

$$W(\vartheta_1, \vartheta_2, \mathbf{k}_1, \mathbf{n}_2) \sin \vartheta_1 \sin \vartheta_2 d\vartheta_1 d\vartheta_2 d\mathbf{k}_1 d\Omega_{2k} = \frac{2|A_p|^2}{3k_0/E_0} \frac{(E_0 - E_1)|k_0 - k_1|^3 \sqrt{(E_0 - E_1)^2 - m^2}}{(2\pi)^7} \{2|I^-|^2 + |I^+|^2\} \\ \times \frac{\sin \vartheta_1 \sin \vartheta_2}{(\cos \vartheta_1 + \cos \vartheta_2)^4} \sin \vartheta_1 \sin \vartheta_2 d\vartheta_1 d\vartheta_2 d\mathbf{k}_1 d\Omega_{2k}, \quad (41)$$

$$\rho_{1,2} = |k_0 - k_1| \sin \vartheta_{2,1} / (\cos \vartheta_1 + \cos \vartheta_2). \quad (42)$$

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<sup>2</sup>B. L. Ioffe and I. M. Shmushkevich, Dokl. Akad. Nauk SSSR **82**, 869 (1952).

<sup>3</sup>N. V. Dushin, Dokl. Akad. Nauk SSSR **106**, 977 (1956), Soviet Phys. "Doklady" **1**, 128 (1956).

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72

## PHENOMENOLOGICAL THEORY OF SUPERFLUIDITY NEAR THE $\lambda$ POINT

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A complete set of phenomenological equations is derived to describe the behavior of superfluid helium near the  $\lambda$  point. The normal part of the liquid is described in the usual way, while the superfluid part is described by an "effective" wave function.

IN the present work, equations obtained earlier by V. L. Ginzburg and the author to describe the behavior of helium II in the immediate vicinity of the  $\lambda$  point<sup>1</sup> are extended to include the nonstationary case. In contrast with the ordinary hydrodynamics of helium II, in the scheme considered here the density of the superfluid part  $\rho_s$  is not assumed a given function of  $p$  and  $T$ , but is determined from the equations; these equations characterize the approach of  $\rho_s$  to its equilibrium value. As in reference 1, the superfluid part of the liquid is described by a complex function  $\psi(x, y, z, t) = \eta e^{i\varphi}$  which is so defined that

$$\rho_s = m|\psi|^2, \quad \mathbf{v}_s = \frac{\hbar}{m} \nabla \varphi \quad (1)$$

( $m$  is the mass of the helium atom). This function is introduced to take account of the quantum nature of the effect; its role in this scheme is the

same as that of the expansion parameter in the usual theory of second-order phase changes.<sup>2</sup> The helium state is characterized by the density of the liquid  $\rho$ , the velocity of the normal part  $\mathbf{v}_n$ , and the entropy per unit volume  $S$  in addition to  $\psi$ .

### 1. BASIC EQUATIONS

In reference 1 an equation was obtained to determine the equilibrium values of  $\psi$  for  $\mathbf{v}_n = 0$ . Here, we first examine the equilibrium condition for the case  $\mathbf{v}_n \neq 0$ . At low values of the velocities  $\mathbf{v}_n$  and  $\mathbf{v}_s$ , the energy per unit volume of the liquid can be written in the form

$$E = (\rho - m|\psi|^2) \frac{v_n^2}{2} + \frac{\hbar^2}{2m} |\nabla \psi|^2 + \varepsilon(\rho, S, |\psi|^2). \quad (1.1)$$

This expression is the first term in the energy expansion in terms of the velocity  $\mathbf{v}_n$  and the

gradient of  $\psi$ . Expressing  $\psi$  in terms of  $\rho_S$  and  $\mathbf{v}_S$  we rewrite Eq. (1.1) in the form

$$E = \rho_n \frac{v_n^2}{2} + \rho_s \frac{v_s^2}{2} + \varepsilon + \frac{\hbar^2}{8m^2} \frac{(\nabla \rho_s)^2}{\rho_s}; \quad (1.1a)$$

$$\rho_n = \rho - m|\psi|^2.$$

The first three terms in (1.1a) are the usual energy density, while the last term is a specific quantum energy which can be attributed to the uncertainty principle. The equilibrium condition can be obtained by minimizing (1.1) with respect to  $\psi^*$  or  $\psi$ . However, this procedure must be carried out for the total momentum  $\mathbf{j}$  per unit volume of the liquid; if this procedure were not followed we would also be minimizing the kinetic energy of the liquid as a whole, a quantity which bears no relation to the thermodynamic properties. Thus  $\mathbf{v}_n$  is a function of  $\psi$  and  $\mathbf{j}$ . The expression for  $\mathbf{j}$  follows immediately from the definition of  $\psi$  [Eq. (1)]:

$$\mathbf{j} = (\rho - m|\psi|^2) \mathbf{v}_n + \rho_s \mathbf{v}_s$$

$$= (\rho - m|\psi|^2) \mathbf{v}_n + \frac{i\hbar}{2} (\psi \nabla \psi^* - \psi^* \nabla \psi). \quad (1.2)$$

