuming a model for the spin structure of the ground state. Instead, purely phenomenological assumptions are made in regard to the existence of exchange interaction with a positive exchange integral and, consequently, the presence of a spontaneous magnetic moment at $T = 0$.

In order to study kinetic processes in ferromagnetics, it is necessary to know not only the energy spectrum, which determines all the thermodynamic quantities[3] but also the wave functions of the spin waves; it is by means of these that the probabilities of transition between different states of the system are calculated. Therefore we shall here use a systematic quantum-mechanical method to find the energy levels related to the motion of the magnetic moment in a ferromagnetic. We shall take account both of the large exchange interaction and of the small relativistic contribution (the anisotropy energy and the magnetic interaction). At very low temperatures, as was shown in reference 3, the

magnetic interaction plays an essential role in the spectrum.

We shall start with the Hamiltonian

$$\mathscr{H} = \int dv \left\{ - \mathbf{H}_0 \mathbf{M} + \frac{1}{2} \beta M_\perp^2 + \frac{1}{2} \alpha_{ik} \frac{\partial M_l}{\partial x_i} \frac{\partial M_l}{\partial x_k} \right\} + \iint \frac{dv\,dv'}{2R^5} [\mathbf{M}(\mathbf{r}) \mathbf{M}(\mathbf{r}') R^2 - 3(\mathbf{R}\mathbf{M}(\mathbf{r}))(\mathbf{R}\mathbf{M}(\mathbf{r}'))],$$

(1)

where the first term in curly brackets describes the interaction of the magnetic moment of unit volume, $\mathbf{M} = \mathbf{M}(\mathbf{r})$,* with a constant magnetic field $\mathbf{H}_0$; $\mathbf{H}_0$ coincides with the external magnetic field if the demagnetizing factor is negligible. The second term is the anisotropy energy; $M_\perp$ is the projection of the magnetic moment on a plane perpendicular to the axis of easiest magnetization, and β is the magnetic anisotropy constant at $T = 0$.† The third term describes an isotropic exchange interaction; the quantities α_{ik} are the constants of this interaction, and in order of magnitude they are equal to $\Theta_c a^2/\mu M_0$, where a is the lattice constant and μ the Bohr magneton.[5]

The second term in (1) (the double integral) describes the magnetic interaction; $\mathbf{R} = \mathbf{r} - \mathbf{r}'$.

The integration extends over all space (the ferromagnet is assumed to be infinite).

Let the magnetic field $\mathbf{H}_0$ be directed at angle φ to the axis of easiest magnetization. Then the magnetic moment in the ground state, $\mathbf{M}_0$, makes some angle ψ with the axis of easiest magnetization. The angle ψ is determined by the condition

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*$\mathbf{M}(\mathbf{r})$ is the magnetic moment density. The total magnetic moment of the ferromagnetic is $\mathfrak{M} = \int \mathbf{M}(\mathbf{r}) dv$.

†The form of the anisotropy energy that we have chosen is correct for small angles of inclination of the magnetic moment to the axis of easiest magnetization.[4] In the general case the anisotropy energy is a complicated function of M_x, M_y, and M_z. As will be clear from what follows, spin waves can always be introduced in the manner described here.

that the magnetic energy in the ground state be a minimum:

$$\sin(\varphi - \psi) = (\beta M_0 / 2H_0)\sin 2\psi. \qquad (2)$$

We note that in the ground state (**M** = const.) the magnetic interaction energy is equal to zero.*

We choose a system of coordinates in the following manner: with the z axis along the direction of the magnetic moment $\mathbf{M_0}$, and with the x axis in the plane of the vectors $\mathbf{M_0}$ and $\mathbf{H_0}$. Then the anisotropy energy $\mathcal{3C}_A$ has the form

$$\mathcal{H}_A = \int dv\, \{M_x \cos\psi + M_z \sin\psi)^2 + M_y^2\}. \qquad (3)$$

As regards the other terms in the Hamiltonian (1), their form, by virtue of invariance, is independent of the choice of the coordinate system.

