ON THE THEORY OF THE FERMI LIQUID

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Submitted to JETP editor February 5, 1958

J. Exptl. Theoret. Phys. (U.S.S.R.) **35**, 97-103 (July, 1958)

A study is made of the zero-angle scattering in collisions of quasiparticles in a Fermi liquid. It is shown that the scattering amplitude for zero angle depends on the limit approached by the ratio of the momentum and energy transfers in the collision as both these quantities go to zero. It is ascertained which of these limits is connected with the interaction energy of the quasiparticles that occurs in the general theory of the Fermi liquid developed earlier by the writer.

A general theory of the Fermi liquid* has been developed in previous papers by the writer.[1,2] One of the quantities that plays an important part in this theory in characterizing the properties of the liquid is the function $f(\mathbf{p}, \mathbf{p}')$ which determines the interaction energy of the quasiparticles, i.e., the variation of the energy $\epsilon(\mathbf{p})$ of the quasiparticles arising from a variation of their distribution function:

$$\delta\varepsilon(p) = \mathrm{Sp}_{\sigma'} \int f(\mathbf{p}, \mathbf{p}')\, \delta n(\mathbf{p}')\, d\tau' \tag{1}$$

(where $d\tau = d^3\mathbf{p}/(2\pi)^3$; here and below we take $\hbar = 1$).

In reference 1 it was shown that the function $f(\mathbf{p}, \mathbf{p}')$ is related in a definite way to the scattering amplitude of the quasi-particles in the liquid for their mutual collisions. The formulation given in reference 1 for this connection is not, however, quite accurate, as will be shown in the present paper.

We used below methods borrowed from quantum field theory; as is well known, these methods have recently been used with success by various authors in the study of the properties of quantum many-particle systems.

The main part in these methods is played by the Green's function G and the "vertex part" Γ. Let us recall the definitions and basic properties of these functions.

The function G is defined as the average value in the ground state of the system of the chronological product of two ψ operators:

$$G_{12} = -i\langle T(\psi_1 \overset{+}{\psi}_2)\rangle. \tag{2}$$

The indices 1, 2 denote sets of values of the three coordinates and the time, and also of the spin index. As usual, we shall use below instead of the space-time representation (2) the Fourier expansion of this function. The only components different from zero are those with identical values of the two momenta and the two energies (that is, of the wave vectors and frequencies): $P_1 = P_2 \equiv P$; we denote by P the "four-momentum", i.e., the combination of the momentum $\mathbf{p}$ and the energy ϵ. In respect to the spin indices (which we denote by Greek letters) the Fourier components $G_{\alpha\beta}(P) = \int G_{\alpha\beta}(X_1 - X_2)\, e^{-iP(X_1-X_2)} d^4(X_1 - X_2)$ are proportional to $\delta_{\alpha\beta}$; we shall write

$$G_{\alpha\beta}(P) = G(P)\,\delta_{\alpha\beta}. \tag{3}$$

As is well known, the poles of the function G(P) give the energies of the quasiparticles (the elementary excitations). In accordance with this, for p close to the boundary momentum p_0 and ϵ close to the boundary energy μ, G(P) has the form

$$G(P) \rightarrow \frac{a}{\varepsilon - \mu - v(p - p_0) + i\delta} \tag{4}$$

(μ is the chemical potential of the gas, and v is the speed of the quasiparticles at the Fermi boundary). This expression has a pole at

$$\varepsilon - \mu = v(p - p_0), \tag{5}$$

and the small constant δ is introduced in the usual way to specify the rule for going around the singularity in integrating; the sign of δ agrees with the sign of $\epsilon - \mu$ (or, what is the same thing near the pole, with the sign of $p - p_0$). The "renormalization" factor a is positive and, as has been shown by Migdal,[3] is smaller than unity:

$$a < 1. \tag{6}$$

*To avoid misunderstanding we emphasize that we are concerned not with simply a liquid composed of Fermi particles; it is also postulated that this liquid has an energy spectrum of the Fermi type, i.e., that it is not a superfluid.

