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ON THE ELECTRON CAPTURE MECHANISM AND THE CURRENT LIMIT IN BETATRONS

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Mechanism of electron capture in betatrons is discussed. Its basis is the Coulomb interaction of the electrons in the beam and the losses of electrons to the walls of the doughnut. The problem is treated exactly for a simplified model. It is shown that the considered capture mechanism has a high effectiveness which is in agreement with experiment. An expression for the limiting current, valid also for relativistic energies, is given.

1. INTRODUCTION

No existing theory explains the capture of electrons in betatrons, nor has a satisfactory physical picture of this process been developed. However, several important experiments[1-6] performed during the last years have clarified this problem to a considerable extent. Thev have confirmed that the capture of electrons in betatrons is due to their collective interactions. It is therefore unnecessary to consider one-electron theories of electron capture.[7-8] Their applicability is limited to rather small injection currents.

One can subdivide the capture mechanisms based on collective interactions into three groups: (a) mechanisms connected with the action of self-induction of the non-stationary electron current in the doughnut;[9-10] (b) mechanisms based on the interaction of the electrons with the Coulomb field of the space charge;[10, 11, 13-15] (c) statistical capture mechanisms.[16]

Experimental models have shown that the mechanism based on the self induction of the non-stationary current[4] cannot explain, at the actual strengths of the injection currents, the observed effectiveness of the capture and does not play an important part in the capture process. Its effectiveness is similar to that of the adiabatic contraction of the orbit and of the adiabatic damping of the betatron oscillations. Thus, one can consider it to be sufficently well established at present that the induction-type mechanism does not play an essential part in the overall picture of the electron capture.

As to the statistical capture mechanism, it has been shown earlier[17] that it can work only at sufficiently small injection currents, in the region between the single-electron capture and the collective capture. At such injection currents where the capture process is particularly effective, this mechanism does not play an essential part.

It appears thus that the most likely injection mechanism is that which takes into account the Coulomb interaction of the electrons. The different effects associated with the Coulomb interaction at injection time are discussed in references 10, 11, and 13 to 15. The mechanism treated by Wideroe is based on the energy lost by electrons passing through a space-charge cloud whose charge density decreases in time. This process cannot be decisive since it does not explain the capture on the leading side of the injection pulse. Bardeen has proposed a mechanism based on the use of the azimuthal inhomogeneity of the space charge. This mechanism is in disagreement with the well-known fact that the capture works equally well with injection from the outside (where $n \approx 1$) as from the inside (where $n \approx 0.5$). It also is in disagreement with the fact that if several injectors are used and the first harmonic of the azimuthal inhomogeneity of the space charge disappears the injection does not get worse. Therefore this mechanism also cannot play an important part.

Rodimov[15] has investigated the capture mechanism which is based on the Coulomb interaction between electron beams that have made a different number of revolutions after injection. We believe that this mechanism correctly describes several important elements of the actual capture process. However, he has assumed that the density of the injected electrons roughly equals the final equilibrium density and states that there will be no capture of the electrons if the injection density differs strongly from the equilibrium value. This last deduction does not agree with the experimental results, which show that the capture is highly effective also at injection densities much higher than at equilibrium. The assumption that the injection and equilibrium densities are equal seems to be too restrictive. It is rather obvious that this condition is almost never fulfilled in practice, since it can be obtained for each betatron only with a very special choice of the parameters and of the quantities that characterize the injection conditions. Another important shortcoming of Rodimov's work is his neglect of the loss of electrons to the walls and on the injector. It is well known that a large part of the injected electrons strike either the walls or the injector and only a small fraction survives to be finally accelerated. It appears therefore that the loss of electrons is actually an essential part of the general capture process.

In this paper we investigate a capture process based on consideration of the Coulomb interaction between the electrons and of the role played by the loss of electrons to the walls and the injector.

2. COMMENTS ON THE FORMULATION OF THE PROBLEM

For an accurate formulation and solution of this problem it would be necessary to use the transport equation. However, because of the extreme nonstationarity of the process and the involved boundary conditions, which furthermore depend on the time, this approach does not seem feasible. We shall therefore use a different approach, which, with some simplifying assumptions, makes a treatment possible.

We shall assume first that the duration of the injection pulse equals the period of one revolution of the injected particle. To simplify the calculations, we take the beam to be of infinite extent along the z axis and neglect the axial asymmetry of the problem when considering the acting forces. This is equivalent to neglecting small quantities proportional to the ratio of the radial width of the doughnut to the radius of the equilibrium orbit.