In quantum theory, the projections of the magnetic moment must be treated as operators with the commutation rules

$$M^-(\mathbf{r})M^+(\mathbf{r}') - M^+(\mathbf{r}')M^-(\mathbf{r}) = 2g\hbar M_z\delta(\mathbf{r} - \mathbf{r}'), \qquad (4)$$

where

$$M^{\pm} = M_x \pm iM_y, \qquad (5)$$

and where g is the gyromagnetic ratio (g > 0). The commutation rule for the components of $\mathbf{M(r)}$ follows from the commutation rule for the components of the total moment,

$$\mathfrak{M}_x\mathfrak{M}_y - \mathfrak{M}_y\mathfrak{M}_x = -igh\mathfrak{M}_z.$$

Following Holstein and Primakoff,[6] we introduce operators a(**r**) and a*(**r**) that satisfy the commutation rule

$$a(\mathbf{r})a^*(\mathbf{r}') - a^*(\mathbf{r}')a(\mathbf{r}) = \delta(\mathbf{r} - \mathbf{r}'), \qquad (6)$$

and we express in terms of them the operators M^+, M^-, and M_z:

$$M^+ = (2g\hbar M_0)^{1/2}a^*(1 - g\hbar a^* a/2M_0)^{1/2},$$
$$M^- = (2g\hbar M_0)^{1/2}(1 - g\hbar a^* a/2M_0)^{1/2}a, \qquad (7)$$
$$M_z = M_0 - g\hbar a^* a.$$

In the neighborhood of the ground state, $M_z \approx M_0$, and M_x and M_y are much smaller than M_0. We may therefore treat the operators a and a* as small and expand the integrand in the Hamiltonian (1) in powers of a and a*. For finding the spectrum, we may limit ourselves to terms of the second order in a and a* (the first-order terms

*The Hamiltonian (1) takes no account of the dependence of the magnitude of the magnetic moment in the ground state upon the magnetic field. This is legitimate up to fields of order $\varepsilon_a/\mu \sim 10^8$, where ε_a is the energy of an electron in the atom ($\varepsilon_a \sim 10^{-12}$ erg).

drop out by virtue of the choice of the ground state).

After simple transformations we get $\mathcal{3C} = \mathcal{3C}_0 + \mathcal{3C}'$, where $\mathcal{3C}'$ contains the terms of third and higher orders in a and a*, and where

$$\mathcal{H}_0 = (a^*\hat{A}a) + \tfrac{1}{2}(a\hat{B}a) + \tfrac{1}{2}(a^*\hat{B}^*a^*). \qquad (8)$$

In the expression (8), the parentheses denote integration over the volume of the ferromagnetic, and the operators $\hat{A}$ and $\hat{B}$ are defined by the equations.

$$\hat{A}a(\mathbf{r}) = \Big[g\hbar H_0\cos(\varphi - \psi) + \tfrac{1}{2}\beta g\hbar M_0(\cos 2\psi + \cos^2\psi)\Big]a(\mathbf{r})$$
$$- g\hbar M_0\alpha_{ik}\frac{\partial^2 a(\mathbf{r})}{\partial x_i \partial x_k} - g\hbar M_0\int \frac{dv'}{R^5}\Big(\frac{3}{2}R^+R^- - R^2\Big)a(\mathbf{r}').$$
$$ \qquad (9) $$

$$\hat{B}a(\mathbf{r}) = \frac{1}{2}\beta g\hbar M_0(\cos^2\psi - 1)a(\mathbf{r})$$
$$- \frac{3}{2}g\hbar M_0\int\frac{dv'}{R^5}(R^+)^2 a(\mathbf{r}').$$

$$R^{\pm} = R_x \pm iR_y.$$

The finding of the spectrum is related, as is well known, to the diagonalization of the Hamiltonian (8). For this purpose it is convenient to go over to the equations of motion of the operators a:[7,8]

$$\dot{a} = (i/\hbar)(\mathcal{H}_0 a - a\mathcal{H}_0). \qquad (10)$$

By use of (8) and (6), we get from (10)

$$\dot{a} = -(i/\hbar)(\hat{A}a + \hat{B}^*a^*). \qquad (11)$$

Since Eqs. (11) are linear, and since the operators $\hat{A}$ and $\hat{B}$ have difference-dependent kernels, we may seek a solution of (11) in the form of a Fourier series

$$a(\mathbf{r}, t) = \sum_\lambda \{u_\lambda(\mathbf{r})a_\lambda(t) + v_\lambda^*(\mathbf{r})a_\lambda^*(t)\}, \qquad (12)$$