The vertex part Γ is defined by means of the four-particle average value

$$\Phi_{1234} = \langle T(\psi_1\psi_2\psi_3^+\psi_4^+)\rangle. \quad (7)$$

The Fourier components of this function contain a part that is expressed in terms of functions $G(P)$ only, and a remainder that gives the definition of the Fourier component of the vertex part by the following formula:

$$\begin{aligned}&\Phi_{\alpha\beta,\gamma\delta}(P_1, P_2; P_3, P_4)\\ &= (2\pi)^8 G(P_1) G(P_2) [\delta(P_1 - P_3)\delta(P_2 - P_4)\delta_{\alpha\gamma}\delta_{\beta\delta}\\ &\quad - \delta(P_1 - P_4)\delta(P_2 - P_3)\delta_{\alpha\delta}\delta_{\beta\gamma}] \quad (8)\\ &\quad + iG(P_1)G(P_2)G(P_3)G(P_4)\\ &\times\Gamma_{\alpha\beta,\gamma\delta}(P_1, P_2; P_3, P_4)(2\pi)^4\delta(P_1 + P_2 - P_3 - P_4).\end{aligned}$$

Here the values of the arguments are connected by the relation

$$P_1 + P_2 = P_3 + P_4. \quad (9)$$

On interchange of the indices 1 and 2 (or 3 and 4) the function (7) changes sign; thus it follows from the definition (8) that Γ has the symmetry property:

$$\Gamma_{\alpha\beta,\gamma\delta}(P_1, P_2; P_3, P_4) = -\Gamma_{\beta\alpha,\gamma\delta}(P_2, P_1; P_3, P_4). \quad (10)$$

In the formation of the vertex part intermediate states occur that correspond to different values of the total number of particles in the system: the unchanged number N and the numbers $N \pm 2$. The latter arise from such arrangements of the ψ operators in the T-product as, for example, $\psi_1\psi_2\psi_3^+\psi_4^+$; the former correspond to arrangements such as, for example, $\psi_1\psi_3^+\psi_2\psi_4^+$. In accordance with this the contributions to the function Γ connected with these intermediate states have different characters in regard to their singularities. Namely, terms due to states that appear with the addition or removal of two particles have singularities with respect to the variables $P_1 + P_2$; terms corresponding to intermediate states with unchanged number of particles have singularities with respect to the variables $P_1 - P_3$ or $P_2 - P_4$.

The probability of scattering of quasi-particles with the transition

$$P_1\alpha, P_2\beta \rightarrow P_3\gamma, P_4\delta \quad (11)$$

is given in terms of the function Γ by the formula

$$\begin{aligned}&dW_{\alpha\beta,\gamma\delta}(P_1, P_2; P_3, P_4)\\ &= 2\pi|a^2\Gamma_{\alpha\beta,\gamma\delta}(P_1, P_2; P_3, P_4)|^2\delta(\varepsilon_1 + \varepsilon_2 - \varepsilon_3 - \varepsilon_4) \quad (12)\\ &\times n_1n_2(1 - n_3)(1 - n_4)\,d\tau_1 d\tau_2 d\tau_3\end{aligned}$$

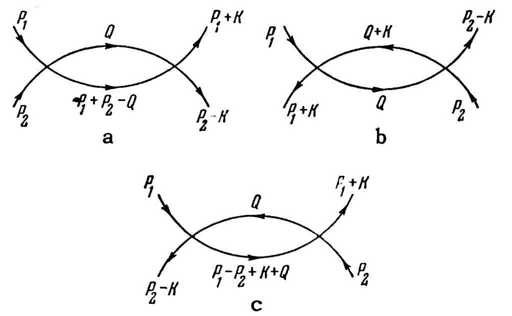
[where $n_1, n_2, \ldots.$ are the values of the distribution function for $P_1\alpha$, $P_2\beta$, and so on, and a is the renormalization constant from Eq. (4)]. The sign of Γ is defined in such a way that it corresponds to a positive scattering amplitude for repulsion and a negative amplitude for attraction.

Below we shall consider the function Γ for nearly equal values of the pairs of variables P_1, P_3 and P_2, P_4, i.e., we set $P_3 = P_1 + K$, $P_4 = P_2 - K$ with small K, and agree to write

$$\Gamma(P_1, P_2; P_1 + K, P_2 - K) \equiv \Gamma(P_1 P_2; K). \quad (13)$$

In terms of the scattering process (11) this means that we are considering collisions of quasiparticles giving nearly "forward scattering."

