ON THE CASCADE THEORY OF SHOWERS

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Submitted to JETP editor February 4, 1958

J. Exptl. Theoret. Phys. (U.S.S.R.) **35**, 132-136 (July, 1958)

The equation for the average number of particles with energy greater than a given value, produced in a layer of finite thickness, has been solved with account of collision losses. The solutions obtained are compared with those of the usual cascade theory. An expression has been obtained for the mean energy of particles of a cascade shower at any stage of its development.

EQUATIONS for the distribution function of the number of particles produced in a layer of matter between t and t + dt with energy between E and E + dE, and their solution for the mean number of particles $\epsilon\{P(E_0, E, t)\}$ neglecting ionization losses, have been given in reference 1.

It is interesting to solve the equations for the mean number of particles produced in a layer of thickness dt, taking the ionization losses into account, and to compare the results with those of the usual cascade theory[2] dealing with the number of particles $P(E_0, E, t)$ with energy between E and E + dE at the distance t to t + dt from the considered layer.

The equations in question are (we consider a shower initiated by a primary electron with energy E_0):

$$\varepsilon\{P(E_0, E, t)\} = 2\int_E^{E_0} \Gamma(E_0, E', t)\, W_p(E, E')\, dE', \quad \varepsilon\{\Gamma(E_0, E_0, t)\} = \int_E^{E_0} P(E_0, E', t)\, W_e(E'-E, E')\, dE', \tag{1}$$

where Γ denotes the corresponding photon distributions. Equation (1) can be solved easily by means of the Mellin transformation. For further calculations it is convenient to use the Laplace transforms of the solutions of Eq. (1). We obtain the following expression for the number of electrons produced in the layer between 0 and t with energy greater than E (at the place of production):

$$\varepsilon\{N_p(E_0, E, t)\} = -\frac{1}{4\pi}\int_{\delta-i\infty}^{\delta+i\infty} ds \int_{d-i\infty}^{d+i\infty} d\lambda\, \varepsilon\{P(E_0, s, \lambda)\}\, E^{-s}(e^{\lambda t}-1)/s\lambda, \tag{2}$$

where

$$\varepsilon\{P(E_0, s, \lambda)\} = \Gamma(E_0, s, \lambda)\, B(s), \quad \varepsilon\{\Gamma(E_0, s, \lambda)\} = P(E_0, s, \lambda)\, C(s);$$

the values of $B(s)$, $C(s)$, and $P(E_0, s, \lambda)$ have been determined in reference 2. The explicit expression for $\epsilon\{N\ (E_0, E, t)\}$, accounting for ionization losses, can be written in the form*

$$\varepsilon\{N_p(E_0, E, t)\} = -\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} ds\, 2\sigma_0 D(s)\, H_1'(s) \exp\{\lambda_1(s)\, t + ys - \ln(-\lambda_1(s))\} \tag{3}$$

$$\times\, G_k(s, \varepsilon)/s + \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} ds\, 2\sigma_0 D(s)\, H_1'(s) \exp\{ys - \ln(-\lambda_1(s))\}\, G_k(s, \varepsilon)/s.$$

Here

$$H_1'(s) = 1/(\lambda_1(s) - \lambda_2(s)); \quad \varepsilon = Ef(\lambda_1(s))/\beta; \quad G_k(s, \varepsilon) = \int_0^1 x^{s-1}(x+\varepsilon)^{-s}\left\{\sum_{\substack{n=0\\ n\neq 1}}^{\infty} (-1)^n x^n/n!\,(1-n) - (1-C)\,x\right\} dx;$$

C is the Euler constant, and other values are given in reference 2.

For $\epsilon > 1$, the function $G_k(s, \epsilon)$ can be written in the form

*We use the simplified expressions for B(s) and C(c) given reference 3. These do not differ by more than 3% from the more accurate values, and greatly simplify the calculations.

$$G_k(s, \varepsilon) = \sum_{\substack{n=0 \\ n\neq 1}}^{\infty} (-1)^n \varepsilon^{-s} F(s, s+n, s+n+1, -1/\varepsilon)/n!(1-n)(s+n) + (C-1)\varepsilon^{-s} F(s, s+1, s+2, -1/\varepsilon)/(s+1), \quad (4)$$

where F(a, b, c, x) is the hypergeometric function; the integral can be evaluated numerically for $\epsilon \leq 1$ and noninteger s. It should be noted that the expression written is true for $y = \ln(E_0/\beta) > 1$ and $t > 1$.