where

$$u_\lambda(\mathbf{r}) = u_\lambda e^{i\mathbf{k}_\lambda\mathbf{r}}; \quad v_\lambda(\mathbf{r}) = v_\lambda e^{i\mathbf{k}_\lambda\mathbf{r}},$$

$$a_\lambda(t) = a_\lambda\exp\{-i\varepsilon_\lambda t/\hbar\}, \qquad (13)$$

and the operators a_λ and a_λ^* satisfy the commutation rules

$$a_\lambda a_\mu^* - a_\mu^* a_\lambda = \delta_{\lambda\mu}. \qquad (14)$$

On substituting (12) in (11) and comparing coefficients of a_λ and a_λ^*, we get

$$\varepsilon_\lambda u_\lambda = A_\lambda u_\lambda + B_\lambda^* v_\lambda, \quad -\varepsilon_\lambda v_\lambda = A_\lambda v_\lambda + B_\lambda u_\lambda. \qquad (15)$$

Here

$$A_\lambda = g\hbar H_0\cos(\varphi - \psi) + \tfrac{1}{2}\beta g\hbar M_0(\cos 2\psi + \cos^2\psi)$$
$$+ g\hbar M_0\alpha_{ij}k_{\lambda i}k_{\lambda j} + 2\pi g\hbar M_0(k_{\lambda x}^2 + k_{\lambda y}^2)/k_\lambda^2; \qquad (16)$$

$$B_\lambda = -\tfrac{1}{2}\beta g\hbar M_0\sin^2\psi + 2\pi g\hbar M_0(k_{\lambda x} + ik_{\lambda y})^2/k_\lambda^2.$$

From Eqs. (15) we find

$$\varepsilon_\lambda = \sqrt{A_\lambda^2 - |B_\lambda|^2}. \qquad (17)$$

The homogeneous equations (15) determine the values of u_λ and v_λ except for an arbitrary factor, whose value may be obtained from the normalization condition

$$|u_\lambda|^2 - |v_\lambda|^2 = 1/V, \qquad (18)$$

where V is the volume of the ferromagnet. Condition (18) guarantees conformity to the commutation rules (14) and (6); it can be obtained by using the orthogonality relation for the solutions of the system (15):

$$\int \{u_\lambda(\mathbf{r}) u_\mu^*(\mathbf{r}) - v_\lambda(\mathbf{r}) v_\mu^*(\mathbf{r})\} dv = 0, \quad (\lambda \neq \mu),$$
$$\int \{u_\lambda(\mathbf{r}) u_\mu(\mathbf{r}) - u_\lambda(\mathbf{r}) v_\mu(\mathbf{r})\} dv = 0. \qquad (19)$$

From (15) and (18) we have, except for a phase factor,

$$u_\lambda = \frac{1}{\sqrt{2V}} \left(\frac{A_\lambda + \varepsilon_\lambda}{\varepsilon_\lambda}\right)^{1/2}, \quad v_\lambda = \frac{1}{\sqrt{2V}} \frac{B_\lambda}{|B_\lambda|} \left(\frac{A_\lambda - \varepsilon_\lambda}{\varepsilon_\lambda}\right)^{1/2}. \qquad (20)$$

By use of the equations of motion (11), the Hamiltonian $\mathcal{H}_0$ can be put into the following form:

$$\mathcal{H}_0 = (i\hbar/2)\{(a^*, \dot{a}) - (\dot{a}^*, a)\}. \qquad (21)$$

On substituting a and a* from the Fourier series (12), we get

$$\mathcal{H}_0 = \sum_\lambda \varepsilon_\lambda (a_\lambda^* a_\lambda + \tfrac{1}{2}) + G; \quad G = -\tfrac{1}{2} \sum_\lambda A_\lambda; \qquad (22)$$

G is a constant, which may always be omitted in the Hamiltonian.

The quantities $a_\lambda^* a_\lambda$, as is evident from (14) and (22), have the meaning of occupancy numbers (n_λ) for spin waves. Thus

$$\mathcal{H}_0 = \sum_\lambda \varepsilon_\lambda (n_\lambda + \tfrac{1}{2}), \qquad (23)$$

where ε_λ is determined by formulas (17) and (16).