We use the Lagrangian-multiplier method to express  $\mathbf{v}_n$  in terms of  $\mathbf{j}$  and  $\psi$  from Eq. (1.2). Multiplying (1.2) by an undetermined multiplier  $\mu$  and adding the result to (1.1) we obtain the expression

$$(\rho - m|\psi|^2) \frac{v_n^2}{2} + \frac{\hbar^2}{2m} |\nabla \psi|^2 + \varepsilon$$

$$+ \mu \left\{ \mathbf{j} - (\rho - m|\psi|^2) \mathbf{v}_n - \frac{i\hbar}{2} (\psi \nabla \psi^* - \psi^* \nabla \psi) \right\}. \quad (1.3)$$

Varying (1.3) independently with respect to  $\psi^*$  and  $\mathbf{v}_n$  (with fixed values of  $\rho$ ,  $\mathbf{j}$ ,  $S$  and  $\psi$ ), and eliminating  $\mu$  we obtain the required condition

$$\frac{1}{2} \left( -\frac{i\hbar}{m} \nabla - \mathbf{v}_n \right)^2 \psi + \left( \frac{\partial \varepsilon}{\partial \rho_s} \right)_{\rho, S} \psi = 0, \quad (1.4)$$

this expression takes the place of the basic equation in reference 1.

We turn now to the immediate problem, that of deriving a complete system of equations for  $\psi$ ,  $\mathbf{v}_n$ ,  $S$ , and  $\rho$ . As is well known,<sup>3,4</sup> in ordinary two-velocity hydrodynamics the equations are determined uniquely by the conservation laws. There is an analogous situation in the present case. In order to simplify the analysis we shall formulate the equations using heuristic reasoning and then demonstrate that these equations do in fact satisfy the conservation laws.

The scheme being developed here is based on the assumption that the state of the system at any

instant of time is completely determined by  $\psi$ ,  $\mathbf{v}_n$ ,  $\rho$ , and  $S$ . In particular, the derivative  $\partial\psi/\partial t$  must also be determined by these quantities at a given time. If the above assumptions are valid the change of  $\psi$  with time must be described by a first-order equation which, by analogy with quantum mechanics, can be written in the form

$$i\hbar \partial\psi/\partial t = \hat{L}\psi, \quad (1.5)$$

where  $\hat{L}$  is a nonlinear operator which acts on  $\psi$ . It will be apparent that the operator  $\hat{L}$  cannot be Hermitian. If  $\hat{L}$  were Hermitian, it would follow from Eq. (1.5), in the usual way, that

$$\frac{d}{dt} \int |\psi|^2 dV = \frac{d}{dt} \frac{1}{m} \int \rho_s dV = 0, \quad (1.6)$$

and we would obtain the usual conservation relation for  $\rho_s$ . But Eq. (1.6) cannot hold if we consider relaxation, i.e., the fact that  $\rho_s$  is "converted" into  $\rho_n$  up to the point at which equilibrium is reached. Thus, the anti-Hermitian part of  $\hat{L}$  is associated with the process by which thermodynamic equilibrium is approached; it becomes larger the more rapidly the system relaxes.

As in (1.1) we limit ourselves to the first term in the expansion in the gradient of  $\psi$ , writing the Hermitian part of  $\hat{L}$  in a form similar to the right-hand side of the Schrödinger equation

$$-(\hbar^2/2m) \Delta\psi + U\psi, \quad (1.7)$$

where the function  $U$  will be determined below. At the outset it is convenient to write  $U$  in the form

$$U = (\partial\varepsilon/\partial|\psi|^2)_{\rho_n, S} - gm$$

$$= (\partial\varepsilon/\partial\rho)_{\rho_s, S} m + (\partial\varepsilon/\partial\rho_s)_{\rho, S} m - gm. \quad (1.8)$$

Separation of the term containing the derivative  $(\partial\varepsilon/\partial|\psi|^2)$  corresponds to the physical meaning of this quantity as the "potential energy" of the superfluid part. The quantity  $g$  is to be determined from the conservation relation.

