In a viscous liquid with finite conductivity, the harmonic current with frequency ω_0 [in contrast to (1)] excites hydromagnetic waves with different frequencies $\omega = ku_1$. The spectral distribution of the hydromagnetic waves radiated is determined from (18).

Making use of the formula

$$\delta(x) = \lim \frac{1}{\pi} \frac{\alpha}{\alpha^2 + x^2},$$

it is easy to show that for $\nu \rightarrow 0$ and $\sigma \rightarrow \infty$, Eq. (18) goes over into (10).

6. We now compare the intensity of the excitation of hydromagnetic waves by currents with the intensity of excitation of hydromagnetic waves by mechanical means, in which case, for a certain plane perpendicular to the magnetic field $\mathbf{H}_0$ (the plane $z = 0$), the velocity of the liquid perpendicular to the magnetic field is given by

$$\mathbf{v} = \mathbf{v}_0 e^{-i\omega_0 t}, \quad \mathbf{v}_0 \perp \mathbf{H}_0, \quad z = 0.$$

Assuming $\mathbf{v}, \mathbf{h} \sim \exp\{-i\omega_0(t - z/V_0)\}$, and taking the liquid to be incompressible, we get, by (3) and (5),

$$\mathbf{E} = -\frac{\mu}{c}[\mathbf{v} \times \mathbf{H}_0], \quad \mathbf{h} = -\sqrt{\frac{4\pi\rho_0}{\mu}}\,\mathbf{v}. \tag{19}$$

The energy flow is determined primarily by the flow of electromagnetic energy

$$I = \frac{c}{8\pi} \operatorname{Re}\,[\mathbf{E} \times \mathbf{h}^*].$$

Substituting this expression in (19), we get

$$I = {}^1/_2\, \rho_0 V_0 v_0^2. \tag{20}$$

A comparison of (20) with Eq. (12) shows that the surface current j_S is equivalent to a velocity

$$v_0 = \sqrt{2\pi\mu/\rho_0}\, j_S / c$$

from the viewpoint of the excitation of hydromagnetic waves.

[1] H. Alfvén, Cosmical Electrodynamics (Oxford, 1950).

[2] S. Lundquist, Phys. Rev. 76, 1805 (1949).

Translated by R. T. Beyer
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QUANTUM-MECHANICAL PROBABILITIES AS SUMS OVER PATHS

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A new formulation of nonrelativistic quantum mechanics is proposed, namely a general definition of the probability of any event. The physical content of quantum mechanics is reduced to a single principle similar to the principle of Gibbs; this makes it possible to solve problems without resorting to the use of wave functions and operators.

THE idea that there may exist in quantum mechanics a general expression for the probability amplitude of any event is due to Feynman.[1] These amplitudes are multiplied and combined like classical probabilities; this leads to the idea of constructing quantum mechanics according to the model of classical statistical physics. In statistical physics the probability of finding a system to have some given property is equal to the sum over all configurations having this property; each configuration is used with the weight assigned by Gibbs. In quantum mechanics the role of the configurations is played by the paths of the particle; according to Feynman's idea the probabilities are replaced by amplitudes. A simple and complete "atomistic" description is obtained (see Sec. 8).

This program has not, however, been completely carried out. According to Feynman, the amplitude of any state must be the sum over all paths consistent with the conditions of the experiment, but, as is stated in reference 1 (Sec. 11), "which properties [of the paths] correspond to which physical measurements has not been formulated in a general way."

This problem is solved in the present paper, but in a different way. We have to give up certain of Feynman's ideas. The question of a general expression for the amplitude does not have to be dealt with, since, as will be shown below, something more is available — a general expression for the probability of any event.

1. A FORMULATION OF QUANTUM MECHANICS

Any experiment can be completely analyzed by means of the following rule:

The probability for finding a particle possessing any given property is equal to the sum over all paths that have this property; each path occurs with the weight $\cos(S/h)$; S is the change of the classical action along the path, and h is Planck's constant.

The probability for finding a given value a for the physical quantity $a\{x(t)\}$, which depends on the path $x(t)$ (a can be a coordinate, a velocity, the energy at a particular time, two coordinates at different times, etc.), is given symbolically by

$$W(a) = \int_a \cos\frac{S}{h}\, d\Gamma; \tag{1}$$

$\int_a d\Gamma$ denotes the integral over all paths for which $a\{x(t)\} = a$.