We consider the limiting cases:

(1) Magnetic field H_0 parallel to the axis of easiest magnetization. In this case $\varphi = \psi = 0$;

$$A_\lambda = g\hbar(H_0 + \beta M_0) + g\hbar M_0 \alpha_{ij} k_{\lambda i} k_{\lambda j} + 2\pi g\hbar M_0 \sin^2 \theta_\lambda,$$

$$B_\lambda = 2\pi g\hbar M_0 \sin^2 \theta_\lambda e^{2i\varphi_\lambda} \qquad (24)$$

(θ_λ and φ_λ are the polar angles of the vector $\mathbf{k}_\lambda$): If we treat α_{ij} as an isotropic tensor, $\alpha_{ij} = \alpha \delta_{ij}$, we arrive at the well known spectrum.[2] Here the quantity βM_0 plays the role of a magnetic anisotropy field. However, in the general case (H_0 not parallel to the axis of easiest magnetization),

the magnetic anisotropy density cannot be expressed in the form $-MH_A$, since the constant β enters in a different way in the expression for A_λ and B [cf. (16)].

(2) For large k_λ ($\alpha_{ij} k_{\lambda i} k_{\lambda j} \gg 1$),* $A_\lambda \gg |B_\lambda|$, and

$$\varepsilon_\lambda \approx A_\lambda \approx g\hbar H_0 + g\hbar M_0 \alpha_{ij} k_{\lambda i} k_{\lambda j}. \qquad (25)$$

(We recall that $\beta \sim 1$.)

(3) For small k_λ ($\alpha_{ij} k_{\lambda i} k_{\lambda j} \ll 1$), A_λ and B_λ are of the same order of magnitude. In this case the expressions for A_λ and B_λ do not simplify.

2. INTERACTION OF SPIN WAVES WITH EACH OTHER

In the processes of interaction of spin waves with each other, a fundamental role is played by the terms of third and fourth order with respect to the operators a and a*. The third-order terms arise from expansion of the anisotropic relativistic terms in the Hamiltonian (1) (anisotropy energy and magnetic interaction energy):

$$\mathcal{H}'_a = -\frac{1}{4}(2g\hbar M_0)^{1/2} \beta g\hbar \sin 2\psi \int (a^* + a) a^* a \, dv$$
$$- \frac{3}{2}(2g\hbar M_0)^{1/2} g\hbar \iint \frac{R_z R^+}{R^5} a^*(\mathbf{r}) a(\mathbf{r}) a(\mathbf{r}') \, dv \, dv' + \text{c.c.} \qquad (26)$$

(c.c. denotes the complex conjugate terms).

The fourth-order terms arise from expansion of the exchange interaction energy

$$\mathcal{H}'_{\text{exch}} = \frac{(g\hbar)^2 \alpha}{4} \int \{2 [\nabla (a^* a)]^2 - \nabla a^* \nabla (a^* a a)$$
$$- \nabla (a^* a^* a) \nabla a\} dv. \qquad (26')$$

(a) We consider first the terms of third order. Henceforth we shall suppose that the magnetic field is applied in the direction of the axis of easiest magnetization.† Therefore $\psi = 0$, and

$$\mathcal{H}'_a = -\frac{3}{2}(2g\hbar M_0)^{1/2} g\hbar \iint \frac{R_z R^+}{R^5} a^*(\mathbf{r}) a(\mathbf{r}) a(\mathbf{r}') \, dv \, dv'$$
$$+ \text{c.c.} \qquad (27)$$

On substituting in (27) the expansion (12) of the operators a and a*, we get

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*This corresponds to a temperature $T \gg 2\pi\mu M_0 \sim 1°K$.

†In the general case ($\psi \neq 0$), the terms related to the anisotropic energy do not change the magnitude of the relaxation time to any essential degree, since the structure of the first term in (26) is similar to that of the second and since the coefficient $\beta \sim 1$.

$$\mathcal{H}'_a = \sum_{\lambda\mu\nu} \Phi_{\lambda\mu\nu} a_\lambda a_\mu a_\nu^* + \text{c.c.}, \qquad (28)$$

where

$$\Phi_{\lambda\mu\nu} = -2\pi g\hbar \,(2g\hbar M_0)^{1/2} V \left\{ \frac{k_\nu^z}{k_\nu^2} v_\lambda u_\mu \,(v_\nu^* k_\nu^+ + u_\nu^* k_\nu^-) \right.$$
$$\left. + \frac{k_\lambda^z}{k_\lambda^2}\,(u_\lambda k_\lambda^+ + v_\lambda k_\lambda^-)(u_\mu u_\nu^* + v_\mu v_\nu^*) \right\}, \qquad (29)$$

and where the sum extends over all wave vectors k_λ, k_μ, and k_ν satisfying the law of conservation of quasi-momentum

$$k_\lambda + k_\mu = k_\nu. \qquad (30)$$

We do not consider transfer processes, since they play no role in the problems being solved here.