The walls of the doughnut do not exert any forces on the electron beam: no charges will accumulate on the inside wall since there is no electric field present; charges will accumulate on the outside wall but they have no influence on the electron beam.

Thus stated, the problem can be solved exactly. There is no electron capture in the following three cases: (1) when the electrons do not interact, (2) when the electrons interact but there are no walls, and (3) when the density of the injected electrons is less than a certain value even if the electrons interact and the presence of the walls is taken into account. On the other hand, if the electrons interact, the presence of the walls and of the injector is taken into account, and the density of the injected electrons exceeds a certain value, the electrons are effectively captured.

Thus, rigorous calculations with the approximate model show that the capture mechanism is due to the Coulomb interaction of the electrons and to the electron loss on the walls and in the injector. The removal of the simplifying assumptions and the change of the model to a more realistic situation do not change these conclusions.

3. EQUILIBRIUM DENSITY OF THE ELECTRONS

An electron moving in a betatron is acted upon by the Lorenz forces

$$\mathbf{F} = e\left(\mathbf{E} + \frac{1}{c}[\mathbf{v} \times \mathbf{H}]\right). \tag{1}$$

If a certain volume element is to be in equilibrium, i.e., if it is to move as a whole and if there are to be no forces trying to change the internal arrangement of the particles inside this volume element, then the following condition must be satisfied.

$$\int \mathbf{F} ds = 0, \tag{2}$$

where the integral is to be taken over any arbitrary closed surface inside the volume element. We thus obtain

$$\operatorname{div}\left(\mathbf{E} + \frac{1}{c}[\mathbf{v}\times\mathbf{H}]\right) = 0. \tag{3}$$

We consequently obtain for the equilibrium density ρ_e

$$\rho_e = -(1/4\pi c)\,\mathrm{H\,curl\,v}\,(\mathcal{E}/m_0c^2)^2, \tag{4}$$

which becomes at the center of the beam, i.e., on the equilibrium orbit R_0,

$$\rho_e(R_0) = \frac{1}{4\pi}\frac{e\beta^2}{r_0R_0^2}\left(\frac{\mathcal{E}}{m_0c^2}\right)^3,\quad \beta = \frac{v}{c},\quad r_0 = \frac{e^2}{m_0c^2}. \tag{5}$$

4. PULSATION OF THE ELECTRON BEAM

Let the electron density upon leaving the iector be ρ_0 and let the width of the injector be l_0. We assume that the density is constant over the width of the injector and that the injected electrons have no radial velocity. The beam will pulsate in a coordinate system tied to its center. Electrons located at a distance x from the center of the beam will be repelled electrostatically from the beam center with a force

$$F_{\text{rep}} = 4\pi e\rho_0x_0, \tag{6}$$

where x_0 is the distance of these electrons from the center of the beam at the time of injection. The magnetic focusing forces are given in this coordinate system by

$$F_{\text{att}}(x) = -m_0(v/R_0)^2(1-n)\,x, \tag{7}$$

where R_0 is the equilibrium orbit radius, n the field index, v the electron velocity ($v \ll c$), and m_0 the electron mass. No pulsations occur if $|F_{\text{rep}}| = |F_{\text{att}}(x_0)|$. This corresponds to the following electron density at injection

$$\rho_{e1} = \frac{1-n}{4\pi}\frac{e\beta^2}{r_0R_0^2}. \tag{8}$$

Let us put $\alpha = \rho_0/\rho_{e1}$. Then for $\alpha = 1$ there will be no beam pulsations. For $1/2 < \alpha < 1$ the beam will pulsate with a frequency $(v/R_0)\sqrt{1-n}$. The maximum beam width is given by the width at injection, and the electrons do not cross the center of the beam. For $0 < \alpha < 1/2$ the electrons cross the beam center; the maximum beam width is still the width at injection. In both cases the beam density is minimum at injection. For $\alpha > 1$ the frequency of the pulsation is $(v/R_0)\sqrt{1-n}$ and the beam width at the point of largest dimension is given by

$$l = l_0(2\alpha - 1) \tag{9}$$

and is reached after one-half a pulsation period. The beam width has its minimum at injection. In this case the density is maximum at the instant of injection.

It is easy to see that at each particular time the electron density is constant over the beam cross section.