If we specify the path by a succession of coordinates x_k and momenta p_k at times t_k, and if $t_k - t_{k-1} = \epsilon_k$ is much smaller than the characteristic time of the system, then

$$W(a) = \sum_{\pm}\iint_a\cdots\int \cos\frac{S}{h}\prod_k \frac{d^3x_k d^3p_k}{(2\pi)^3}; \tag{2}$$

$\sum_{\pm}$ denotes the sum over all signs of the ϵ_k; S is the change of the action along the broken-line path with the vertices x_k, p_k:

$$S(\ldots x_k p_k \ldots) = \int p^\nu dx^\nu = \sum_k \mathbf{p}_k(\mathbf{x}_k - \mathbf{x}_{k-1}) - H(\mathbf{p}_k, \mathbf{x}_k)\,\varepsilon_k.$$

The integration is taken over the region $a(\ldots x_k p_k \ldots) = a$. All the paths begin and end at the times $-T$ and T, and the index k runs through the finite number of values $-N < k < N$. For $N \to \infty$ and $T \to \infty$ one gets all possible paths. It is assumed that the entire history of the particle is known, i.e., all macrofields.

If the physical quantities are expressed in terms of coordinates and velocities (not momenta), the condition $a(\ldots x_k p_k \ldots) = a$ is replaced by $a(\ldots x_k \ldots) = a$. We can integrate over all the p_k, and there remains a sum over purely spatial paths.

Thus a simple construct (though one poorly described by usual notations) — the sum over all paths, i.e., over all ways of realizing the event a — contains in itself all the principles and rules of quantum mechanics, and provides in general form a way of writing down the solution of any quantum mechanical problem.

2. THE UNIVERSAL DISTRIBUTION OF PATHS

Like the Gibbs distribution, the distribution (1) can in some measure be proved.

We start from the classical ideas about the microscopic world. Particles move along paths $x(t)$, and each path occurs with a probability $W\{x(t)\}$. Knowing $W\{x(t)\}$, one can find the probability of any event a:

$$W(a) = \int_a W\{x(t)\}\, d\Gamma. \tag{3}$$

We assume that $W\{x(t)\}$ depends only on the change of the action along the path. In classical mechanics the sign of the action can be arbitrary; therefore $W\{x(t)\} = W(S) + W(-S)$. For two noninteracting particles the probability must fall apart into a product of probabilities, i.e., we must have the relation:

$$\begin{aligned} &W(S_1+S_2) + W(-S_1+S_2) + W(S_1-S_2) \\ &\quad + W(-S_1-S_2) \\ &= [W(S_1) + W(-S_1)]\,[W(S_2) + W(-S_2)]. \end{aligned}$$

From this we have $W(S) = \cos\alpha S$, where α is a constant; comparing Eq. (3) with quantum mechanics (or with experiment) we get $\alpha = 1/h$. The quantity $\cos(S/h)$ can be negative (our assumptions are in part untrue), but the probabilities measured in experiments turn out to be positive. These are probabilities associated with the measurement of commuting quantities, in which it is not necessary to take into account the reaction of the instruments.

It is interesting that many formulas are simplified and "given meaning" (see Sec. 3) if we assume that the two signs of the action correspond to two ways of traversing the path (not at all related to the actual direction of motion of the particle along a path in the classical limit).

Repeating the above considerations, we get

$$W(a) = \int_a e^{iS/h} d\Gamma. \tag{4}$$

Here the integration is taken over directed paths, i.e., paths with different directions of traversal are distinguished. According to Eq. (4) the quantity $\exp[iS\{x(t)\}/h]$ is the formal probability (for short we shall call it the probability) for the particle to pass along the path $x(t)$, and $\cos[S\{x(t)\}/h]$ in Eq. (1) is the probability for finding the path $x(t)$.

For the further discussion we introduce the notations: $(xt \rightarrow x't')$ is the sum over all directed paths (each with the factor $\exp iS$) passing from x,t to x',t'; $(xt \sim x't')$ is the sum over all closed paths (each with the factor $\exp iS$) passing through x,t and x',t'; $(xt \underset{a}{\sim} x't')$ is the sum over all closed paths passing through x,t and x',t' and possessing the property a.

3. THE PRINCIPLES OF QUANTUM MECHANICS

To make clear the connection of Eq. (1) with the usual formalism, let us calculate $W(x,t)$, the probability for finding the particle at the point x at the time t. This is the sum over all paths passing through the point x at the time t; it can be obtained if we multiply the sum over all paths (each with the factor $\exp iS$) arriving at the point x,t [we denote this sum by $\psi(x,t)$] by the sum over all paths that come out from the point x,t. The second sum will be equal to the complex conjugate expression $\psi^+(x,t)$, since for each departing path there exists one just like it but oppositely directed. Thus $W = \psi\psi^+$. As can be verified, the quantity ψ satisfies the Schrödinger equation, and consequently is identical with the wave function, although the definition given above contains some arbitrariness and will be made more precise in Sec. 4.

According to Eq. (4) $\psi(x,t)$ is the probability of arriving at the point x,t; $\psi^+(x,t)$ is the probability of emerging from the point x,t; $W(x,t)$ is the probability of passage through x,t, obtained by multiplication of the probabilities of these two events, which always appear together.