The corresponding formulae, obtained neglecting ionization losses, are

$$\varepsilon\{N_p(E_0, E, t)\} = -\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} ds 2\sigma_0 H_1'(s) \exp\{\lambda_1(s)t + ys \quad (5)$$

$$-\ln(-\lambda_1(s))\}/s^2(1+s) + \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} ds 2\sigma_0 H_1'(s) \exp\{ys - \ln(-\lambda_1(s))\}/s^2(1+s).$$

The function $\epsilon\{N_p(E_p, E, t)\}$ should not be compared with $N_p(E_0, E, t)$ of the usual cascade theory, as it has been done in reference 2, but with the function

$$N_{ps}(E_0, E, t) = \int_0^t N_p(E_0, E, \tau)\, d\tau, \quad (6)$$

which represents the area under the curve $N_p(E_0, E, t)$ from 0 to t. Carrying out the necessary computations, we obtain (neglecting the ionization losses):

$$N_{ps}(E_0, E, t) = -\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} ds H_1(s) \exp\{\lambda_1(s)t + ys - \ln(-\lambda_1(s))\}/s + \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} ds H_1(s) \exp\{ys - \ln(-\lambda_1(s))\}/s \quad (7)$$

and, taking the ionization losses into account:

$$N_{ps}(E_0, E, t) = -\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} ds H_1(s) D(s) \exp\{\lambda_1(s)t + ys - \ln(-\lambda_1(s))\} \quad (8)$$

$$\times G(s, \varepsilon)/s + \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} ds H_1(s) D(s) \exp\{ys - \ln(-\lambda_1(s))\} G(s, \varepsilon)/s.$$

where

$$G(s, \varepsilon) = e^{\varepsilon}\int_{\varepsilon}^{\varepsilon+1}\left(1 - \frac{\varepsilon}{x}\right)^s e^{-x} dx,$$

and other values have been given in reference 2. The values of the function $G_k(s, \epsilon)$, calculated according to Eq. (4) for several values of s and ϵ, are given in Table I.

It should be noted that the second term in Eqs. (3) and (5) gives the number of particles produced in the layer $(0-\infty)$ and the first term gives the number of particles produced in the layer $(t-\infty)$. In Eqs. (7) and (8), the second term represents the area under the cascade curve from $t = 0$ to $t = \infty$, and the first term — from t to ∞. The results of computation of the functions $\overline{N}_{ps}(E_0, E, t)$ and $\epsilon\{\overline{N}_p(E_0, E, t)\}$, neglecting and accounting for ionization losses, given in Table II, represent, therefore, the area under the cascade curve and the number of electrons produced in the layer t to ∞, respectively. For the functions calculated neglecting ionization losses, $y = \ln(E_0/E)$; for those accounting for ionization losses, $y = \ln(E_0/\beta)$. In the latter case, particle energy is measured in units of $\beta/f(\lambda_1(s))$.

If the range of electrons is determined solely

TABLE I. The function $G_k(s, \epsilon)$

s	ε = 1.5	3.18	6.36	12.7	25.4
1.00	0.329	0.180	0.0950	0.0489	0.0248
1.04	0.303	0.162	0.0835	0.0418	0.0207
1.10	0.268	0.139	0.0690	0.0332	0.0158
1.15	0.244	0.123	0.0590	0.0275	0.0126
1.25	0.200	0.0958	0.0434	0.0190	0.00815
1.35	0.167	0.0754	0.0321	0.0132	0.00530
1.45	0.139	0.0597	0.0240	0.00918	0.00347

by radiation processes, then the values of $\overline{N}_{ps}(y,t)$ and $\epsilon\{\overline{N}_p(y,t)\}$ should be equal. It can be seen from Table II that, within the accuracy of calculations (< 5%), such is the case. It can therefore be maintained that if ionization losses are taken into account, the functions $N\Gamma_s$ and $\epsilon\{N\Gamma_s(E_0, E, t)\}$ for photons should coincide as well.

In the energy region where the range of electrons is determined by radiation and ionization processes, it is necessary to assume that $\epsilon\{\overline{N}_p(E_0, E, t)\} > \overline{N}_{ps}(E_0, E, t)$. For instance, in the limiting case $E \to 0$, we have $\epsilon\{\overline{N}_p(E_0, E, t)\} \to \infty$, while $\overline{N}_{ps}$ remains finite. For $\epsilon = 1.5$, we have $\epsilon\{\overline{N}_p\} \approx 2\overline{N}_{ps}$, and for $\epsilon = 25.4$, $\epsilon\{\overline{N}_p\} \approx 1.3\,\overline{N}_{ps}$. It is, therefore, necessary to bear in mind the difference between the functions

TABLE II.* The functions $\overline{N}_{ps}(E_0, E, t)$ and $\epsilon\{\overline{N}_p(E_0, E, t)\}$, with and without account of ionization losses, giving the area under the cascade curve, and the number of electrons produced in the layer from t to ∞, respectively.