As we have already indicated, the anti-Hermitian part of  $\hat{L}$  characterizes the approach to equilibrium and must vanish at equilibrium, i.e., when the left-hand side of Eq. (1.4) is zero. Hence, if the deviation from equilibrium is small, the anti-Hermitian part of  $\hat{L}$  can be written in the form

$$-i\Lambda \left\{ \left( -\frac{i\hbar}{m} \nabla - \mathbf{v}_n \right)^2 m\psi + (\partial\varepsilon/\partial\rho_s)_{\rho, S} m\psi \right\}. \quad (1.9)$$

Here  $\Lambda$  is a dimensionless factor which is inversely proportional to the relaxation time.

Finally we obtain the equation for  $\psi$ :

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \Delta \psi + \left\{ \left( \frac{\partial \epsilon}{\partial \rho} \right)_{\rho, s} + \left( \frac{\partial \epsilon}{\partial \rho_s} \right)_{\rho, s} - g \right\} m \psi - i\Lambda \left\{ \frac{1}{2} \left( -\frac{i\hbar}{m} \nabla - \mathbf{v}_n \right)^2 + \left( \frac{\partial \epsilon}{\partial \rho_s} \right)_{\rho, s} \right\} m \psi. \quad (1.10)$$

It does not follow from the above reasoning that  $\Lambda$  must be pure real. We assume to begin with that  $\Lambda$  is complex:  $\Lambda = \Lambda_1 + i\Lambda_2$ . Then, multiplying (1.10) by  $\psi^*$  and subtracting the complex conjugate equation, we easily verify that with  $\Lambda_2 \neq 0$  part of the superfluid mass is carried with the normal velocity rather than with the superfluid velocity; obviously this contradicts the physical meaning of these quantities. On this basis, which admittedly is not a rigorous one, we take  $\Lambda_2 = 0$ , i.e., we assume that  $\Lambda$  is real.

In carrying out the calculations it is sometimes useful to write Eq. (1.10) in symbolic form

$$i\hbar \partial \psi / \partial t = \hat{A} \psi - i\Lambda \hat{B} \psi \quad (1.10a)$$

using the properties of  $\hat{A}$  and  $\hat{B}$ :

$$\begin{aligned} \psi^* \hat{A} \psi - \psi \hat{A}^* \psi^* &= -(i\hbar/m) \operatorname{div} \rho_s \mathbf{v}_s, \\ \psi^* \hat{B} \psi - \psi \hat{B}^* \psi^* &= -(i\hbar/m) \operatorname{div} (\mathbf{j} - \rho \mathbf{v}_n). \end{aligned} \quad (1.11)$$

The present theory must satisfy the conservation laws in addition to Eq. (1.10). The conservation-of-mass relation is of the obvious form

$$\partial \rho / \partial t + \operatorname{div} \mathbf{j} = 0, \quad (1.12)$$

where  $\mathbf{j}$  is determined by Eq. (1.2).

In the momentum flux  $\Pi_{ik}$ , which satisfies the conservation law

$$\partial j_i / \partial t + \partial \Pi_{ik} / \partial x_k = 0, \quad (1.13)$$

we can immediately separate the terms

$$\begin{aligned} \Pi_{ik} &= \frac{\hbar^2}{4m^2} \left\{ \frac{\partial \psi}{\partial x_i} \frac{\partial \psi^*}{\partial x_k} - \psi \frac{\partial^2 \psi^*}{\partial x_i \partial x_k} + \text{c.c.} \right\} \\ &+ \rho_n v_{ni} v_{nk} + p \delta_{ik} - \tau_{ik} \end{aligned} \quad (1.14)$$

( $p$  is the pressure and c.c. denotes the complex conjugate of a given expression). The separation of the first term is due to the fact that Eq. (1.10) is analogous to the Schrödinger equation and this term acts as the expression for momentum flux which follows from the usual Schrödinger equation. Here  $\tau_{ik}$  is a symmetric tensor which is to be determined; it can be given in the form

$$\tau_{ik} = \tau \delta_{ik} + \mu_{ik}, \quad (1.15)$$

where  $\mu_{ik}$  is a tensor whose trace is zero ( $\mu_{ii} = 0$ ).

