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Translated by M. Hamermesh
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SOVIET PHYSICS JETP VOLUME 35(8), NUMBER 2 FEBRUARY, 1959

EFFECT OF THERMOELECTRIC FORCES ON THE SKIN EFFECT IN A METAL

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Submitted to JETP editor March 26, 1958

J. Exptl. Theoret. Phys. (U.S.S.R.) **35**, 474-478 (August, 1958)

The surface impedance of a metal is calculated with thermoelectric forces taken into account.

1. In the calculation of the surface resistance (impedance) of metals, one usually starts with Ohm's law,

$$j_i = \sigma_{ik} E_k,$$

where σ_{ik} is the conductivity tensor, and **E** and **J** are the vector electric field intensity and current density.* One thereby neglects the effect of heat waves that are produced in the metal by passage of an electromagnetic wave through it. As will be evident later, this is correct only in isotropic metals in the absence of a magnetic field, and in anisotropic metals when the surface of the metal coincides with a principal plane of the resistivity tensor.

The complete system of equations describing the propagation of waves in a metal, with heat flow taken into account, has the form

$$\mathrm{curl}\,\mathbf{H} = \frac{4\pi}{c}\mathbf{j}; \mathrm{curl}\,\mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{H}}{\partial t}; \quad C\frac{\partial\Theta}{\partial t} + \mathrm{div}\,\mathbf{q} = 0;$$
$$E_i = \rho_{ik} j_k + \alpha_{ik}\,\partial\Theta/\partial x_k; \quad q_i = T\alpha_k\, j_k - \varkappa_{ik}\,\partial\Theta/\partial x_k. \qquad (1)$$

Here Θ is the high-frequency addition to the mean temperature T of the specimen; C is the heat capacity of unit volume of the metal; **q** is the heat current; ρ_{ik} is the resistivity tensor; κ_{ik} is the heat conductivity tensor; and α_{ik} is the tensor of thermoelectric coefficients.

The tensor α_{ik}, in general, is not symmetric. However, all metals possess lattice symmetry that excludes an antisymmetric part of the tensor α_{ik}. Therefore we shall hereafter suppose, in the absence of a magnetic field, that $\alpha_{ik} = \alpha_{ki}$.

Besides the usual boundary conditions of continuity of the tangential components of the vectors **E** and **H**, we must add to the system of equations (1) boundary conditions for the temperature. We shall consider two limiting cases:

(a) Heat current equal to zero at the surface:

$$\mathbf{q}\cdot\mathbf{n} = 0 \qquad (2a)$$

(**n** = unit vector normal to the surface).

(b) Surface temperature maintained constant and equal to T:

$$\Theta = 0. \qquad (2b)$$

2. We consider normal incidence of a plane monochromatic electromagnetic wave, of frequency ω, on the surface of a uniaxial metal, whose principal axis (1 in the figure) makes an angle φ with the normal to the surface of the metal; we choose for the z axis the direction of the normal. Then all quantities (**E**, **H**, Θ) depend on the coordinate z alone. With the x and y axes chosen as shown in the figure, it is easy to show that $\rho_{xy} = \rho_{yx} = \rho_{xz} = 0$ and $\alpha_{xy} = \alpha_{yx} = \alpha_{zx} = 0$. It follows furthermore from Maxwell's equations, in this case, that $j_z = 0$

*We are interested here in the range of frequencies and temperatures in which there is a normal skin effect.

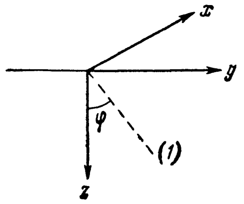
and $H_z = 0$. Therefore the complete system of equations takes the form

$$-\frac{\partial H_y}{\partial z} = \frac{4\pi}{c} j_x, \quad \frac{\partial E_x}{\partial z} = \frac{i\omega}{c} H_y, \quad E_x = \rho_{xx} j_x, \qquad (3)$$

$$\frac{\partial H_x}{\partial z} = \frac{4\pi}{c} j_y, \quad -\frac{\partial E_y}{\partial z} = \frac{i\omega}{c} H_x,$$

$$E_y = \rho_{yy} j_y + \alpha_{yz} \frac{\partial \Theta}{\partial z}, \quad -i\omega C\Theta + \frac{\partial q_z}{\partial z} = 0, \qquad (4)$$

$$q_z = T\alpha_{yz} j_y - \varkappa_{zz} \frac{\partial \Theta}{\partial z}$$



As is evident from (3) and (4), the complete system of equations separates into two independent systems. If the incident electromagnetic wave is so polarized that the vector E is parallel to the x axis, the field in the metal is described by the system (3). In this case no heat flow occurs, and the surface impedance is determined by the usual formula*

$$\zeta_x = \zeta_{0x} = \sqrt{\omega\rho_{xx}/4\pi i} = \sqrt{\omega\rho_\perp/4\pi i}, \qquad (5)$$

where $\rho_\perp$ is one of the principal values of the tensor ρ_{ik} (the principal values of the tensor ρ_{ik} are equal to $\rho_{22} = \rho_{33} = \rho_\perp$ and $\rho_{11} = \rho_\parallel$).