$y = 15.9$

	ε	t = 0	4.8	7.9	12.7	18.4	23.2	27.8
$N_{ps}(y,t)$		3.61(6)	3.36(6)	3.07(6)	2.29(6)	9.88(5)	3.09(5)	7.06(4)
$\varepsilon\{N_p(y,t)\}$		3.57(6)	3.31(6)	3.02(6)	2.23(6)	9.58(5)	2.99(5)	6.85(4)
N_{ps}	0	8.45(6)	7.93(6)	7.30(6)	5.53(6)	2.48(6)	8.02(5)	1.90(5)
N_{ps}	1.5	2.26(6)	2.12(6)	2.10(6)	1.53(6)	6.26(5)	1.85(5)	4.02(4)
εN_p	1.5	5.31(6)	4.89(6)	4.44(6)	3.25(6)	1.36(6)	4.10(5)	9.12(4)
N_{ps}	3.18	1.64(6)	1.51(6)	1.36(6)	9.68(5)	3.85(5)	1.10(5)	2.28(4)
εN_p	3.18	2.81(6)	2.56(6)	2.30(6)	1.70(6)	6.49(5)	1.86(5)	3.91(4)
N_{ps}	6.36	9.84(5)	8.94(5)	7.99(5)	5.62(5)	2.20(5)	6.27(4)	1.33(4)
εN_p	6.36	1.42(6)	1.30(6)	1.14(6)	7.87(5)	2.94(5)	7.91(4)	1.57(4)
N_{ps}	12.7	5.18(5)	4.53(5)	3.98(5)	2.70(5)	9.71(4)	2.53(4)	4.84(3)
εN_p	12.7	7.04(5)	6.26(5)	5.51(5)	3.67(5)	1.28(5)	3.25(4)	6.01(3)
N_{ps}	25.4	2.60(5)	2.30(5)	2.00(5)	1.31(5)	4.45(4)	1.09(4)	1.97(3)
εN_p	25.4	3.43(5)	3.03(5)	2.61(5)	1.69(5)	5.52(4)	1.30(4)	2.27(3)

$y = 11.7$

	ε	t = 0	2.94	7.36	12.3	16.1	19.7
$N_{ps}(y,t)$		5.35(4)	4.99(4)	3.98(4)	2.06(4)	8.25(3)	2.61(3)
$\varepsilon\{N_p(y,t)\}$		5.27(4)	4.90(4)	3.89(4)	2.00(4)	7.98(3)	2.54(3)
N_{ps}	0	1.26(5)	1.19(5)	9.63(4)	5.17(4)	2.14(4)	7.03(3)
N_{ps}	1.5	3.69(4)	3.42(4)	2.66(4)	1.30(4)	4.95(3)	1.49(3)
εN_p	1.5	7.80(4)	7.22(4)	5.66(4)	2.83(4)	1.09(4)	3.38(3)
N_{ps}	3.18	2.37(4)	2.21(4)	1.68(4)	8.02(3)	2.93(3)	8.44(2)
εN_p	3.18	4.08(4)	3.74(4)	2.85(4)	1.35(4)	4.96(3)	1.45(3)
N_{ps}	6.36	1.42(4)	1.30(4)	9.79(3)	4.58(3)	1.67(3)	4.93(2)
εN_p	6.36	2.07(4)	1.86(4)	1.37(4)	6.12(3)	2.11(3)	5.82(2)
N_{ps}	12.7	7.20(3)	6.48(3)	4.70(3)	2.02(3)	6.75(2)	1.79(2)
εN_p	12.7	9.98(3)	8.95(3)	6.39(3)	2.68(3)	8.67(2)	2.23(2)
N_{ps}	25.4	3.66(3)	3.25(3)	2.29(3)	9.28(2)	2.91(2)	7.31(1)
εN_p	25.4	4.84(3)	4.24(3)	2.94(3)	1.15(3)	3.48(2)	8.41(1)