We may consider the Hamiltonian (28) as a perturbation that causes transitions between stationary states of the system of spin waves.

The probability of transition (per second) is, as is well known,

$$W_{if} = (2\pi/\hbar)\,|\mathcal{H}'^{i}_{af}|^2\,\delta\,(E_i - E_f),$$

where $\mathcal{H}'^{i}_{af}$ is the matrix element for a transition from the initial state (i) to the final state (f), and $E_i(E_f)$ is the energy of the initial (final) state.

According to the commutation rule (14), the nonvanishing matrix elements of the operators a_λ and a_λ^* are

$$(a_\lambda)^{n_\lambda}_{n_\lambda+1} = \sqrt{n_\lambda + 1}\; e^{-i\varepsilon_\lambda t/\hbar}.$$
$$(a_\lambda^*)^{n_\lambda}_{n_\lambda-1} = \sqrt{n_\lambda}\; e^{i\varepsilon_\lambda t/\hbar}. \qquad (31)$$

Therefore, according to (28), the nonvanishing matrix elements of the operator $\mathcal{H}'$ are the following:

$$(\mathcal{H}')^{n_\lambda,\,n_\mu,\,n_\nu}_{n_\lambda-1,\,n_\mu-1,\,n_\nu+1}\,, \qquad (\mathcal{H}')^{n_\lambda,\,n_\mu,\,n_\nu}_{n_\lambda+1,\,n_\mu-1,\,n_\nu+1}\,,$$

$$(\mathcal{H}')^{n_\lambda,\,n_\mu,\,n_\nu}_{n_\lambda-1,\,n_\mu+1,\,n_\nu+1}$$

together with the matrix elements of the opposite transitions.

The corresponding transition probabilities have the form:

$$W^{n_\lambda,\,n_\mu,\,n_\nu}_{n_\lambda-1,\,n_\mu-1,\,n_\nu+1} = \frac{2\pi}{\hbar} A_{\lambda\mu\nu} n_\lambda n_\mu\,(n_\nu+1)\,\delta\,(\varepsilon_\lambda + \varepsilon_\mu - \varepsilon_\nu);$$

$$W^{n_\lambda,\,n_\mu,\,n_\nu}_{n_\lambda+1,\,n_\mu-1,\,n_\nu+1} = \frac{2\pi}{\hbar} A_{\lambda\nu\mu}\,(n_\lambda+1)\,n_\mu\,(n_\nu+1)$$
$$\times\,\delta\,(\varepsilon_\lambda - \varepsilon_\mu + \varepsilon_\nu); \qquad (32)$$

$$W^{n_\lambda,\,n_\mu,\,n_\nu}_{n_\lambda-1,\,n_\mu+1,\,n_\nu+1} = \frac{2\pi}{\hbar} A_{\nu\mu\lambda} n_\lambda\,(n_\mu+1)\,(n_\nu+1)$$
$$\times\,\delta\,(\varepsilon_\lambda - \varepsilon_\mu - \varepsilon_\nu).$$

Here

$$A_{\lambda\mu\nu} = |\,\Phi_{\lambda\mu\nu} + \Phi_{\mu\lambda\nu}\,|^2. \qquad (33)$$

Using the expressions obtained for the transition probabilities, we write the collision integral for n_ν:

$$(\dot{n}_\nu)^a_{\text{coll}} = \frac{2\pi}{\hbar} \sum_{\lambda\mu} \{ A_{\lambda\mu\nu}\,[n_\lambda n_\mu\,(n_\nu+1)$$
$$- (n_\lambda+1)\,(n_\mu+1)\,n_\nu]\,\delta\,(\varepsilon_\lambda + \varepsilon_\mu - \varepsilon_\nu) \qquad (34)$$
$$+ A_{\lambda\nu\mu}\,[(n_\lambda+1)\,n_\mu\,(n_\nu+1) - n_\lambda\,(n_\mu+1)\,n_\nu]\,\delta\,(\varepsilon_\lambda - \varepsilon_\mu + \varepsilon_\nu)$$
$$+ A_{\nu\mu\lambda}\,[n_\lambda\,(n_\mu+1)\,(n_\nu+1) - (n_\lambda+1)\,n_\mu n_\nu]\,\delta\,(\varepsilon_\lambda - \varepsilon_\mu - \varepsilon_\nu) \}.$$

We shall calculate the mean relaxation time of a gas of spin waves with the processes under consideration taken into account.