Now let the electric field of the incident wave be directed along the y axis. On eliminating the magnetic field from the system (4) and representing $-\alpha_{yz}\partial\Theta/\partial z$ by E_T, we get

$$\left(k_\rho^2 + \frac{d^2}{dz^2}\right) E_y + k_\rho^2 E_T = 0,$$

$$-k_\rho^2 \frac{a}{1+a} E_y + \left(\frac{d^2}{dz^2} + k_\rho^2 \frac{b-a}{1+a}\right) E_T = 0, \qquad (6)$$

$$a = T\alpha_{yz}^2/\rho_{yy}\varkappa_{zz}, \quad b = c^2 C\rho_{yy}/4\pi\varkappa_{zz}.$$

Here $k_\rho = (4\pi i\omega/c^2\rho_{yy})^{1/2}$ is the complex wave vector of the electromagnetic wave in the metal without allowance for thermoelectric forces (with $\alpha_{yz} = 0$).

From the system (6) we get for the half-space ($z > 0$) occupied by the metal

$$E_y = Ae^{ik_1 z} + Be^{ik_2 z},$$
$$E_T = -(1 - k_1^2/k_\rho^2)Ae^{ik_1 z} - (1 - k_2^2/k_\rho^2)Be^{ik_2 z}, \qquad (7)$$

where

$$k_{1,2} = k_\rho \left[\frac{1 + b \pm \sqrt{(1-b)^2 - 4ab}}{2(1+a)}\right]^{1/2}, \qquad (8)$$

and where A and B are constants, a relation between which is determined by the boundary conditions for the temperature. The sign of the root in formula (8) must be so chosen that the imaginary parts of the wave vectors k_1 and k_2 are positive.

According to the definition of the impendance we have

$$\zeta_y = (\omega/c)(A+B)/(k_1 A + k_2 B). \qquad (9)$$

In the case in which the heat current vanishes on the surface (2a),

$$B = -\frac{(a+1)(k_1/k_\rho)^2 - 1}{(a+1)(k_2/k_\rho)^2 - 1} A, \qquad (10)$$

$$\zeta_y^{\text{ad}} = \zeta_{0y}\sqrt{\frac{1+a}{2}} \qquad (11a)$$

$$\times \frac{(1+a+\sqrt{(1-b)^2-4ab})^{1/2} + (1+a-\sqrt{(1-b)^2-4ab})^{1/2}}{1+\sqrt{(a+1)b}},$$

where $\zeta_{0y} = \sqrt{\omega\rho_{yy}/4\pi i}$

Under the condition (2b) (isothermal surface), we obtain similarly

$$\zeta_y^{\text{is}} = \zeta_{0y}\sqrt{2} \qquad (11b)$$

$$\times \frac{\sqrt{b} + \sqrt{1+a}}{(1+b+\sqrt{(1-b)^2-4ab})^{1/2} + (1+b-\sqrt{(1-b)^2-4ab})^{1/2}}.$$

For $a \ll 1$, formulas (11a) and (11b) simplify considerably:

$$\zeta_y^{\text{ad}} \approx \zeta_{0y}\left\{1 + \frac{a}{2}\,\frac{1+2\sqrt{b}}{(1+\sqrt{b})^2}\right\}; \qquad (12a)$$

$$\zeta_y^{\text{is}} \approx \zeta_{0y}\left\{1 + \frac{a}{2}\,\frac{1}{(1+\sqrt{b})^2}\right\}. \qquad (12b)$$

We remark that the dependence of the impedance ζ_y on the angle between the normal to the metal surface and the crystal axis, in contrast to ζ_{0y}, is determined not solely by the quantity

$$\rho_{yy} = \rho_\perp \cos^2\varphi + \rho_\parallel \sin^2\varphi, \qquad (13)$$

but also by the parameters a and b, which vary with the angle as follows:

$$a = \frac{T(\alpha_\perp - \alpha_\parallel)^2 \sin^2\varphi\cos^2\varphi}{(\rho_\perp\cos^2\varphi + \rho_\parallel\sin^2\varphi)(\varkappa_\perp\sin^2\varphi + \varkappa_\parallel\cos^2\varphi)};$$

$$b = \frac{c^2 C(\rho_\perp\cos^2\varphi + \rho_\parallel\sin^2\varphi)}{4\pi(\varkappa_\perp\sin^2\varphi + \varkappa_\parallel\cos^2\varphi)}. \qquad (14)$$