We write the entropy equation in the form

$$\partial S / \partial t + \operatorname{div} \{ S \mathbf{v}_n - \mathbf{q} \} = R / T. \quad (1.16)$$

In writing Eq. (1.16) we have used the fact that the entropy is carried with the normal part of the liquid. In Eq. (1.16)  $\mathbf{q}$  is the entropy flux, as yet unknown, and  $R$  is the dissipation function for the liquid, i.e., the energy dissipation per unit time per unit volume. The problem now is essentially that of choosing the quantities  $g$ ,  $\tau_{ik}$ ,  $\mathbf{q}$  and  $R$  in such a way as to satisfy energy conservation. This requirement turns out to be a sufficient condition for the unique determination of all these quantities.

## 2. EQUATION OF ENERGY CONTINUITY

In view of Eqs. (1.10), (1.12), (1.13), and (1.16), energy conservation requires that the following relations be satisfied identically:

$$\partial E / \partial t + \operatorname{div} \mathbf{Q} = 0 \quad (2.1)$$

( $\mathbf{Q}$  is the energy loss vector). Now we differentiate the right-hand side of Eq. (1.1) with respect to time, expressing  $\dot{\psi}$ ,  $\dot{\mathbf{v}}_n$ ,  $\dot{\rho}$ , and  $\dot{S}$  from Eqs. (1.10), (1.12), (1.13), and (1.16) and choosing  $g$ ,  $\tau_{ik}$ ,  $\mathbf{q}$  and  $R$  in such a way as to cancel terms which cannot be written in the form of a divergence. Expressing  $\mathbf{v}_n$  in Eq. (1.1) in terms of  $\rho$ ,  $\mathbf{j}$ , and  $\psi$ , and differentiating and collecting terms with derivatives of the same order we have:

$$\frac{\partial E}{\partial t} = \mathbf{v}_n \frac{\partial \mathbf{j}}{\partial t} + \left[ \left( \frac{\partial \epsilon}{\partial \rho} \right)_{s, \psi} - \frac{v_n^2}{2} \right] \frac{\partial \rho}{\partial t} + \left( \frac{\partial \epsilon}{\partial S} \right)_{\rho, s, \psi} \frac{\partial S}{\partial t} + \left\{ \frac{\partial \psi}{\partial t} \left[ \left( \frac{\partial \epsilon}{\partial \rho_s} \right)_{\rho, s} + \frac{v_n^2}{2} - \frac{i\hbar}{2m} \mathbf{v}_n \nabla \right] m \psi^* + \left( \frac{\hbar^2}{2m} \nabla \psi^* + \frac{i\hbar}{2} \psi^* \mathbf{v}_n \right) \frac{\partial \nabla \psi}{\partial t} + \text{c.c.} \right\}. \quad (2.2)$$

Assuming that  $\partial \epsilon / \partial S = T$  and substituting the time derivatives of the equations, and after some lengthy calculations which are carried out most conveniently in symbolic form (1.10a), we finally obtain from (1.11):

$$\begin{aligned} \frac{\partial E}{\partial t} + \operatorname{div} \mathbf{Q} &= \mathbf{v}_n \left( S \nabla T + \rho \nabla \frac{\partial \epsilon}{\partial \rho} + \rho_s \nabla \frac{\partial \epsilon}{\partial \rho_s} - \nabla p \right) + R - \frac{2\Lambda}{\hbar} |\hat{B} \psi|^2 - \tau \operatorname{div} \mathbf{v}_n - g \operatorname{div} (\mathbf{j} - \rho \mathbf{v}_n) - \mathbf{q} \nabla T \\ &- \frac{1}{2} \mu_{ik} \left( \frac{\partial v_{ni}}{\partial x_k} + \frac{\partial v_{nk}}{\partial x_i} - \frac{2}{3} \delta_{ik} \operatorname{div} \mathbf{v}_n \right), \end{aligned} \quad (2.3)$$

where

$$Q_i = \left( \frac{i\hbar}{2m} \frac{\partial \psi^*}{\partial x_i} \hat{A}\psi + \frac{\Lambda\hbar}{2m} \frac{\partial \psi^*}{\partial x_i} \hat{B}\psi + \text{c.c.} \right) + \left[ TS + \frac{1}{2} \rho_n v_n^2 + \left( \frac{\partial \varepsilon}{\partial \rho} \right)_{\rho, S} \rho_n - \tau + \rho_s g + \frac{\hbar\Lambda}{2m} \text{div}(\mathbf{j} - \rho \mathbf{v}_n) \right] v_{ni} - Tq_i - v_{nh} \mu_{ih}. \quad (2.4)$$