For small deviations of the system from the state of statistical equilibrium, the mean occupancy numbers n_ν can be expressed in the form

$$n_\nu = n_\nu^0 + \Delta n_\nu, \quad n_\nu^0 = 1/(e^{\varepsilon_\nu/T} - 1), \qquad (35)$$

where Δn_ν is a small correction to the Bose equilibrium distribution

Upon substituting (35) in (34), we shall limit ourselves to linear terms in the expansion with respect to the correction tó the equilibrium distribution function.

The coefficient of Δn_ν in the linearized collision integral, averaged over all the equilibrium states, may be regarded as the reciprocal of the mean relaxation time.[1] After simple transformations we get

$$\frac{1}{\tau_a} = \frac{2\pi}{\hbar} \sum_{\lambda\mu\nu} (n_\nu^0 + 2n_\lambda^0 n_\mu^0)\,\delta\,(\varepsilon_\lambda + \varepsilon_\mu - \varepsilon_\nu)\,A_{\lambda\mu\nu} \,\Big/ \sum_\nu n_\nu^0, \qquad (36)$$

where $A_{\lambda\mu\nu}$ is determined by formulas (33), (29), (20), (16), and (17).

We consider first the case of high temperatures, $T \gg 2\pi\mu M_0$. As was pointed out in Sec. 1, under these conditions the spectrum is considerably simplified. If we suppose that $\alpha_{ik} = \alpha\delta_{ik} = (\Theta_c a^2/\mu M_0) \times \delta_{ik}$, then

$$\varepsilon_\lambda = \mu H_{\text{eff}} + \Theta_c\,(ak_\lambda)^2. \qquad (37)$$

Here $H_{\text{eff}} = H_0 + \beta M_0$. In this approximation

$$u_\lambda = 1/\sqrt{V}; \quad v_\lambda = 0. \qquad (38)$$

From (33), (29), and (38), we get

$$A_{\lambda\mu\nu} = \frac{(2\pi\mu)^2\,2\mu M_0}{V} \left| \frac{k_\lambda^z k_\lambda^+}{k_\lambda^2} + \frac{k_\mu^z k_\mu^+}{k_\mu^2} \right|^2. \qquad (39)$$

In expression (36) we go over from summation to integration; after integrating over angles and

changing to dimensionless variables, we get

$$\frac{1}{\tau_a} = \frac{\pi}{5} \, g M_0 \, \frac{\mu^2}{a^3 \Theta_c} \left(\frac{T}{\Theta_c} \right)^{1/2} F(\eta), \qquad (40)$$

where

$$F(\eta) = \iint\limits_{(xy > \eta^2)} \left\{ \frac{1}{e^{x+y} - 1} + \frac{2}{(e^x - 1)(e^y - 1)} \right\}$$

$$\times \left\{ \frac{1}{3} + \frac{\eta^2}{xy} \right\} dx \, dy \Big/ \int_0^\infty \frac{x^2 \, dx}{e^{x+2\eta} - 1}, \qquad (41)$$

$$\eta = \mu H_{eff} / 2T. \qquad (42)$$

On calculating the asymptotic behavior of the function $F(\eta)$, we get

$$F(\eta) \approx \begin{cases} (2 / 3\zeta(3)) \ln^2 \eta, & (\eta \ll 1) \\ 2\sqrt{\pi\eta}. & (\eta \gg 1). \end{cases} \qquad (43)$$

Thus for $\Theta_c \gg T \gg 2\pi\mu M_0$:

$$\frac{1}{\tau_a} = \begin{cases} \dfrac{2\pi}{15\zeta(3)} \, g M_0 \, \dfrac{\mu^2}{a^3 \Theta_c} \left(\dfrac{T}{\Theta_c} \right)^{1/2} \ln^2 \dfrac{\mu H_{eff}}{2T}, & (\mu H_{eff} \ll T) \\ \dfrac{\sqrt{2}\,\pi^{3/2}}{5} \, g M_0 \, \dfrac{\mu^2}{a^3 \Theta_c} \left(\dfrac{\mu H_{eff}}{\Theta_c} \right)^{1/2}, & (\mu H_{eff} \gg T). \end{cases} \qquad (44)$$