In the lowest order of perturbation theory contributions to the function $\Gamma(P_1, P_2; K)$ are made by the diagrams shown in the figure (a, b, c).



The internal parts of these diagrams correspond to the following propagation functions:

$$\text{(a) } G(Q)G(P_1 + P_2 - Q), \text{ (b) } G(Q)G(K + Q),$$
$$\text{(c) } G(Q)G(P_1 - P_2 + K + Q),$$

where Q is the intermediate four-momentum over which one integrates. With arbitrary P_1 and P_2 there is nothing to distinguish the value $K = 0$ for the functions (a) and (c), and for small K we can put $K = 0$. In the case (b), on the other hand, for $K \rightarrow 0$ the poles of the two factors come together, so that diagrams of this type require special consideration.

To calculate Γ one must sum the entire series of perturbation theory. Since in doing this our purpose is to separate out the parts having a singularity at $K = 0$, we must first single out the contribution from all the diagrams that do not have any parallel pairs of lines with nearly equal (differing by K) values of the four-momentum. We denote by $\Gamma^{(1)}$ this part of the function Γ, which has no singularity at $K = 0$; in it we can simply put $K = 0$, so that $\Gamma^{(1)}$ will be a function of the variables P_1 and P_2 only: $\Gamma^{(1)} = \Gamma^{(1)}(P_1, P_2)$. The entire series that has to be summed can be written symbolically in the form

$$(:\Gamma:) = (:\Gamma_1:) + (:\Gamma_1:\Gamma_1:) + (:\Gamma_1:\Gamma_1:\Gamma_1:) + \cdots, \quad (14)$$

where the colons replace pairs of lines in the diagram with nearly equal values of the four-momentum, and Γ_1 denotes the set of all possible diagram elements that do not have such pairs.

The problem of summing this series (so-called "ladder" summation) reduces to the solution of an integral equation, to obtain which we "multiply" the series (14) by Γ_1, i.e., replace it by the series

$$(:\Gamma_1:\Gamma:) = (:\Gamma_1:\Gamma_1:) + (:\Gamma_1:\Gamma_1:\Gamma_1:) + \ldots.$$

Comparison of this with Eq. (14) leads to the equation

$$(:\Gamma:) - (:\Gamma_1:) = (:\Gamma_1:\Gamma:),$$

which, when written out in explicit form, is the desired integral equation

$$\Gamma_{\alpha\beta,\gamma\delta}(P_1, P_2; K) = \Gamma^{(1)}_{\alpha\beta,\gamma\delta}(P_1, P_2) - \frac{i}{(2\pi)^4}\int \Gamma^{(1)}_{\alpha\varepsilon,\gamma\zeta}(P_1, Q)\, G(Q)\, G(Q+K)\, \Gamma_{\zeta\beta,\varepsilon\delta}(Q, P_2; K)\, d^4Q \tag{15}$$

(in the first factor of the integrand we should, strictly speaking, have $Q + K$ instead of the argument Q; but in view of the absence of singularities in $\Gamma^{(1)}$ we can here set $K = 0$). To investigate this equation we examine the product $G(Q)G(Q + K)$ that occurs in the integrand. On substituting here $G(P)$ in the form (4) we get

$$a^2/[\varepsilon - \mu - v(q - p_0) + i\delta_1] \times [\varepsilon + \omega - \mu - v(|\mathbf{q} + \mathbf{k}| - p_0) + i\delta_2]. \tag{16}$$

Here ϵ and $\mathbf{q}$ are the energy and momentum corresponding to the four-momentum Q, and $\epsilon + \omega$ and $\mathbf{q} + \mathbf{k}$ are those corresponding to $Q + K$.