From the first formula (14) it is clear that a vanishes when $\varphi = 0$ or $\pi/2$. Then heat current is absent, and the expression for the surface impedance ζ_y coincides with the ordinary ζ_{0y}. We

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*We define the impedance as follows:

$$\zeta_x = E_x(0)/H_y(0); \quad \zeta_y = -E_y(0)/H_x(0).$$

write an expression for the impedance near these values of the angle. For $\varphi \ll 1$,

$$\zeta_y^{ad} \approx \sqrt{\frac{\rho_\perp \omega}{4\pi i}}\Big\{1 + \frac{\varphi^2}{2}\Big[\frac{\rho_\parallel}{\rho_\perp} - 1 + \frac{T(\alpha_\parallel - \alpha_\perp)^2}{\rho_\perp \varkappa_\parallel}\frac{1+\sqrt{c^2 C\rho_\perp/\pi\varkappa_\parallel}}{(1+\sqrt{c^2 C\rho_\perp/4\pi\varkappa_\parallel})^2}\Big]\Big\}; \quad (15a)$$

$$\zeta_Y^{is} \approx \sqrt{\frac{\rho_\perp \omega}{4\pi i}}\Big\{1 + \frac{\varphi^2}{2}\Big[\frac{\rho_\parallel}{\rho_\perp} - 1 + \frac{T(\alpha_\parallel - \alpha_\perp)^2}{\rho_\perp \varkappa_\parallel}(1+\sqrt{c^2 C\rho_\perp/4\pi\varkappa_\parallel})^{-2}\Big]\Big\} \quad (15b)$$

For $\psi \ll 1$ $(\psi = \pi/2 - \varphi)$,

$$\zeta_y^{ad} \approx \sqrt{\frac{\omega\rho_\parallel}{4\pi i}}\Big\{1 + \frac{1}{2}\psi^2\Big[\frac{\rho_\perp}{\rho_\parallel} - 1 + \frac{T(\alpha_\perp - \alpha_\parallel)^2}{\rho_\parallel \varkappa_\perp}\frac{1+\sqrt{c^2 C\rho_\parallel/\pi\varkappa_\perp}}{(1+\sqrt{c^2 C\rho_\parallel/4\pi\varkappa_\perp})^2}\Big]\Big\}; \quad (16a)$$

$$\zeta_y^{is} \approx \sqrt{\frac{\omega\rho_\parallel}{4\pi i}}\Big\{1 + \frac{1}{2}\psi^2\Big[\frac{\rho_\perp}{\rho_\parallel} - 1 + \frac{T(\alpha_\perp - \alpha_\parallel)^2}{\rho_\parallel \varkappa_\perp}\Big(1+\sqrt{\frac{c^2 C\rho_\parallel}{4\pi\varkappa_\perp}}\Big)^{-2}\Big]\Big\}. \quad (16b)$$

3. The effect under consideration can be observed even in anisotropic metal (polycrystal) if the latter is placed in a constant magnetic field. In this case the most general possible linear relation between the electric field and heat current, on the one hand, and the electric current and temperature gradient, on the other, has the form

$$\mathbf{E} = \rho\mathbf{j} + R[\mathbf{H}_0\times\mathbf{j}] + \beta\mathbf{H}_0(\mathbf{H}_0\cdot\mathbf{j}) + \alpha\nabla T + N[\mathbf{H}_0\nabla T] + \gamma\mathbf{H}_0(\mathbf{H}_0\nabla T),$$
$$\mathbf{q} = T\alpha\mathbf{j} - \varkappa\nabla T + NT[\mathbf{H}_0\times\mathbf{j}] + L[\mathbf{H}_0\nabla T] + T\gamma\mathbf{H}_0(\mathbf{H}_0\cdot\mathbf{j}) - \delta\mathbf{H}_0(\mathbf{H}_0\nabla T).$$

The phenomenological constants introduced (ρ, R, α, β, etc.) may, in general, depend on the magnitude of the magnetic field.