In order to satisfy energy conservation, the right-hand side of Eq. (2.3) must vanish. Then  $\mathbf{Q}$  is the energy loss vector. Setting the right-hand side of Eq. (2.3) equal to zero, we first obtain an expression for the pressure in terms of the variables  $\rho$ ,  $S$  and  $\rho_S$  (it is assumed that the pressure can be determined in terms of the thermodynamic variables without their coordinate derivatives).

$$\nabla p = \rho \nabla \left( \frac{\partial \varepsilon}{\partial \rho} \right)_{\rho, S} + S \nabla T + \rho_s \nabla \left( \frac{\partial \varepsilon}{\partial \rho_s} \right)_{\rho, S} = \nabla \left[ \rho \left( \frac{\partial \varepsilon}{\partial \rho} \right)_{\rho, S} + \rho_s \left( \frac{\partial \varepsilon}{\partial \rho_s} \right)_{\rho, S} + TS - \varepsilon \right]. \quad (2.5)$$

Thus

$$p = -\varepsilon + TS + \rho \left( \frac{\partial \varepsilon}{\partial \rho} \right)_{\rho, S} + \rho_s \left( \frac{\partial \varepsilon}{\partial \rho_s} \right)_{\rho, S}. \quad (2.6)$$

Further, for  $R$  we find:

$$\begin{aligned} R &= \frac{2\Lambda}{\hbar} |\hat{B}\psi|^2 + \mathbf{q} \nabla T + \frac{1}{2} \mu_{ih} \left( \frac{\partial v_{ni}}{\partial x_h} + \frac{\partial v_{nh}}{\partial x_i} - \frac{2}{3} \delta_{ih} \text{div} \mathbf{v}_n \right) + \tau \text{div} \mathbf{v}_n + g \text{div}(\mathbf{j} - \rho \mathbf{v}_n) \\ &= \frac{\Lambda}{2\hbar |\psi|^2} (\psi^* \hat{B}\psi + \psi \hat{B}^* \psi^*) + \mathbf{q} \nabla T + \frac{1}{2} \mu_{ih} \left( \frac{\partial v_{ni}}{\partial x_h} + \frac{\partial v_{nh}}{\partial x_i} - \frac{2}{3} \delta_{ih} \text{div} \mathbf{v}_n \right) + \tau \text{div} \mathbf{v}_n + \left[ \frac{\Lambda\hbar}{2m^2 |\psi|^2} \text{div}(\mathbf{j} - \rho \mathbf{v}_n) + g \right] \text{div}(\mathbf{j} - \rho \mathbf{v}_n). \end{aligned} \quad (2.7)$$

Reasoning as in reference 5, we see that in small deviations from the equilibrium state the following equality must be satisfied when  $R$  is quadratic in the velocity derivatives and thermodynamic variables:

$$\mathbf{q} = \frac{\kappa}{T} \nabla T, \quad \mu_{ih} = \eta \left( \frac{\partial v_{ni}}{\partial x_h} + \frac{\partial v_{nh}}{\partial x_i} - \frac{2}{3} \delta_{ih} \text{div} \mathbf{v}_n \right). \quad (2.8)$$

It is apparent from Eq. (2.7) that  $\tau$  and  $g$  contain those terms with the second viscosities of helium which are not related to the relaxation  $\rho_S$ . However, close to the  $\lambda$  point the relaxation of  $\rho_S$  makes the chief contribution in the second viscosities. Hence, terms with  $\tau$  and  $g$  are set equal to zero.

It is obvious that  $\kappa$  is the thermal conductivity and  $\eta$  is the first viscosity of the normal part.

We now have the final system of equations (the extremely complicated expression for  $\mathbf{Q}$  is not of general interest and is not given here):

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \Delta \psi + \left[ \left( \frac{\partial \varepsilon}{\partial \rho} \right)_{\rho, S} + \left( \frac{\partial \varepsilon}{\partial \rho_s} \right)_{\rho, S} \right] m \psi - i\Lambda \left[ \frac{1}{2} \left( -\frac{i\hbar}{m} \nabla - \mathbf{v}_n \right)^2 + \left( \frac{\partial \varepsilon}{\partial \rho_s} \right)_{\rho, S} \right] m \psi; \quad (2.9)$$

$$\frac{\partial \rho}{\partial t} + \text{div}(\rho - m |\psi|^2) \mathbf{v}_n + \frac{i\hbar}{2} (\psi \Delta \psi^* - \psi^* \Delta \psi) = 0; \quad (2.10)$$