For small $\mathbf{k}$ and ω the expression (16), as a function of ϵ and q, behaves like δ functions of the arguments $\epsilon - \mu$ and $q - p_0$; that is, it has the form

$$A\delta(\varepsilon - \mu)\,\delta(q - p_0), \tag{17}$$

where the coefficient A depends on the angle θ between the vectors $\mathbf{k}$ and $\mathbf{q}$. Comparing Eqs. (16) and (17), we see that this coefficient is given by the integral

$$A = \iint \frac{a^2 d\varepsilon\, dq}{[\varepsilon - \mu - v(q - p_0) + i\delta_1]\,[\varepsilon + \omega - \mu - v(|\mathbf{q} + \mathbf{k}| - p_0) + i\delta_2]}. \tag{18}$$

Let us first carry out the integration with respect to $d\epsilon$. The result of the integration depends essentially on the value of q. If the two differences $q - p_0$ and $|\mathbf{q} + \mathbf{k}| - p_0$ have the same sign, then we must also assign like signs to the quantities δ_1 and δ_2. The poles of the integrand then lie in one half-plane of the complex variable ϵ, and by closing the path of integration through the other half-plane we can see that the integral vanishes. Thus the integral is nonvanishing only for opposite signs of the differences $q - p_0$ and $|\mathbf{q} + \mathbf{k}| - p_0$. Let us first suppose that $\mathbf{qk} > 0$, i.e., $\cos\theta > 0$. Then the integral is nonvanishing for $q < p_0$, $|\mathbf{q} + \mathbf{k}| > p_0$, which, because of the smallness of $\mathbf{k}$, is equivalent to the condition

$$p_0 - k\cos\theta < q < p_0. \tag{19}$$

In addition we must have for the quantities δ that $\delta_1 < 0$, $\delta_2 > 0$, so that the poles of the integrand lie in different half-planes. Closing the path of integration through one half-plane and calculating the integral from the residue at the corresponding pole, we find

$$A = \int \frac{2\pi i a^2 dq}{\omega - v(|\mathbf{q} + \mathbf{k}| - q)}.$$

Since by Eq. (19) q is nearly equal to p_0 and varies over a range $k\cos\theta$, we can put $|\mathbf{q} + \mathbf{k}| - q = k\cos\theta$, so that

$$A = \frac{2\pi i a^2 k\cos\theta}{\omega - vk\cos\theta}.$$

Let us note the peculiar character of this expression: its limit for $k \to 0$, $\omega \to 0$ depends on the limit approached by the ratio ω/k.

It is easy to show in the same way that for $\cos\theta < 0$ (in which case the integration must be taken over the region $q > p_0$, $|\mathbf{q} + \mathbf{k}| < p_0$) one gets the same expression for $A(\theta)$. Thus we have

$$G(Q)\,G(Q+K) = \frac{2\pi i a^2\, \mathbf{l}\cdot\mathbf{k}}{\omega - v\,\mathbf{l}\cdot\mathbf{k}}\,\delta(\varepsilon - \mu)\,\delta(q - p_0) + g(Q), \tag{20}$$

where $l\mathbf{k}$ has been written instead of $k\cos\theta$ (l is the unit vector in the direction of $\mathbf{q}$), and $g(Q)$ does not contain any δ-function part (for small K), so that in it we can put $K = 0$.

Substituting Eq. (20) into Eq. (15), we get the fundamental integral equation in the form

$$\Gamma_{\alpha\beta,\gamma\delta}(P_1, P_2; K) = \Gamma^{(1)}_{\alpha\beta,\gamma\delta}(P_1, P_2) - \frac{i}{(2\pi)^4}\int_{\varepsilon} \Gamma^{(1)}_{\alpha\varepsilon,\gamma\zeta}(P_1, Q)\, g(Q)\, \Gamma_{\zeta\beta,\varepsilon\delta}(Q, P_2; K)\, d^4Q + \frac{a^2 p_0^2}{(2\pi)^3}\int \Gamma^{(1)}_{\alpha\varepsilon,\gamma\zeta}(P_1, Q)\, \Gamma_{\zeta\beta,\varepsilon\delta}(Q, P_2, K)\, \frac{\mathbf{l}\cdot\mathbf{k}}{\omega - v\,\mathbf{l}\cdot\mathbf{k}}\, do. \tag{21}$$

In the last term we have put $d^4Q = q^2\, dq\, do\, d\epsilon$, where do is an element of solid angle in the direction of l, and have carried out the integration of the δ functions in the integrand with respect to $dq\, d\epsilon$. In the arguments of the functions $\Gamma^{(1)}$ and Γ in this term Q is taken on the Fermi surface, i.e., it consists of the momentum $\mathbf{q} = p_0 l$

and the constant energy μ.