Let us now calculate the average value of the momentum of the particle at the time t. According to Eq. (2)

$$\bar{p}(t_l) = \sum_{\pm}\iint\cdots\int p_l \cos S \prod_k \frac{dx_k dp_k}{2\pi},$$

and we now replace $\cos S$ by $(e^{iS} + e^{-iS})/2$ and set $\epsilon_l = 0$. Then

$$\bar{p}(t_l) = \frac{1}{2}\int \psi(x_l, t_l) p_l e^{ip_l(x_l - x_{l-1})}\psi^+(x_{l-1}, t_l)\, dx_l dp_l dx_{l-1}$$
$$+ \text{Compl. conj.} = \frac{1}{2}\int \psi(x, t_l)\frac{1}{i}\frac{\partial\psi^+}{\partial x}(x, t_l)\, dx + \text{Compl. conj.} \tag{5}$$

In a similar way one can show that Eq. (2) gives the correct values for the first powers of such quantities as momentum, energy, and angular momentum. In calculating the average values of higher powers of most physical quantities, however, it is necessary to take precautions against possible inaccuracies associated with the replacement of continuous paths by broken ones. In order that the value of the physical quantity $f(x,p)$ at the time t shall not depend on the way the time axis is broken up into intervals ϵ_k, one must replace

$f(t)$ by $\lim\limits_{\epsilon\to 0}\int\limits_{t-\epsilon}^{t+\epsilon} f\, dt$; $\epsilon \rightarrow 0$, $\epsilon_k \rightarrow 0$, but $\epsilon \gg \epsilon_k$.

Or, more simply, inside the interval $(t-\epsilon, t+\epsilon)$ we take N intermediate time intervals t_k and set $f = \frac{1}{N}\sum\limits_{k=1}^{N} f_k$ (f_k is the value of f at the time t_k).

Then

$$\bar{f^n} = \lim_{N\to\infty}\int\left(\frac{\Sigma f_k}{N}\right)^n \cos S\, d\Gamma.$$

Here $|t_k - t_{k-1}|$ remains of the order of magnitude of ϵ, and the sign of the difference $|t_k - t_{k-1}|$ can be arbitrary.

For $N \rightarrow \infty$ there remain only those terms in $\left(\sum f_k\right)^n$ in which the f_k are taken at different times, but, as can easily be shown, for $t_k \rightarrow t_{k-1}$ the quantity $\int f_{k_1}\ldots f_{k_n} \cos S\, d\Gamma$ becomes equal to $\int \psi^+(x,t)\, f^n(x, -i\partial/\partial x)\, \psi(x,t)\, dx$. If Eq. (1) gives the correct value for all a^n, then the correct value is also obtained for $W(a)$. This can be verified by direct calculation of the probabilities (see Sec. 5).

We note that since every measurement reduces in the final analysis to measurements of coordinates, it is sufficient that Eq. (2) give the correct value for $W(x,t)$.

We shall now show that $(xt \rightarrow x't')$ is the same as the propagation function of Schrödinger's equation. Unlike the corresponding Feynman expression, $(xt \rightarrow x't')$ contains paths with change of the sign of the time [because of the operation $\sum\limits_{\pm}$ in Eq. (2)]. In the case in question, however, these paths make no contribution to the sum.

In fact, let us compare $(xt \rightarrow x't')$ — the sum over all paths (each with the factor $\exp iS$) connecting the points x,t and x',t' without change of the sign of the time — and $(xt \underset{1}{\rightarrow} x't')$ — the sum over the paths connecting x,t and x',t' with a single change of the sign of the time at the time

t'' (t'' lies outside the interval (t, t')):

$$(xt \underset{1}{\rightarrow} x't') = \int (xt \underset{0}{\rightarrow} x''t'')\,(x''i'' \underset{0}{\rightarrow} x't')\,dx''.$$

As is well known,[1] the sum over the paths connecting the points x, t and x', t' without change of the sign of the time is the propagation function of Schrödinger's equation, so that $(xt \underset{1}{\rightarrow} x't') = (xt \underset{0}{\rightarrow} x't')$.

Let us now consider the probability of this event: at the time t_0 a measurement was made of the complete set of quantities $L_i(x, p)$ and they were found to be equal to the numbers L_i, and the time t the quantity $M_1(x, p)$ is equal to M_1. The quantities L_i form a complete set if $W(L_1 \ldots L_n t)$ comes apart into the product of two functions: ψ, the sum over the paths arriving, and ψ^+, the sum over the paths departing. Let $M_2 \ldots M_n$ be quantities that make a complete set together with M_1.