5. MOTION OF THE ELECTRON BEAM

The center of the beam moves only under the influence of the magnetic focusing forces. When $0 \le \alpha \le 1$, no electrons are lost to the walls. In this case the entire motion does not differ essentially from the one-electron case. After the first few revolutions all electrons hit the rear part of the injector and are lost.* We are therefore interested in the case $\alpha > 1$.

We shall characterize the width of the vacuum chamber by the dimensionless parameter $\beta = (L_0/l_0) > 1$ (L_0 is the width of the vacuum chamber.) If $1 < \alpha < \beta$ the beam will start to expand immediately after injection but will not touch the wall on the injector side. In terms of the variable

$$\tau = (v/R_0)\sqrt{1-n}\,t$$

the period of either the pulsations or the oscillations of the center of the beam is 2π. It is convenient to introduce, in lieu the distance y between the center of the beam and the wall of the vacuum chamber, the dimensionless quantity $\xi = y/l_0$. The coordinates of the walls are then given by $\xi = \pm\beta/2$ and the coordinates of the edges of the injector are $\xi_1^{(0)} = -\beta/2$; $\xi_2^{(0)} = -\beta/2 + 1$. In these dimensionless variables the width of the injector equals unity. Let $\tau = 0$ denote the time at which a particular group of electrons is being emitted. Denoting the coordinates of the boundaries of the beam by ξ_1 and ξ_2, we have the following equations of motion

$$\begin{aligned} &\xi_1'' + \xi_1 = -\alpha/2, \qquad \xi_1(0) = -\beta/2,\quad \xi_1'(0) = 0,\\ &\xi_2'' + \xi_2 = \alpha/2, \qquad \xi_2(0) = -\beta/2 + 1,\quad \xi_2'(0) = 0, \end{aligned} \tag{10}$$

which have the solutions

$$\begin{aligned} \xi_1 &= -\frac{\alpha}{2} + \left(\frac{\alpha-\beta}{2}\right)\cos\tau,\\ \xi_2 &= \frac{\alpha}{2} + \left(1 - \frac{\alpha+\beta}{2}\right)\cos\tau. \end{aligned} \tag{11}$$

The condition $\alpha < \beta$ insures that the boundary of the beam, ξ_1, does not touch the wall on the injector side right after emission. At the instant τ_0, given by the condition

$$\xi_2(\tau_0) = \beta/2, \tag{12}$$

the beam will hit the wall opposite of the injector and the loss of electrons will commence. This loss will change the dynamics of the beam, and two competing processes will take place. On the one hand the loss of electrons to the walls eliminates the repulsive forces on the remaining

*If the injection pulse is longer than the time of one revolution, collective capture occurs also for $0 < \alpha < 1$.

electrons from the side where the electrons hit the wall. This is equivalent to an additional force attracting the remaining electrons to the wall. Thus, for example, the boundary of the beam ξ_1, will be closer to the wall in the presence of electron loss than if there were no loss. On the other hand, the electron loss causes the center of the beam to move away from the wall. This process reduces the amplitude of the oscillations of the center of the beam.

For $\tau > \tau_0$, i.e., after the start electron loss to the walls, the equations of motion for ξ_1 are

$$\xi_1'' + \left(1 - \frac{\alpha}{2f}\right)\xi_1 = -\frac{\alpha\beta}{4f}, \qquad f = \alpha(1 - \varepsilon\cos\tau),\ \varepsilon = \frac{\alpha - 1}{\alpha} < 1,\ \tau > \tau_0. \tag{13}$$

The initial conditions are given by the values of $\xi_1(\tau_0)$ and $\xi_1'(\tau_0)$ obtained from (11). Utilizing the particular solution of (13),

$$\xi_1 = -\beta/2 + \beta\varepsilon\cos\tau,$$

the problem reduces to solving Hill's equation. The theory of this equation is well developed. However, bearing in mind that the quantity $\alpha/2f$ does not change much during the interval of interest, $\tau > \tau_0$, it is easier to use approximation methods.