Because of the special character of the kernel of the integral equation as noted above, its solution also has just the same character: the limit of the function $\Gamma(P_1, P_2; K)$ for $K \to 0$ depends on the way in which $\mathbf{k}$ and ω go to zero, i.e., on the limit of the ratio ω/k.

Let us denote by $\Gamma^{\omega}(P_1, P_2)$ the limit

$$\Gamma^{\omega}_{\alpha\beta,\gamma\delta}(P_1, P_2) = \lim_{K\to 0} \Gamma_{\alpha\beta,\gamma\delta}(P_1, P_2; K) \quad \text{for } k/\omega \to 0 \tag{22}$$

[we shall see below that it is just this quantity with which the function $f(\mathbf{p}, \mathbf{p}')$ of Eq. (1) is related]. With this way of approaching the limit the kernel of the last term in Eq. (21) goes to zero, so that Γ^{ω} satisfies the equation

$$\Gamma^{\omega}_{\alpha\beta,\gamma\delta}(P_1, P_2) = \Gamma^{(1)}_{\alpha\beta,\gamma\delta}(P_1, P_2) - \frac{i}{(2\pi)^4}\int \Gamma^{(1)}_{\alpha\varepsilon,\gamma\zeta}(P_1, Q)\, g(Q)\, \Gamma^{\omega}_{\zeta\beta,\varepsilon\delta}(Q, P_2)\, d^4Q. \tag{23}$$

We can eliminate $\Gamma^{(1)}$ from (21) and (23). The result of the elimination is

$$\Gamma_{\alpha\beta,\gamma\delta}(P_1, P_2; K) = \Gamma^{\omega}_{\alpha\beta,\gamma\delta}(P_1, P_2) + \frac{a^2p_0^2}{(2\pi)^3}\int \Gamma^{\omega}_{\alpha\varepsilon,\gamma\zeta}(P_1, Q)\, \Gamma_{\zeta\beta,\varepsilon\delta}(Q, P_2; K)\, \frac{\mathbf{lk}}{\omega - v\mathbf{lk}}\, do. \tag{24}$$

In fact, if we formally write Eq. (23) in the form

$$\Gamma^{(1)}_{\alpha\beta,\gamma\delta}(P_1, P_2) = \hat{L}\Gamma^{\omega}_{\alpha\beta,\gamma\delta}(P_1, P_2), \tag{25}$$

then Eq. (21) is written

$$\hat{L}\Gamma_{\alpha\beta,\gamma\delta}(P_1, P_2; K) = \Gamma^{(1)}_{\alpha\beta,\gamma\delta}(P_1, P_2) + \frac{a^2p_0^2}{(2\pi)^3}\int \Gamma^{(1)}_{\alpha\varepsilon,\gamma\zeta}(P_1, Q)\, \Gamma_{\zeta\beta,\varepsilon\delta}(Q, P_2; K)\, \frac{\mathbf{lk}}{\omega - v\mathbf{lk}}\, do;$$

and substituting Eq. (25) and then applying the operator $\hat{L}^{-1}$ to both sides, we get Eq. (24).

Let us now introduce the function Γ^k defined by

$$\Gamma^{k}_{\alpha\beta,\gamma\delta}(P_1, P_2) = \lim_{K\to 0} \Gamma^{k}_{\alpha\beta,\gamma\delta}(P_1, P_2; K) \quad \text{for } \omega/k \to 0. \tag{26}$$

This function (multiplied by the renormalization constant a^2) is the "forward" scattering amplitude (i.e., that for the transition $P_1, P_2 \to P_1, P_2$), corresponding to actual physical processes occurring with quasi-particles on the Fermi surface: collisions leaving the quasiparticles on this surface involve changes of momentum without change of energy, so that the passage to the limit of zero momentum transfer $\mathbf{k}$ must be made for energy transfer ω strictly equal to zero. On the other hand the function Γ^{ω} introduced above corresponds to the nonphysical limiting case of "scattering" with small energy transfer and momentum transfer strictly equal to zero.