$$\begin{aligned} \frac{\partial}{\partial t} \left\{ (\rho - m |\psi|^2) v_{ni} + \frac{i\hbar}{2} \left( \psi \frac{\partial \psi^*}{\partial x_i} - \psi^* \frac{\partial \psi}{\partial x_i} \right) \right\} &= -\frac{\partial}{\partial x_h} \left\{ (\rho - m |\psi|^2) v_{ni} v_{nh} + \frac{\hbar^2}{2m} \left( \frac{\partial \psi}{\partial x_i} \frac{\partial \psi^*}{\partial x_h} - \psi \frac{\partial^2 \psi^*}{\partial x_i \partial x_h} + \text{c.c.} \right) \right. \\ &\quad \left. + \rho \delta_{ih} - \eta \left( \frac{\partial v_{ni}}{\partial x_h} + \frac{\partial v_{nh}}{\partial x_i} - \frac{2}{3} \delta_{ih} \text{div} \mathbf{v}_n \right) \right\}; \end{aligned} \quad (2.11)$$

$$\frac{\partial S}{\partial t} + \text{div}(S \mathbf{v}_n) = \frac{1}{T} \text{div}(\kappa \nabla T) + \frac{1}{T} \left\{ \frac{2\Lambda}{\hbar} \left[ \frac{1}{2} \left( -\frac{i\hbar}{m} \nabla - \mathbf{v}_n \right)^2 + \left( \frac{\partial \varepsilon}{\partial \rho_s} \right)_{\rho, S} \right] m |\psi|^2 + \frac{1}{2} \eta \left( \frac{\partial v_{ni}}{\partial x_h} + \frac{\partial v_{nh}}{\partial x_i} - \frac{2}{3} \delta_{ih} \text{div} \mathbf{v}_n \right)^2 \right\}. \quad (2.12)$$

Making use of Eq. (2.9) we can write Eq. (2.11) in another somewhat more convenient form:

$$\frac{\partial \rho_n v_{ni}}{\partial t} + i\Lambda \left\{ \frac{\partial \psi}{\partial x_i} \hat{B}^* \psi^* - \text{c.c.} \right\} + v_{ni} \text{div}(\rho_n \mathbf{v}_n) + \rho_n (\mathbf{v}_n \nabla) v_{ni} + \frac{\partial}{\partial x_h} \left\{ \rho \delta_{ih} - \eta \left( \frac{\partial v_{ni}}{\partial x_h} + \frac{\partial v_{nh}}{\partial x_i} - \frac{2}{3} \delta_{ih} \text{div} \mathbf{v}_n \right) \right\}. \quad (2.11a)$$

### 3. CASES IN WHICH THE GRADIENT OF $\rho_S$ IS SMALL

We now consider the equations in the limit of a small gradient for  $\rho_S$ , in which case the quantum energy terms in Eq. (1.1a) are small. Again we replace the operators  $-(i\hbar/m)\nabla$  (which act on  $\psi$  in  $\hat{L}\psi$ ),  $\Pi_{jk}$  and  $\mathbf{j}$  by  $\mathbf{v}_S$ . Making this substitution and applying the gradient operator to (2.9), from we have Eqs. (2.9) to (2.12)

$$\mathbf{v}_s + \nabla \left\{ \frac{v_s^2}{2} + \left( \frac{\partial \epsilon}{\partial \rho} \right)_{\rho_s, S} + \left( \frac{\partial \epsilon}{\partial \rho_s} \right)_{\rho, S} - \frac{\hbar \Lambda}{2m\rho_s} \operatorname{div} \rho_s (\mathbf{v}_s - \mathbf{v}_n) \right\} = 0; \quad (3.1)$$

$$\partial \rho / \partial t + \operatorname{div} (\rho_s \mathbf{v}_s + \rho_n \mathbf{v}_n) = 0; \quad (3.2)$$

$$\frac{\partial}{\partial t} (\rho_n v_{ni} + \rho_s v_{si}) + \frac{\partial}{\partial x_k} \left\{ \rho_n v_{ni} v_{nk} + \rho_s v_{si} v_{sk} + p \delta_{ik} - \eta \left( \frac{\partial v_{ni}}{\partial x_k} + \frac{\partial v_{nk}}{\partial x_i} - \frac{2}{3} \delta_{ik} \operatorname{div} \mathbf{v}_n \right) \right\} = 0; \quad (3.3)$$