If $\xi \ll 1$, we obtain for the maximum excursion $\xi_{1\max}$, at which the electron loss ceases, the following expression:

$$\xi_{1\max} = \frac{\beta}{2} - \alpha + \frac{\alpha^2}{2(\beta - 1)}\varepsilon^2 + O(\varepsilon^3). \tag{14}$$

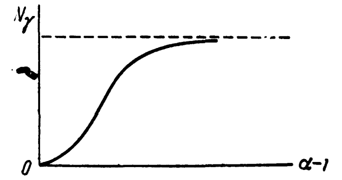
The new center of the beam formed of the remaining electrons is located at the time of maximum beam excursion at a distance $\alpha/2$ from the wall. This means that the amplitude of the beam-center oscillations has been reduced. From now on the electrons will no longer hit the walls, but only the back of the injector. However, since the amplitude of the oscillations has decreased, a certain fraction of the electrons will miss the injector and will survive. One can calculate from (14) the number of electrons that will miss the injector and will be accelerated. This way we obtain

$$N_\gamma = N_0(\alpha - 1)^2/(2\alpha - 1) \approx N_0(\alpha - 1)^2, \quad \alpha \approx 1. \tag{15}$$

Here N_0 is the number of injected electrons and N_γ the number of electrons captured in acceleration.

If α increases, the number of captured electrons increases. If $\alpha > \beta$, Eq. (10) must be replaced from the beginning with equations analogous to (13), to account for the loss of electrons to the

FIG. 1



wall on the injector side. Proceeding in this manner, one can calculate the number of captured electrons as a function of the number of injected electrons. The curve shown in Fig. 1 indicates the character of this dependence. As the density of the injected electrons increases, the number of captured electrons approaches asymptotically a value which, when spread iniformly over the cross section of the doughnut, would give a density $\rho_{e1}/2$, where ρ_{e1} is given by (8).

This behavior of the number of captured electrons can be understood from the following considerations, which explain the capture process at high electron dinsities, $\alpha \gg \beta$. Right after emission, the electrons will begin hitting the wall on the side of the injector. After a short time, less than the time of one revolution, the other boundary of the beam will reach the opposite wall. Hereafter the complete cross section of the doughnut will be filled with space charge and electrons will be lost to all walls. The electron density will now steadily decrease. At the instant when the expansion of the beam ceases and the loss of electrons to the wall terminates, the electron density is given by

$$\rho = \rho_{e1}\alpha/(2\alpha - 1) \approx \rho_{e1}/2, \quad \alpha \gg 1, \tag{16}$$

and the beam fills the entire doughnut cross section. The beam center is located close to the center of the doughnut and continues to oscillate about it with a very small amplitude. Thus almost all electrons distributed with equilibrium density over the full doughnut cross section will be captured to be accelerated.

6. THE CASE OF A ROUND ELECTRON BEAM

The results above are based on a rigorous treatment of a simplified model. The main simplifying assumption, which must be analyzed, is that the beam has a infinite extent along the z axis. If the beam is assumed finite along the z axis, the exact solution becomes rather involved. However, we can estimate the changes that will be introduced into the above picture of the capture process by the finite length of the beam in the z direction.

We assume the focusing magnetic field to be approximately symmetrical. This does not seem to be an overidealization. We can thus assume the

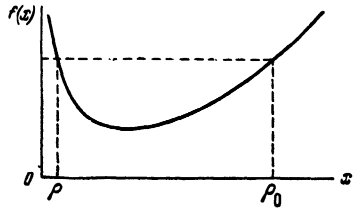


FIG. 2

electron beam to have a round cross section. Then its pulsations will be axially symmetrical with respect to its center. Let r denote the radius of that layer of electrons which had a radius r_0 at injection. We then have the following equation of motion for r

$$r'' + \frac{v^2}{2R_0^2} r = \frac{v^2}{2R_0^2} q \frac{r_0^2}{r}, \quad q = \frac{\rho_0}{\rho_e} > 1, \quad r(0) = r_0, \quad r'(0) = 0, \tag{17}$$

where ρ_0 is the electron density at injection time and ρ_e is the density given by Eq. (5) (in the present nonrelativistic case $\mathscr{E}/m_0c^2 \approx 1$). The solution of (17) is given by the quadrature

$$t = \frac{R_0}{v} \int_1^{r/r_0} d\xi / \sqrt{q \ln \xi - (\xi^2 - 1)/2}. \tag{18}$$

The solution has therefore the form

$$r = r_0 \varphi(t). \tag{19}$$

This means that the period of oscillation is the same for all points within the beam. The density thus remains uniform over the beam. The period of oscillation can be obtained from Eq. (18) in an obvious way. One sees immediately from (17) that for small amplitudes this period is smaller than the period of oscillation of the center of the beam about the equilibrium orbit.