Setting $\omega = 0$ in Eq. (24) going to the limit $K \to 0$, and multiplying both sides of the equation by a^2, we get

$$a^2\Gamma^{k}_{\alpha\beta,\gamma\delta}(P_1, P_2) = a^2\Gamma^{\omega}_{\alpha\beta,\gamma\delta}(P_1, P_2) - \frac{p_0^2}{v(2\pi)^3}\int a^2\Gamma^{\omega}_{\alpha\varepsilon,\gamma\zeta}(P_1, Q)\cdot a^2\Gamma^{k}_{\zeta\beta,\varepsilon\delta}(Q, P_2)\, do. \tag{27}$$

Thus there exists a general relation connecting the two limiting forms of the forward scattering amplitude.

Let us now turn to the study of the poles of $\Gamma(P_1, P_2; K)$ as function of K. As was already pointed out at the beginning of this paper, the poles with respect to the variable $K = P_3 - P_1$ are due to contributions to Γ associated with intermediate states in which the number of particles in the system is not changed. Therefore these poles correspond to elementary excitations of the liquid without change of the number of quasiparticles in it. It is obvious that these are the excitations which can be described as sonic excitations in the gas of quasiparticles (phonons of the "zeroth sound").

Near a pole of the function $\Gamma(P_1, P_2; K)$ the left side and the integral on the right side of the equation (24) are arbitrarily large; the term $\Gamma^{\omega}(P_1, P_2)$, on the other hand, remains finite and therefore can be dropped. We note further that the variables P_2 and also the indices β and δ are not affected by the operations applied to the function Γ in Eq. (24), i.e., they here play the role of parameters. Finally, we shall consider Γ close to the Fermi surface, i.e., we shall consider the energy of the quasiparticle, which is one of the variables P_1, to be equal to μ, and the momentum to be equal to p_0, so that we write it in the form $p_0\mathbf{n}$, where $\mathbf{n}$ is a variable unit vector. Keeping all this in mind, we conclude that the determination of the sonic excitations in the liquid reduces to the problem of the eigenvalues of the integral equation

$$\chi_{\alpha\gamma}(\mathbf{n}) = \frac{a^2p_0^2}{(2\pi)^3}\int \Gamma^{\omega}_{\alpha\varepsilon,\gamma\zeta}(\mathbf{n},\mathbf{l})\, \chi_{\zeta\varepsilon}(\mathbf{l})\, \frac{\mathbf{lk}}{\omega - v\mathbf{lk}}\, do, \tag{28}$$

where $\chi_{\alpha\gamma}(\mathbf{n})$ is an auxiliary function.

We transform this equation, introducing instead of χ a new function, by the substitution

$$\nu_{\alpha\gamma}(\mathbf{n}) = \frac{\mathbf{nk}}{\omega - v\mathbf{nk}}\, \chi_{\alpha\gamma}(\mathbf{n}). \tag{29}$$

Then Eq. (28) takes the form

$$(\omega - v\mathbf{nk})\, \nu_{\alpha\gamma}(\mathbf{n}) = (\mathbf{k}\cdot\mathbf{n})\frac{p_0^2a^2}{(2\pi)^3}\int \Gamma^{\omega}_{\alpha\varepsilon,\gamma\zeta}(\mathbf{n},\mathbf{l})\, \nu_{\zeta\varepsilon}(\mathbf{l})\, do. \tag{30}$$

This equation agrees precisely in form with equation (11) found in reference 2 for the distribution function ν in the zeroth sound, and moreover a comparison of the two equations (using the definition of F by Eq. (6) of reference 2) leads to the following correspondence between the function $f(\mathbf{p}, \mathbf{p}')$* and the function Γ^{ω}:

$$f_{\alpha\beta,\gamma\delta}(\mathbf{n},\mathbf{l}) = a^2\Gamma^{\omega}_{\alpha\beta,\gamma\delta}(\mathbf{n},\mathbf{l}) \quad (31)$$

This is the desired relation between f and the properties of the scattering of the quasiparticles. For clarity we point out that the four spin indices on this function correspond to the fact that $f(\mathbf{p},\mathbf{p}')$, or more explicitly $f(\mathbf{p},\boldsymbol{\sigma};\ \mathbf{p}',\boldsymbol{\sigma}')$, depends on the spin operators (two-row matrices) $\boldsymbol{\sigma}$ and $\boldsymbol{\sigma}'$ of the two particles; thus to the two particles (momenta $p_0\mathbf{n}$ and $p_0\mathbf{l}$) there correspond the pairs of indices α,γ and β,δ (in the function $\Gamma_{\alpha\beta,\gamma\delta}(P_1,P_2;\ P_3,P_4)$ these pairs correspond to the pairs of nearly equal four-momenta P_1, P_3 and P_2, P_4).