$y = 9.25$

	ε	t = 0	4.20	8.68	11.9	14.8
$N_{ps}(y,t)$		4.27(3)	3.55(3)	2.06(3)	9.64(2)	3.71(2)
$\varepsilon\{N_p(y,t)\}$		4.19(3)	3.47(3)	2.00(3)	9.33(3)	3.60(2)
N_{ps}	0	1.01(4)	8.60(3)	5.18(3)	2.50(3)	9.98(2)
N_{ps}	1.5	2.92(3)	2.37(3)	1.31(3)	5.78(2)	2.12(2)
εN_p	1.5	6.17(3)	5.06(3)	2.83(3)	1.28(3)	4.79(2)
N_{ps}	3.18	1.88(3)	1.50(3)	8.03(2)	3.43(2)	1.20(2)
εN_p	3.18	3.20(3)	2.54(3)	1.35(3)	5.80(2)	2.06(2)
N_{ps}	6.36	1.11(3)	8.75(2)	4.59(2)	1.96(2)	7.00(1)
εN_p	6.36	1.59(3)	1.47(3)	6.13(2)	2.47(2)	8.26(1)
N_{ps}	12.7	5.53(2)	4.19(2)	2.03(2)	7.89(1)	2.55(1)
εN_p	12.7	7.65(2)	7.34(2)	2.68(2)	1.01(2)	3.16(1)
N_{ps}	25.4	2.08(2)	2.04(2)	9.29(1)	3.40(1)	1.04(1)
εN_p	25.4	3.63(2)	2.62(2)	1.15(2)	4.07(1)	1.19(1)

TABLE II. (Continued)*

$y = 5.94$

	ε	t = 0	3.84	6.32	8.43
$N_{ps}(y, t)$		1.45(2)	9.94(1)	5.79(1)	2.75(1)
$\varepsilon\{N_p(y, t)\}$		1.43(2)	9.63(1)	5.61(1)	2.67(1)
N_{ps}	0	3.51(2)	2.50(2)	1.50(2)	7.41(1)
N_{ps}	1.5	9.68(1)	6.29(1)	3.47(1)	1.57(1)
εN_p	1.5	2.06(2)	1.36(2)	7.69(1)	3.56(1)
N_{ps}	3.18	6.14(1)	3.87(1)	2.06(1)	8.88
εN_p	3.18	1.04(2)	6.52(1)	3.48(1)	1.53(1)
N_{ps}	6.36	3.57(1)	2.21(1)	1.17(1)	5.19
εN_p	6.36	4.99(1)	29.53	14.84	6.13
N_{ps}	12.7	1.71(1)	9.76	4.74	1.89
εN_p	12.7	2.33(1)	1.29(1)	6.09	2.34
N_{ps}	25.4	8.33	4.48	2.04	0.769
εN_p	25.4	10.70	5.55	2.45	0.89

$y = 3.32$

	ε	t = 0	1.89	3.35
$N_{ps}(y, t)$		9.15	6.50	4.15
$\varepsilon\{N_p(y, t)\}$		8.87	6.30	4.03
N_{ps}	0	23.0	16.9	11.2
N_{ps}	1.5	5.80	3.90	2.37
εN_p	1.5	12.6	8.64	5.37
N_{ps}	3.18	3.56	2.31	1.34
εN_p	3.18	6.01	3.91	2.30
N_{ps}	6.36	2.04	1.32	0.78
εN_p	6.36	2.72	1.67	0.92
N_{ps}	12.7	0.900	0.53	0.28
εN_p	12.7	1.19	0.68	0.35

*The number in parenthesis indicates the power of 10, a factor by which the corresponding value should be multiplied.

$N_{ps}(E_0, E, t)$ and $\epsilon\{N_p(E_0, E, t)\}$ in the interpretation of emulsion and diffusion-chamber measurements.

It should be noted that while $N_{ps}(E_0, 0, t) \to E_0/\beta$ for $t \to \infty$[2], direct computations according to Eq. (2) yield a value differing from E_0/β by less than 5% for $y > 5$, which makes it possible to appreciate the influence of the approximations made in the calculation. For $y = 3.32$, the value of $N_{ps}(0, \infty)$ differs from E_0/β by 20%, which indicates that the integrals should be computed more accurately, i.e., the integrand should be expressed in the form $\exp\{\Phi(s, y, t)\}$ and the integral evaluated by the saddle-point method.

It should be noted, furthermore, that the expressions for the function $\overline{N}_{ps}(E_0, E, t)$ make it possible to find easily the mean energy of shower particles at a given depth:

$$\overline{E(t)} = \overline{N}_{ps}(E_0, 0, t)/N_p(E_0, 0, t). \qquad (9)$$

Analogous expressions for the distribution function of electrons in showers initiated by a single primary photon, as well as for the photon distribution function, can be obtained easily.

The author wishes to thank T. V. Klopkova for help in the computations.

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Translated by H. Kasha
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