$$\begin{aligned} \frac{\partial S}{\partial t} + \operatorname{div} (S \mathbf{v}_n) &= \frac{1}{T} \operatorname{div} (\mathbf{x} \nabla T) + \frac{1}{T} \left\{ \frac{\hbar \Lambda}{2m\rho_s} [\operatorname{div} \rho_s (\mathbf{v}_s - \mathbf{v}_n)]^2 \right. \\ &+ \left. \frac{2\Lambda m}{\hbar} \left[ \frac{1}{2} (\mathbf{v}_s - \mathbf{v}_n)^2 + \left( \frac{\partial \epsilon}{\partial \rho_s} \right)_{\rho, S} \right]^2 \rho_s + \frac{\eta}{2} \left( \frac{\partial v_{ni}}{\partial x_k} + \frac{\partial v_{nk}}{\partial x_i} - \frac{2}{3} \delta_{ik} \operatorname{div} \mathbf{v}_n \right)^2 \right\}; \end{aligned} \quad (3.4)$$

$$\frac{\partial \rho_s}{\partial t} + \operatorname{div} \rho_s \mathbf{v}_s = -\frac{\Lambda m}{2\hbar} \left\{ \frac{(\mathbf{v}_n - \mathbf{v}_s)^2}{2} + \left( \frac{\partial \epsilon}{\partial \rho_s} \right)_{\rho, S} \right\} \rho_s. \quad (3.5)$$

Formally Eqs. (3.1) to (3.4) are similar to the usual equations for the hydrodynamics of the superfluid liquid, in which the role of the chemical potential is played by the quantity

$$(\partial \epsilon / \partial \rho)_{\rho_s, S} + (\partial \epsilon / \partial \rho_s)_{\rho, S},$$

with this difference, that the density  $\rho_s$  is not given in these equations. This density is taken as an unknown quantity whose approach to its equilibrium value is described by the auxiliary equation (3.5).

In order to find the parameter  $\Lambda$  we make a brief analysis of the propagation of sound in helium II near the  $\lambda$  point. (The absorption of first sound near the  $\lambda$  point has been computed by Landau and Khalatnikov.<sup>6</sup>) Linearizing Eqs. (3.1) to (3.5) and eliminating  $\rho_s$  from the first three equations, we arrive at the usual equation for the propagation of sound in a superfluid. In this case the second viscosities, which are responsible for the absorption of sound, are expressed in terms of  $\Lambda$  (the definition of the coefficients  $\zeta$  are those used by Khalatnikov<sup>7</sup>):

$$\zeta_1 = \zeta_4 = \frac{2\hbar}{\Lambda m} \frac{\rho}{\rho_s} \frac{\left( \frac{1}{\rho} \frac{\partial \rho}{\partial T} \right)_p \left( \frac{1}{\rho} \frac{\partial \rho_s}{\partial T} \right)_p - \left( \frac{\partial \rho_s}{\partial \rho} \right)_T \frac{c_p}{T}}{\left( \frac{\partial \rho}{\partial \rho} \right)_T \frac{c_p}{T} - \left( \frac{1}{\rho} \frac{\partial \rho}{\partial T} \right)_p^2} \left\{ 1 + \frac{\left( \frac{1}{\rho} \frac{\partial \rho_s}{\partial T} \right)_p \left( \frac{1}{\rho} \frac{\partial \rho}{\partial T} \right)_p - \frac{c_p}{T} \left( \frac{\partial \rho_s}{\partial \rho} \right)_T + \sigma \left( \frac{1}{\rho} \frac{\partial \rho_s}{\partial T} \right)_p \left( \frac{\partial \rho}{\partial \rho} \right)_T - \sigma \left( \frac{1}{\rho} \frac{\partial \rho}{\partial T} \right)_p \left( \frac{\partial \rho_s}{\partial \rho} \right)_T} \right\}; \quad (3.6)$$

$$\zeta_2 = \frac{2\hbar}{\Lambda m \rho_s} \rho^2 \left[ \frac{\frac{1}{\rho} \left( \frac{\partial \rho}{\partial T} \right)_p \frac{1}{\rho} \left( \frac{\partial \rho_s}{\partial T} \right)_p - \left( \frac{\partial \rho_s}{\partial \rho} \right)_T \frac{c_p}{T}}{\left( \frac{\partial \rho}{\partial \rho} \right)_T \frac{c_p}{T} - \frac{1}{\rho^2} \left( \frac{\partial \rho}{\partial T} \right)_p^2} \right]^2; \quad (3.7)$$