The beam radius r_{max} at the widest point is given by the root of the equation

$$q \ln \frac{r_{max}}{r_0} - \frac{1}{2}\left[\left(\frac{r_{max}}{r_0}\right)^2 - 1\right] = 0. \tag{20}$$

The beam density ρ at maximum beam expansion is evidently

$$\rho = \rho_0 (r_0 / r_{max})^2. \tag{21}$$

Transforming (20) algebraically and utilizing (21) we obtain an equation for the density ρ:

$$\rho_0 \exp(\rho_e/\rho_0) = \rho \exp(\rho_e/\rho). \tag{20a}$$

This equation can easily be solved graphically. After plotting the function $f(x) = x \exp(\rho_e/x)$, we find the connection between ρ_0 and ρ by the construction indicated in Fig. 2.

If we rewrite (20a) in the form

$$\frac{\rho}{\rho_e} \ln \frac{\rho_e}{\rho} - 1 = \frac{\rho}{\rho_e} \ln \frac{\rho_e}{\rho_0} - \frac{\rho}{\rho_0},$$

We can express the solution for the case $\rho_0/\rho_e >> 1$ as

$$\rho \approx \rho_e / \ln(\rho_0/\rho_e). \tag{22}$$

It should be noted here that there is an essential difference in the behavior of the electron density at the maximum of the beam expansion compared with the previously-treated case, when this density approached $\rho_{e1}/2$ with increasing density of the injected electrons. In the present case, on the other hand, the density at maximum beam expansion approaches zero asymptotically with increasing density of the injected electrons. As a result, the curve analogous to Fig. 1 has here a different trend: at very large densities of the injected electrons, the number of captured electrons decreases with increasing injection density. At low injection densities the picture is not qualitatively different from Fig. 1. With increasing density the number of captured electrons increases, owing to the increase in the fraction of the doughnut volume filled by the electron beam at the time when the losses to the wall cease. If the beam fills at that time the whole effective doughnut volume, the saturation of the captured beam current has been reached. If the injection density is further increased, the useful doughnut volume does not increase further but the electron density in this cross section continues to decrease and the number of captured electrons, $N\gamma$, starts to decline. The dependence of the number of captured electrons on the injection density is thus represented by a curve of the form shown in Fig. 3.

As can be seen from (22), the curve decreases very slowly in the region past the maximum, since it has only a logarithmic dependence on the injection density. One therefore can utilize Eq.(22) near the maximum. Denoting the area of the injector by S and the effective area of the doughnut by S_{eff}, then if

$$\rho_0/\rho_e \gg S_{eff}/S > 1 \tag{23}$$

we obtain for the maximum number of captured electrons, $N\gamma_{max}$, the equation

$$N_{\gamma\, max} = \rho_e V_{eff} / e \ln(\rho_{0\, max}/\rho_e), \tag{24}$$

where V_{eff} is the effective doughnut volume and

FIG. 3

$\rho_{0\,max}$ is the injection electron density at saturation.

7. CONCLUSIONS

The above calculations and considerations give a clear physical picture of the processes that lead to the capture of electrons at an efficiency that is in agreement with the experiment. This physical picture allows in principle a rigorous mathematical formulation of the problem. However, the solution of this problem is very involved and depends in general on many parameters (form and dimensions of the injector, density distribution of the electrons over the area of the injector, etc.) Evidently it is not worth while to solve this problem in general. The functional dependence and the general character of the dependence of the capture on the different factors can be understood from the general physical picture without a detailed quantitative solution of the problem.

In the present paper we have assumed the injection pulse to have a duration equal to the time of one electron revolution around the betatron orbit. If the injection pulse length exceeds this value one has to take into account the Coulomb interaction between electron beams which have completed a different number of revolutions. The role of this kind of interaction has been investigated by Rodimov[15] for the case $q \approx 1$. For arbitrary values of q this problem becomes considerably more involved. However, the physical picture of the capture process remains essentially unchanged.

In the special case of very high injection electron densities, when conditions (23) are fulfilled and when the duration of the injection pulse exceeds the time of one revolution (these conditions apply in the example of reference 16 on which we have made comments in reference 17, one can make the definite qualitative and quantitative findings which have been given in reference 17. We only add that in this case the main contribution to the electron capture will obviously come from the trailing edge of the injection pulse, which must lie within the acceptance region of the V_{inj} vs. t diagram of the injection voltage.

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