It should be noticed that at strong magnetic fields, the relaxation time slowly decreases with increase of the field ($\tau_a \sim H^{-1/2}$) but is independent of temperature. The relaxation time at small fields agrees in order of magnitude with the result obtained by Akhiezer,[1] if we suppose that $M_0 \approx \mu/a^3$ and if we set μH_{eff} equal to μ^2/a^3.

In the case of low temperatures ($T \ll 2\pi\mu M_0$), the fundamental role in the expression (16) is played by the term related to the magnetic interaction of the spins.* Therefore for small k_λ, A_λ and $|B_\lambda| \gg \epsilon_\lambda$. On taking account of this, we get from (33), (29), and (20)

$$A_{\lambda\mu\nu} = \frac{\pi^2 \mu^3 M_0}{V \epsilon_\lambda \epsilon_\mu \epsilon_\nu} \left\{ \epsilon_\lambda \sqrt{\frac{A_\mu A_\nu}{A_\lambda}} \sin 2\theta_\lambda \cos(\varphi_\mu - \varphi_\nu) \right.$$

$$+ \epsilon_\mu \sqrt{\frac{A_\lambda A_\nu}{A_\mu}} \sin 2\theta_\mu \cos(\varphi_\lambda - \varphi_\nu)$$

$$\left. - \epsilon_\nu \sqrt{\frac{A_\lambda A_\mu}{A_\nu}} \sin 2\theta_\nu \cos(\varphi_\lambda - \varphi_\mu) \right\}^2. \qquad (45)$$

If in the dispersion law we limit ourselves to the lowest power of k_λ, then

$$\epsilon_\lambda = \sqrt{4\pi\mu M_0 \Theta_c} \, a k_\lambda \sin \theta_\lambda.$$

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*At low temperatures the whole treatment is carried out in the absence of a magnetic field. Furthermore it is assumed that $\beta \ll 2\pi$. This permits neglect of the anisotropy energy in (16).

Under these conditions, however, simultaneous fulfilment of the laws of conservation of energy and of momentum leads to a divergence of the integral in the expression for $1/\tau_a$ to the extent of a delta function. On the other hand, $A_{\lambda\mu\nu}$ approaches zero. Therefore in the calculation of the relaxation time, it is necessary to take account, in the spectrum, of further terms in the expansion with respect to k_λ ($\sim k_\lambda^3$).

After rather tedious calculations we get, except for a numerical factor:

$$1/\tau_a \sim g M_0 (T/\Theta_c)^{2/3} (T/\mu M_0)^3, \quad (T \ll 2\pi\mu M_0). \qquad (46)$$

Thus, according to (44) and (46),

$$\tau_a \sim \begin{cases} T^{-3/2} & (T \ll 2\pi\mu M_0) \\ T^{-1/2} \ln^{-2} T & (2\pi\mu M_0 \ll T \ll \Theta_c). \end{cases} \qquad (47)$$

(b) We now calculate the relaxation time with the exchange interaction taken into account.

From (26′) we have, for $T \gg 2\pi\mu M_0$ (for $T \ll 2\pi\mu M_0$, the exchange interaction is quite unimportant for kinetic processes),

$$\mathscr{H}'_{exch} = \sum_{\varkappa\lambda\mu\nu} \Phi_{\varkappa\lambda\mu\nu} a_\varkappa^* a_\lambda^* a_\mu a_\nu, \quad (k_\varkappa + k_\lambda = k_\mu + k_\nu), \qquad (48)$$

where

$$\Phi_{\varkappa\lambda\mu\nu} = (\mu^2 \alpha / 4V) [2 k_\lambda k_\mu - (k_\mu + k_\lambda)(k_\varkappa + k_\nu)]. \qquad (49)$$