Having thus found the connection of the function f with the properties of the scattering of the quasiparticles, let us return to the formula (27) and obtain with its aid explicit relations between the function f and the "physical" amplitude for zero-angle scattering on the Fermi surface, which we write in the form

$$A(\mathbf{n}_1,\sigma_1;\ \mathbf{n}_2,\sigma_2) = a^2\Gamma^k(\mathbf{n}_1,\sigma_1;\ \mathbf{n}_2,\sigma_2). \quad (32)$$

On the Fermi surface the relation (27) takes the form

$$A(\mathbf{n}_1,\sigma_1;\ \mathbf{n}_2,\sigma_2) = f(\mathbf{n}_1,\sigma_1;\ \mathbf{n}_2,\sigma_2) - \frac{1}{4\pi}\frac{d\tau}{d\varepsilon}\mathrm{Sp}_{\sigma'}\int f(\mathbf{n}_1,\sigma_1;\ \mathbf{n}',\sigma')A(\mathbf{n}',\sigma';\ \mathbf{n}_2,\sigma_2)\,do' \quad (33)$$

(where $d\tau/d\epsilon = 4\pi p_0^2/v(2\pi)^3$). The scalar functions A and f depend on all scalar combinations of the four vectors $\mathbf{n}_1$, $\mathbf{n}_2$, $\boldsymbol{\sigma}_1$, $\boldsymbol{\sigma}_2$. If, however, the interaction between the particles is an exchange interaction, then the only admissible scalar products are $\mathbf{n}_1\mathbf{n}_2$ and $\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2$. Then we can expand A and f as functions of $\cos\theta$ in terms of Legendre polynomials:

$$A(\cos\theta) = \sum_l A_l P_l(\cos\theta), \quad f(\cos\theta) = \sum_l f_l P_l(\cos\theta). \quad (34)$$

*In references 1 and 2 we did not write the spin indices explicitly.

Substituting this into Eq. (33) and performing the integration with respect to do', we get

$$A_l(\sigma_1,\sigma_2) = f_l(\sigma_1\sigma_2) - \frac{1}{2l+1}\frac{d\tau}{d\varepsilon}\mathrm{Sp}_{\sigma'} f_l(\sigma_1\sigma')A_l(\sigma'\sigma_2). \quad (35)$$

In the case of an exchange interaction the spin dependence of the function reduces to a term proportional to $\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2$ (cf. reference 1*), so that

$$f_l = \varphi_l + \psi_l\sigma_1\sigma_2, \quad (36)$$

where φ_l, ψ_l do not depend on the spins. Corresponding to this we also set

$$A_l = B_l + C_l\sigma_1\sigma_2. \quad (37)$$

Substituting Eqs. (36) and (37) into Eq. (35), we get without difficulty

$$B_l = \varphi_l - \frac{2}{2l+1}\frac{d\tau}{d\varepsilon}B_l\varphi_l, \quad (38)$$

$$C_l = \psi_l - \frac{1}{2(2l+1)}\frac{d\tau}{d\varepsilon}C_l\psi_l.$$

These formulas give a simple algebraic connection between the coefficients of the expansions of f and A in spherical harmonics. We note that only terms of the same l are related to each other, and that B is related only to the φ's and C only to the ψ's.

In conclusion, I would like to thank A. B. Migdal, who called my attention to the dependence of the forward scattering amplitude on the ratio ω/k, and also E. M. Lifshitz and L. P. Gor'kov for a discussion of this work.

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Translated by W. H. Furry
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*We take occasion to correct a mistake which got into reference 1: Equation (27) should be

$$1/\chi = \beta^{-2}\{4\pi^2k^2/3\alpha + \bar{\psi}_0\}.$$