The greatest change of surface resistance resulting from the effect of the heat wave will occur in a magnetic field parallel to the metal surface. After a calculation similar to that carried out above, we get

$$\zeta_x^{ad} = \zeta_0\sqrt{\frac{1-a'}{2}} \times \frac{(1+b'+\sqrt{(1-b')^2+4a'b'})^{1/2} + (1+b'-\sqrt{(1-b')^2+4a'b'})^{1/2}}{1+\sqrt{(1-a')b'}}; \quad (17a)$$

$$\zeta_x^{is} = \zeta_0\sqrt{2} \times \frac{\sqrt{b'}+\sqrt{1-a'}}{(1+b'+\sqrt{(1-b')^2+4a'b'})^{1/2} + (1+b'-\sqrt{(1-b')^2+4a'b'})^{1/2}} \quad (17b)$$

$$\zeta_y^{ad} = \zeta_y^{is} = \zeta_0 = \sqrt{\omega\rho/4\pi i}.$$

Here

$$a' = (NH_0)^2 T/\rho\varkappa; \quad b' = c^2 C\rho/4\pi\varkappa.$$

It is interesting to note that for $a' = 1$, $\zeta_x^{ad} = 0$; i.e., the metal behaves in this case like an ideal conductor. The vanishing of the impedance (ζ_x^{ad}) is essentially connected with the rigorous vanishing of the heat current on the surface. Actually, when account is taken of heat radiation, q_z always differs from zero. Therefore even for $a' = 0$, $\zeta_x \neq 0$; but it has a minimum there.

4. As is clear from the formulas presented, the magnitude of the expected effect (the dependence of the surface resistance on the heat emission conditions) is determined by the value of the constant

$$a = T\alpha_{yz}^2/\rho_{yy}\varkappa_{zz}.$$

The parameter a can be estimated only in order of magnitude, since there are no direct measurements of the components of the tensor α_{ik}. If we suppose that the quantities α_{ik} have the same order of magnitude as the Thomson coefficient and that $\rho\kappa/T$ has the order of magnitude of the constant in the Wiedemann-Franz law, then a has a value of order 10^{-2} for ordinary metals (Pt, Pd, W). For metals such as Bi, the Thomson coefficient is anomalously large ($\sim 10^3$),[2] and a may be of order unity (or even greater).

In the case of small a, observation of the described effect by reflection is difficult. However, if the length of the heat wave in the metal is much larger than the skin depth (for $a \ll 1$, this corresponds to a small value of the parameter b), then there can be a peculiar "pulling" of the electromagnetic wave by the heat wave. For films whose thickness d satisfies the condition

$$\delta_{sk} \ll d \ll \delta_T$$

($\delta_{sk} = \sqrt{c^2\rho_{yy}/2\pi\omega}$ is the thickness of the skin layer, $\delta_T = \sqrt{2\kappa_{zz}/\omega C}$ is the depth of penetration of the heat wave), the amplitude of the transmitted wave is expressed in terms of the amplitude of the incident wave by the equation

$$E_{tr}/E_{inc} = -(c^2 C\alpha_{yz}^2 T/2\pi\varkappa_{zz}^2)\zeta_{0y}. \quad (18)$$

The last formula was obtained on the assumption that the heat current vanishes on both sides of the film. We have neglected terms of order

$\exp(-d/\delta_{sk})$.

5. We further consider oblique incidence of an electromagnetic wave on an isotropic conductor. In metals the surface impedance is independent of the angle of incidence φ and of the polarization of the wave because of the fact that $4\pi\sigma/\omega \gg 1$. Therefore the following consideration relates to semiconductors, for which the dependence on angle of incidence is appreciable. On representing by $\epsilon = \epsilon' + i\epsilon''$ the complex dielectric constant ($\epsilon'' = 4\pi\sigma/\omega$), we get the known value for the impedance in the case in which the electric field **E** of the incident wave is perpendicular to the plane of incidence:

$$\zeta = 1/\sqrt{\varepsilon - \sin^2\varphi}.$$

In the case in which **E** lies in the plane of incidence, consideration of thermoelectric forces naturally changes the value of the impedance:

$$\zeta^{ad} = \frac{1}{\varepsilon}\left\{\sqrt{\varepsilon - \sin^2\varphi} - \frac{T\sigma\alpha^2}{\varkappa}\,\frac{i\varepsilon''\sin^2\varphi}{\varepsilon + T\sigma\alpha^2\varepsilon'/\varkappa}\left[\frac{ic^2C}{\omega\varkappa}\,\frac{1}{1 + T\alpha^2\sigma\varepsilon'/\varkappa\varepsilon} - \sin^2\varphi\right]^{-1/2}\right\}.$$

This expression corresponds to absence of heat current at the surface. Under isothermal conditions at the surface, no heat wave is produced, and therefore

$$\zeta^{is} = \sqrt{\varepsilon - \sin^2\varphi}/\varepsilon.$$

In closing, the authors express their gratitude to L. D. Landau for helpful discussions.

[1] L. D. Landau and E. M. Lifshitz, Электродинамика сплошных сред (Electrodynamics of Continuous Media), Gostekhizdat, M. 1957, Sect. 25.

[2] Сборник физических констант (Collection of Physical Constants), ONTI, L.-M. 1937.

Translated by W. F. Brown, Jr.
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