According to quantum mechanics

$$\begin{aligned} &W(M_1 t \mid L_1 \ldots L_n t_0) \\ &= \int W(M_1 \ldots M_n t \mid L_1 \ldots L_n t_0)\,dM_2 \ldots dM_n \qquad (6) \\ &= \int |G(M_1 \ldots M_n t \mid L_1 \ldots L_n t_0)|^2\,dM_2 \ldots dM_n; \end{aligned}$$

G is the propagation function in the mixed L, M representation. Since G is the sum over all paths going from t_0 to t, and G^+ is the sum over all paths going from t to t_0, $W(M_1 t \mid L_1 \ldots L_n t_0)$ is the sum over all closed paths (in the x, t space), for which at the time t_0 $L_i(x, p) = L_i$ and at the time t $M_1(x, p) = M_1$, that is,

$$W(M_1 t \mid L_1 \ldots L_n t_0) = \int \cos S\,d\Gamma = (M_1 t \sim L_1 \ldots L_n t_0). \qquad (7)$$

$$M_1 t \rightleftarrows L_1 \ldots L_n t_0.$$

This formula makes it possible to solve both scattering problems and problems relating to stationary states. In the latter case W does not depend on t and t_0.

The closed paths in Eq. (7) have no special physical meaning; their appearance can be understood if one traces through the proof of Eqs. (7) and (1) presented in Sec. 4. The expression (7) does not mean that the particles move along closed paths.

There is one further relation, similar to Eq. (7) and interesting from the point of view of a causal description. A comparison of $W(x, p, t)$ and $W(x, p, t')$ as calculated from Eq. (1) gives

$$W(x, p, t) = \int (xpt \sim x'p't')\,W(x', p', t')\,dx'dp'. \qquad (8)$$

4. EQUIVALENCE TO THE USUAL FORMALISM

The expression (1) is an obvious analogue of the Gibbs distribution; the number N (cf. Sec. 1) plays the role of the number of particles, but the dependence of the expression (1) on N for $N \rightarrow \infty$ is completely different from that existing in statistical physics. Because of this fact, problems arise in quantum mechanics that do not have to be solved in statistical physics.

Let us go back to Eq. (1). Suppose that measurements of some kind are carried out; we must find the relation between them. The probability for each measurement to give a prescribed result is defined by Eq. (1), and at first sight is not related in any way to the results of the other measurements. In actual fact, however, the integral on the right side of Eq. (1) is not completely defined. The time ranges from $t = -\infty$ to $t = +\infty$, and the integral is infinitely manifold even for discrete time, so that it can be defined only as a limit of an N-fold integral for $N \rightarrow \infty$. It is easy to show by examples, however, that such a limit does not exist; the result depends on the assumptions that are made about the positions of the final and initial points of the path. For example, one can assume that at the initial instant the particle is at the point 0, or that the particle can be at any point.

In the general case the behavior depends on two functions which give the distributions of the initial and final points for the two possible directions of motion:

$$\begin{aligned} W(a) = \iint u(x)\,(x, -T \underset{a}{\sim}\ x', T)\,u(x')\,dxdx' \qquad (9) \\ + \iint v(x)\,(x, -T \underset{a}{\sim}\ x', T)\,v(x')\,dxdx'. \end{aligned}$$

Part of the measurements serves to determine the behavior as T goes to infinity, i.e., to determine u and v, and the probability of the other results is calculated just from this limiting behavior. The connection between the mean values with this approach agrees with the quantum-mechanical results (see below), but the hypothesis that the behavior of the paths for $t \rightarrow \pm\infty$ affects the results of measurement is, generally speaking, an abstraction. This is the quantum-mechanical form of Laplace determinism.

In actual fact the past and future histories of the particle are unknown to us; from this point of view, the behavior at infinity is unimportant and one can fix on some single rule,* for example the hypothesis that all particles appear from the point 0 at the time $-\Theta$ and disappear into this point at the time Θ. In this case the probability of an event is found to depend on unknown past and future forces. For definiteness we may suppose that a prescribed external field acts on the particles during the time interval $(-T, T)$, an unknown field X during the interval $(-\Theta, -T)$, and the field Y during the interval (T, Θ):

$$W(a) = (0, -\Theta \underset{a}{\sim} 0, \Theta);$$
$$S = \int_{-\Theta}^{\Theta} L dt = \int_{-\Theta}^{-T} L(X)\, dt + \int_{-T}^{T} L dt + \int_{T}^{\Theta} L(Y)\, dt. \quad (10)$$

To find the connection between observed quantities, one must use Eq. (10) to express them in terms of the unknown fields X and Y, and then eliminate X and Y from the resulting system of equations. The result again agrees with that of quantum mechanics, though the dependence on future forces seems rather strange. This dependence exists also in the classical principle of least action; if the coordinates of a particle are given at the times t_1 and t_2, then the value of a coordinate at a time t $(t_1 < t < t_2)$ depends both on the past and also on the future history.

This formalism can be somewhat simplified if we equate to zero the density of "sources" at $t = \Theta$. There remains only the sum over paths beginning at $t = -\Theta$ and ending at $t = -\Theta$. In this case the probability depends only on the past history:

$$W(a) = (0, -\Theta \underset{a}{\sim} 0, -\Theta)$$
$$S = \int_{-\Theta}^{-T} L(X)\, dt + \int_{-T}^{-T} L dt + \int_{-T}^{-\Theta} L(X)\, dt. \quad (11)$$

Thus the sum in Eq. (1) must be taken either over paths with prescribed behavior for $T \to \pm\infty$, or else over paths in a field of unknown past and future (or only past) forces.