$$\zeta_3 = \frac{\hbar \Lambda}{2m\rho_s} \left\{ 1 + \frac{4}{\Lambda^2} \left[ \frac{\frac{1}{\rho} \left( \frac{\partial \rho_s}{\partial T} \right)_p \frac{1}{\rho} \left( \frac{\partial \rho}{\partial T} \right)_p - \frac{c_p}{T} \left( \frac{\partial \rho_s}{\partial \rho} \right)_T + \sigma \frac{1}{\rho} \left( \frac{\partial \rho_s}{\partial T} \right)_p \left( \frac{\partial \rho}{\partial \rho} \right)_T - \sigma \frac{1}{\rho} \left( \frac{\partial \rho}{\partial T} \right)_p \left( \frac{\partial \rho_s}{\partial \rho} \right)_T}{\left( \frac{\partial \rho}{\partial \rho} \right)_T \frac{c_p}{T} - \frac{1}{\rho^2} \left( \frac{\partial \rho}{\partial T} \right)_p^2} + 1 \right] \right\}. \quad (3.8)$$

(In these expressions  $\sigma = S/\rho$ .)

For simplicity we limit ourselves to the low-frequency case, which is of practical interest; at low frequencies the dispersion properties of the coefficients  $\zeta$  can be neglected.

The absorption coefficients for first sound and second sound are expressed in terms of  $\zeta_1$ ,  $\zeta_2$  and  $\zeta_3$ , the first viscosity and thermal conductivity. In the absorption coefficient for first sound, only the term containing  $\zeta_2$  increases without limit as the  $\lambda$  point is approached. In the absorption coefficients for second sound, however, the terms containing  $\zeta_1$ ,  $\zeta_2$ ,  $\zeta_3$  and  $\kappa$  all increase.

Comparing the computed value of the absorption coefficient for first sound

$$\alpha = (\omega^2 / 2\rho u_1^3) \zeta_2 \quad (3.9)$$

( $u_1$  is the velocity of first sound) with the measured value,<sup>8</sup> it is possible to determine  $\Lambda$ . The

derivative  $(\partial \rho_s / \partial \rho)_T$  which appears in Eq. (3.7) can be found from the pressure dependence of the temperature of the  $\lambda$  point. Differentiating  $\rho_s$  along the  $\lambda$  line, we have

$$(\partial \rho_s / \partial \rho)_T = -(\partial \rho_s / dT)_p dT_\lambda / d\rho. \quad (3.10)$$

The calculations yield a value of approximately 15 for  $\Lambda$ .

In order to carry out the analysis, the function  $\epsilon(\rho, S, \rho_s)$  must be known. It is determined from the experimental data on the specific heat of helium and the density of the superfluid component as functions of temperature and pressure in the vicinity of the  $\lambda$  point.

The equations which have been obtained apply only for temperatures such that  $T_\lambda - T \ll T_\lambda$ , exactly as is the case for the equations obtained in reference 1.

The most interesting application of the system

being described appears in processes which take place in the presence of vortices in which high gradients of  $\rho_S$  exist. Under these conditions the quantum terms in the equations turn out to be important. The equations which have been derived make it possible to compute the coefficients  $B$  and  $B^1$  of Hall and Vinen<sup>9</sup> because of friction between the superfluid part and the normal part in the presence of a vortex; it is proposed to carry out this calculation in a subsequent paper.

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73

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### THE SPECTRAL REPRESENTATION OF THE TWO-MESON GREEN'S FUNCTION

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A spectral representation has been found for the two-meson Green's function which is analogous to that found earlier<sup>1-3</sup> for the vertex part. The analytic properties of this function are examined. It follows from the representations that dispersion relations exist for the scattering amplitude for a fixed scattering angle in the center-of-mass system. These relations are obtained and discussed.

#### 1. INTRODUCTION

IN a previous paper by the writer<sup>3</sup> the causality conditions were used to obtain the spectral representation of the vacuum expectation value of the T-product of three Heisenberg operators. In the present paper this same method is used to obtain the spectral representation of the vacuum expectation value of the T-product of four operators, i.e., for the two-particle Green's function. In order not to encumber the discussion with elab-

orate computations associated with the spinor character of the meson-nucleon Green's function and its possession of poles, we shall confine ourselves in this paper to the derivation of the spectral representation for the two-meson Green's function and the study of the analytical properties of the meson-meson scattering amplitude.

From the spectral representation it follows that the scattering amplitude is an analytic function in the upper half-plane and has no essential singularity at infinity, not only for a fixed pre-