Hence the transition probabilities, which correspond to nonvanishing matrix elements of $\mathscr{H}'_{exch}$, have the form

$$W^{n_\varkappa n_\lambda n_\mu n_\nu}_{n_\varkappa - 1 \, n_\lambda - 1 \, n_\mu + 1 \, n_\nu + 1} = \frac{2\pi}{\hbar} \{ A_{\varkappa\lambda\mu\nu} + A_{\varkappa\lambda\nu\mu} \} n_\varkappa n_\lambda (n_\mu + 1)$$

$$\times (n_\nu + 1) \, \delta(\epsilon_\varkappa + \epsilon_\lambda - \epsilon_\mu - \epsilon_\nu),$$

$$W^{n_\varkappa n_\lambda n_\mu n_\nu}_{n_\varkappa - 1 \, n_\lambda + 1 \, n_\mu - 1 \, n_\nu + 1} = \frac{2\pi}{\hbar} A_{\varkappa\nu\mu\lambda} n_\varkappa (n_\lambda + 1) n_\mu$$

$$\times (n_\nu + 1) \, \delta(\epsilon_\varkappa - \epsilon_\lambda + \epsilon_\mu - \epsilon_\nu),$$

$$W^{n_\varkappa n_\lambda n_\mu n_\nu}_{n_\varkappa + 1 \, n_\lambda - 1 \, n_\mu - 1 \, n_\nu + 1} = \frac{2\pi}{\hbar} A_{\nu\lambda\mu\varkappa} (n_\varkappa + 1) n_\lambda n_\mu \qquad (50)$$

$$\times (n_\nu + 1) \, \delta(\epsilon_\varkappa - \epsilon_\lambda - \epsilon_\mu + \epsilon_\nu).$$

Here

$$A_{\varkappa\lambda\mu\nu} = | \Phi_{\varkappa\lambda\mu\nu} + \Phi_{\lambda\varkappa\mu\nu} |^2. \qquad (51)$$

With the aid of the collision integral corresponding to these transition probabilities, we find the relaxation time in the same manner as before:

$$\frac{1}{\tau_{exch}} = \frac{2\pi}{\hbar} \sum_{\varkappa\lambda\mu\nu} A_{\varkappa\lambda\mu\nu} \{ n_\varkappa^0 n_\lambda^0 (n_\mu^0 + n_\nu^0 + 2)$$

$$+ n_\mu^0 n_\nu^0 (n_\varkappa^0 + n_\lambda^0 + 2) \} \, \delta(\epsilon_\varkappa + \epsilon_\lambda - \epsilon_\mu - \epsilon_\nu) \Big/ \sum_\nu n_\nu^0. \qquad (52)$$

On substituting the value of $A_{\varkappa\lambda\mu\nu}$ and going over

from summation to integration, we get except for a numerical factor

$$1 / \tau_{\mathrm{exch}} \sim (\Theta_c / \hbar)(T / \Theta_c)^4. \qquad (53)$$

By comparison of this expression for $1/\tau_{\mathrm{exch}}$ with $1/\tau_a \sim gM_0 (\mu M_0 / \Theta_c)(T/\Theta_c)^{1/2}$ [cf. formula (44)], we conclude that for

$$T \gg \mu M_0 (\Theta_c / \mu M_0)^{\mathfrak{s}/\mathfrak{z}}, \quad \tau_{\mathrm{exch}} \ll \tau_a,$$

and that for

$$T \ll \mu M_0 (\Theta_c / \mu M_0)^{\mathfrak{s}/\mathfrak{z}}, \quad \tau_{\mathrm{exch}} \gg \tau_a.$$

This means that for $T \gg \mu M_0 (\Theta_c / \mu M_0)^{3/7} \sim 10$ to $30°K$, the relaxation of a spin-wave gas occurs in the following manner: in a time $t = \tau_{\mathrm{exch}}$, the spin waves reach a quasiequilibrium state with a non-equilibrium value of the magnetic moment;* thereafter, the magnetic moment "slowly" (in a time τ_a) relaxes to its equilibrium value. For $T \ll \mu M_0 \times (\Theta_c / \mu M_0)^{3/7}$, the nonequilibrium state of the spin system cannot be described as having a definite value of the moment.

It must further be remembered that for $T \gg \Theta^2/\Theta_c$ (Θ = Debye temperature), a fundamental role is played by processes of interaction of the spin waves with phonons.[1] These processes will be treated from the phenomenological point of view in a separate article.

*This is a consequence of the fact that the exchange interaction does not change the magnetic moment of the system.

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