The agreement of the consequences of Eqs. (9), (10), and (11) with quantum mechanics can be shown easily by relating u, v and X, Y with the wave function. In Eq. (9)

$$\psi(x, -T) = u(x) + iv(x)$$
$$+ \int (x, -T - x', T)\,[u(x') + iv(x')]\, dx'.$$

In Eq. (10)

$$\psi(x, -T) = (0, -\Theta \to x, -T)$$
$$+ \int (x, -T \to x', T)(x', T \to 0, \Theta)\, dx'.$$

In Eq. (11)

$$\psi(x, -T) = (0, -\Theta \to x, -T).$$

Unknown forces and "densities of sources" exist also in statistical physics. For example, the statistical sum for a gas will be different for the cases of free and fixed piston. In statistical physics, however, the forces acting on the piston, or the probability of a given position of the piston are usually known, whereas in quantum mechanics they have to be calculated from the results of measurements.

The operation of eliminating u, v or X, Y is in general complicated, but it is no more complicated than the elimination of ψ in quantum mechanics. For example, the problem of finding the coordinate distribution at time t' for given coordinate distribution and momentum distribution at the time t is difficult also in the usual formalism (it is hard to find ψ at the time t). Just as in the usual formulation, the problem is considerably simplified if part of the measurements gives a complete description of the state (this case includes all known problems of quantum mechanics). Let us consider this case (for simplicity we take a one-dimensional problem).

It is required to determine $W(M,t) = \int_{M,t} \cos S\, d\Gamma$ if it is known that

$$W(L', t_0) = \int_{L', t_0} \cos S d\Gamma = \delta(L' - L).$$

To find $W(M,t)$ we go back to Eq. (1); this formula, however, depends on the unknown past and future history [or only the past, if we use Eq. (11)] of the particle, i.e., on X and Y. We need some sort of information about this history. We get this information from the condition $W(L', t_0) = \delta(L' - L)$.

In other words, we have two equations

$$W(M, t) = \int_{M,t} \cos S d\Gamma, \qquad \delta(L' - L) = \int_{L', t_0} \cos S d\Gamma$$

[with S taken from Eq. (10) or Eq. (11)] from which we must eliminate the unknown forces X and Y. We obtain the conditional probability $W(M, t \mid L, t_0)$.

We shall show that $W(M, t \mid L, t_0)$ agrees with the expression (7). Let $\psi(x, t_0)$ be the sum over all paths that arrive at x, t_0. According to Sec. 3,

$$W(L, t) = \int \psi^+(x, t)\, \delta(L - \hat{L})\, \psi(x, t)\, dt.$$

From the condition

$$W(L', t_0) = \int \psi^+ \delta(L' - \hat{L})\, \psi dx = \delta(L' - L)$$

(this is a condition on the unknown forces or behavior for $T \to \pm\infty$), we get: $\psi(x, t_0) = \psi_L(x, t_0)$, an eigenfunction of the operator L. Similarly,

$$W(M, t) = \int \psi^+ \delta(M - \hat{M}) \psi dx$$

$$= \int \psi^+ \delta(M - M') \psi dM' = |\psi(M, t)|^2.$$

Since

$$\psi(x, t) = \int (xt \leftarrow x't_0) \psi(x', t_0) dx,$$

then $\psi(x, t) = G(x, t \mid L, t_0)$.

In the M-representation $\psi(M,t) = G(M,t \mid L,t_0)$, so that

$$W(M, t) = |G(M, t \mid L, t_0)|^2.$$

In just the same way one obtains Eq. (8). According to Eq. (1) $W(x,p,t)$ and $W(x,p,t')$ depend on the unknown past and future history. Comparing these dependences, we get the relation (8). One has to write out detailed expressions for $W(x,p,t)$ and $W(x,p,t')$, draw the corresponding paths, and note that for $t > t'$ the contribution to $W(x,p,t')$ from the paths $-\infty \to t' \to t \to \infty$ is the same as that from the paths $-\infty \to t \to t' \to t \to \infty$. These last differ from the paths $-\infty \to t \to \infty$, which occur in $W(x,p,t)$, by the loop $t \to t' \to t$. This loop gives $(xpt \sim x'p't')$ in Eq. (8).

In practice, however, there is no need to use the general scheme; Eq. (7), which contains no unknown forces and densities, is sufficient.

5. EXAMPLES

The equivalence of the expression (7) to the usual formalism can be verified by examples.

(a) A scattering problem.

Let us calculate $W(p,t \mid x,t_0)$ — the probability for the particle to have the momentum p at the time t, if at the time t_0 it had the coordinate x. According to Eq. (7),

$$W = (pt \sim xt_0) = \lim_{\varepsilon \to 0} \int (yt \underset{p}{\to} y't + \varepsilon)$$

$$\times (y't + \varepsilon \to xt_0)(xt_0 \to yt) dy dy'$$

$$= \int \frac{e^{ip(y-y')}}{2\pi} G(y't \mid xt_0) G^+(yt \mid xt_0) dy dy'$$

$$= |(2\pi)^{-1/2} \int e^{ipy} G(yt \mid xt_0) dy|^2.$$

(b) The distribution of the coordinate in the stationary state with energy E, $W(x \mid E)$.

It is more convenient to consider a more general quantity, $W(xt \mid Et')$. This is the sum over all paths that pass through the point x at the time t and have the energy E at the time t'.

Prescription of the energy means prescription of the average energy in an infinitely small neighborhood (t_1, t_N) of the point t'. We denote the coordinates at the times t_1 and t_N by x_1 and x_N.

The more precisely written expression for W, with meaning clear from the notations, is

$$W(xt \; Et') = \iint (x_1 t_1 \underset{\bar{E}=E}{\longrightarrow} x_N t_N)$$

$$\times (x_N t_N \to xt)(xt \to x_1 t_1) dx_1 dx_N.$$

Inside the interval (t_1, t_N) we choose $N - 2$ instants of time

$$t_2, t_3, \ldots, t_{N-1} (t_k = t_{k-1} + \varepsilon_k);$$

and replace the continuous paths by broken lines with vertices at x_k, p_k:

$$W = \int_{\bar{E}=E} \exp\left\{i \sum_k (p_k \Delta x_k - H_k \varepsilon_k)\right\}$$

$$\times \prod_k \frac{dx_k dp_k}{2\pi} (x_N t_N \to xt)(xt \to x_1 t_1).$$

The condition

$$\bar{E} = \frac{1}{N} \sum_1^N H_k = \frac{1}{N} \sum_1^N H(p_k x_k) = E$$

is now replaced by a factor $(2\pi)^{-1} \int \exp\{i\tau(E - \bar{E})\} d\tau$:

$$W = \int \exp\left\{i\tau E + i \sum_k [p_k \Delta x_k - H_k(\varepsilon_k + \tau/N)]\right\}$$

$$\times \prod_k \frac{dx_k dp_k}{2\pi} (x_N t_N \to xt)(xt \to x_1 t_1) \frac{d\tau}{2\pi}$$

$$= \int e^{i\tau E} (x_1 t_1 \to x_N t_N + \tau)(x_N t_N \to xt)$$

$$\times (xt \to x_1 t_1) dx_1 dx_N d\tau / 2\pi.$$

Substituting in this the relation

$$(xt \to x't') = G = \sum_E \varphi_E^+(x) \varphi_E(x') e^{iE(t-t')},$$

we get: $W = |\varphi_E(x)|^2$.

(c) Diffraction from two apertures.

We have to calculate $W(x \mid x_0)$ — the probability of finding the particle at the point x if the source is at the point x_0 behind the screen. This is the sum over all closed paths passing through x and x_0 (Sec. 7), each with the weight $\cos S$. The paths for which the time direction does not change at the points of integration can be omitted, and there remain only paths for which the sign of the time changes at the points x and x_0.

The sum over paths falls into three parts:

$$(x_0 \sim x) = (x_0 \overset{1}{\underset{1}{\rightleftarrows}} x) + (x_0 \overset{2}{\underset{2}{\rightleftarrows}} x) + \{(x_0 \overset{1}{\underset{2}{\rightleftarrows}} x) + (x_0 \overset{2}{\underset{1}{\rightleftarrows}} x)\}.$$

The arrows represent the path, and the numbers

1 and 2 refer to the apertures. These three terms correspond to the three terms of the quantum-mechanical expression for the probability:

$$W = |\psi_1 + \psi_2|^2 = \psi_1^+\psi_1 + \psi_2^+\psi_2 + (\psi_1^+\psi_2 + \psi_2^+\psi_1).$$

6. THE PAULI PRINCIPLE

We shall show that imposition of the Pauli principle is equivalent to the exclusion from the sum over paths of paths that have a common part traversed in different directions.

Let us consider $W(x, y, t \mid x', y', t')$ — the probability of finding two particles at the points x and y at the time t, if at the time t' there were two particles at the points x' and y'. There are two possibilities: the transition $x' \rightleftarrows x$, $y' \rightleftarrows y$ and the transition $x' \rightleftarrows y$, $y' \rightleftarrows x$. The sum over paths breaks up into two parts, each of which separates into the product of two sums over paths, i.e.,

$$W(x, y \mid x', y') = W(x \mid x')\, W(y \mid y') + W(x \mid y')\, W(y \mid x').$$

From this we must subtract the sum over all paths that have a common part. Suppose the two paths coincide on the transition $\xi'\tau' \rightarrow \xi\tau$.

The path $x' \rightarrow x \rightarrow \xi \rightarrow \xi' \rightarrow x'$, $y' \rightarrow y \rightarrow \xi' \rightarrow \xi \rightarrow y'$ must be subtracted from the sum over paths. But this path comes in with the same contribution as the path

$$x' \rightarrow x \rightarrow y' \rightarrow y \rightarrow x'.$$

(This becomes clear if one draws the corresponding paths.)

Symbolically we can write:

$$x'y' \sim xy = x' \rightleftarrows x \cdot y' \rightleftarrows y + x' \rightleftarrows y \cdot y' \rightleftarrows x - \qquad (12)$$
$$- x' \rightarrow x \rightarrow y' \rightarrow y \rightarrow x' - x' \rightarrow y \rightarrow y' \rightarrow x \rightarrow x'.$$

Since the sum over each of the transitions represented by an arrow is the same as a quantum-mechanical propagation function,

$$W(x, y \mid x', y') = |\,G(x \mid x')\, G(y \mid y') - G(x \mid y')\, G(y \mid x')\,|^2.$$

This is the same as the quantum-mechanical expression (for Fermi particles).

When we consider Bose particles the last two terms in Eq. (12) must be taken with the plus sign. If a boson is a bound state of two fermions, this rule can be explained. The forbidden paths will be subtracted for each of the fermions, and the twice-forbidden term comes in with a plus sign, but since the paths of the two particles must coincide, only this term remains in Eq. (1) (in addition to the main term). The minus sign in Eq. (12) is replaced by a plus sign.

7. COMPARISON WITH THE FEYNMAN FORMULATION

The approach that has been presented differs from that of Feynman primarily as to results. The Feynman formula for the transition amplitude replaces only the Schrödinger equation, whereas Eq. (1) contains within it, in addition to the Schrödinger equation, relations of the types $W = \psi^+\psi$, $p = -ih\partial/\partial x$, and the whole theory of representations. Equation (1) makes it possible to solve any problem, whereas the Feynman principle is insufficient, for example, for the problems (a) and (b) of Sec. 5.

We note that Eq. (1) cannot be obtained by simple multiplication of the Feynman expressions for the amplitudes, since, firstly, a general expression for the amplitudes is not known, and secondly, even in the special case in which the amplitude is equal to $\psi(x, t)$, multiplication of $\psi(x, t)$, written according to reference 1 as a sum over paths, by $\psi^+(x, t)$ does not give the sum over all possible paths as is required by the meaning of Eq. (1) (see below).

The limited nature of Feynman's results is due to a lack of consistency in the description of the "quantum microscopic world", i.e., of such events as the passage of a particle along a prescribed path. Dirac[2] has shown how, without contradiction, one can associate with each path a formal probability, and how this probability differs from the square of the absolute value of the Feynman amplitude [this contradiction has been noted in a paper by Stratonovich,[5] where there are some indications of the existence of the formula (1)]. It can be shown (as will be done in another place) that the difference is due to a different definition of the concept "path of the particle," and that in sums over paths a path must be taken just in the Dirac sence, not in the Feynman sense.

Thus it is necessary to abandon the belief in the deep physical meaning of the concept of amplitude, as it exists in quantum mechanics and in reference 1. According to Eq. (1), not all probabilities separate ínto products of amplitudes.

Another lack of consistency in Feynman's treatment is the absence of paths with change of sign of the time, whereas the basic idea is to include all conceivable possibilities. No limitations at all are to be imposed on the path; they appear only after the averaging. It can be said in advance that in the majority of cases the paths with changing sign of the time will make no contribution in the nonrelativistic region, but inclusion of both signs turns out to be necessary, for example, to get the right value for the average of any function that depends on the velocity (if purely spatial paths are taken). Because paths with both signs of the time are absent in reference 1 the product of the Feynman ψ and ψ^+ does not give the W of Eq. (1), but the ψ defined in Sec. 3 differs from that of Feynman, since in addition to the paths coming from $t = -\infty$ it contains paths coming from $t = +\infty$.

Whereas in reference 1 $\psi(x, t) = (-\infty \rightarrow xt)$, in Sec. 3

$$\psi(x, t) = (-\infty \rightarrow xt) + (xt \leftarrow \infty).$$

A further necessary addition to reference 1 is the refinement made in Sec. 4 of the concept of the sum over paths for paths going out to $t = \pm\infty$. In quantum mechanics wave functions are prescribed by the value of a complete set of quantities, and from this point of view it is not clear what wave function appears in the Feynman relation $\psi(x, t) = (-\infty \rightarrow xt)$. This is a purely symbolic expression. In Sec. 4 it is assumed that such expressions depend on the way in which T goes to infinity (or on an unknown history); this is equivalent to the quantum-mechanical dependence on a complete set of quantities.

Furthermore, we get a logical source of the Feynman rule that in calculating $\overline{v^n}$ (v is the velocity) all the factors must be taken at times arbitrarily close together, but still distinct. By the meaning of Eq. (1), the sum must be taken over paths with arbitrary values of the higher derivatives at the point x, t, and such a sum is not obtained by the use of broken paths. In order for this not to affect the results, the instantaneous value of a physical quantity along a broken path must be defined as the average in an infinitely small neighborhood of the point t (just such quantities are measured in experiments). With such an approach a particle has at the time t both a coordinate x and a momentum $\tilde{p}$ [$\tilde{p} = \frac{1}{2\epsilon}\int_{t-\epsilon}^{t+\epsilon} p\, dt$, ϵ much smaller than the characteristic time of the system, but much larger than ϵ_k in Eq. (2)]. As can be shown, the sum over the paths with prescribed x and p gives the Wigner[3] distribution W(x, p). This distribution cannot, however, be used for the calculation of the average value of an arbitrary quantity f(x, p), since generally speaking $\widetilde{f}(x, p) \neq f(\tilde{x}, \tilde{p})$. This is one of the objections against the statistical interpretation of quantum mechanics (cf. reference 4); it is removed if we suppose that what is measured is always $\widetilde{f}(x, p)$, and not $f(\tilde{x}, \tilde{p})$.

Thus, unlike the situation in reference 1: (1) the probability of any event is written as the sum of formal probabilities (and not the amplitude as a sum of amplitudes, see beginning of paper); (2) paths with changing time direction are taken into account; (3) the peculiarities of paths receding to infinity are taken into account; (4) the special features of broken paths are more consistently treated.

We note that certain amplitudes agree with the formal probabilities. For example, the probability amplitude for a particle to pass through x, t agrees with the formal probability for the particle to arrive at the point x, t. Just for this reason the amplitudes have the properties of probabilities and the Feynman approach (see beginning of paper) is correct in a certain region.

The present treatment is also directly related to references 2 to 5, which are devoted to the statistical interpretation of quantum mechanics, but its purpose and ideas are quite different from those of these papers.

In references 2 to 5 it is shown how one can, within the framework of quantum mechanics, "restore" the classical picture, if it exists. The distribution of paths has been defined, firstly by the principles and rules of quantum mechanics, secondly by the prescription of the wave function or, what is the same thing, by the prescription of a complete set of physical quantities. The idea of a universal distribution is absent (if the history of the particle is known).

Above, unlike in references 2 to 5: (1) the statistical interpretation is used for the derivation of the principles of quantum mechanics (the approach is in a certain sense an inverse one to that of references 2 to 5; (2) the difference of the quantum distributions from the classical (see beginning of Sec. 4) is interpreted in the framework of the statistical picture.

Two considerations — the possibility of "inverting" the exposition of quantum mechanics (with the Gibbs distribution as a model), and the calculation of conditional probabilities by elimination of the unknown history of the particle — are basic to the present communication.

8. CONCLUSION

Equation (1) is of interest from various points of view.

First of all, the logical structure of quantum mechanics is simplified and clarified. Beyond this, the reduction of all the principles of quantum mechanics to Eq. (1) and the possibility of deriving Eq. (1) from simple and general assumptions about the physical world (this question is not discussed in the present paper) may be of importance for possible generalizations of quantum mechanics to new domains.

It can be expected that in the relativistic domain the language of possible classes of paths will turn out to be the only mode of description (like the Gibbs distribution in statistical physics). For example, it may happen that there is a retarded interaction between particles that is not reducible to a field (cf. reference 1). In this case the probability does not separate into the product of wave functions, the Schrödinger equation cannot be written, and problems can be solved only by means of Eq. (1).

On the mathematical side, Eq. (1) broadens the domain of application of the functional integration (as compared with reference 1), and there is a hope that in time simplicity and generality will appear not only in the way of writing the general principle, but also in the methods of solving all concrete problems. At present there are a few problems that are solved very simply by means of Eq. (1).

Also of interest is the possibility of using Eq. (1) to carry over accustomed classical ideas into the microscopic world. True, a simple imposition of the probabilities on the the classical picture is somewhat hindered by the "negative probabilities," but everything that has been presented can be translated in the most varied ways into the language of ordinary probabilities. It can be shown that the "negative probabilities" are in no way connected with the formalism of wave functions and operators and are not the cause of the sharp difference between classical physics and the canonical quantum description. The difference exists only in the means of describing mathematically analogous objects (for example, ensembles of configurations in classical statistical physics and ensembles of paths of particles), and in the problems that have to be solved.

It can also be shown that Eq. (1) bears the same relation to the canonical formalism that the Gibbs distribution does to thermodynamics (there is a precise and far-reaching analogy). Therefore the success of the atomistic (not phenomenological) approach may possibly repeat itself for Eq. (1). It is interesting that Eq. (4) can be united with the Gibbs principle, if one introduces a formal integration over paths with a complex time and takes the change of the time equal to ih/kT (k is Boltzmann's constant and T the temperature). Equation (4) will contain both the Gibbs principle and Hamilton's principle.

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