$$\tau+\int_0^u \frac{du}{g(u)} \to \tau \exp\int \frac{dz}{g(z)}\,,\quad \frac{1}{g(z)} \to \frac{2}{3}\,\frac{\tau^2}{g(z)} \exp\int \frac{dz}{g(z)}\,,$$

$$1 \gg \left|u-\tfrac{3}{2}\right| \gg \varepsilon(\tau). \qquad \text{(A.15)}$$

In actuality

$$\tau+\int_0^u \frac{du}{g(u)} = \tau\left(1-1\Big/\frac{2}{3}\left(u-\frac{3}{2}\right)\tau\right) = \tau\left(1-\frac{1}{z}\right)$$

$$= \tau \exp\int \frac{dz}{g(z)};\ z \gg 1. \qquad \text{(A.16)}$$

Also, for such values of u we must obtain the previous distribution function in zero approximation:

$$f\,\frac{dz}{du} \to \frac{A}{{}^2/_3\,(u-{}^3/_2)^2} \exp\left\{-\left(\tau-\frac{1}{{}^2/_3\,(u-{}^3/_2)}\right)\right\},\ z \gg 1. \qquad \text{(A.17)}$$

Hence we obtain immediately

$$f = A\,\frac{\tau}{g(z)} \exp\left[\int \frac{dz}{g(z)} - \tau \exp\int \frac{dz}{g(z)}\right]. \qquad \text{(A.18)}$$

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PROJECTION OPERATORS IN THE THEORY OF ELEMENTARY PARTICLES

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Minimal polynomials in the matrices occurring in relativistic wave equations and the spin matrices are used to construct projection matrices, which are dyads describing arbitrary possible states of a free particle of arbitrary spin. The fundamental physical quantities (energy and momentum, charge and current, transition probabilities) are expressed directly in terms of these projection operators in an invariant way (independent of the choice of basis for the representation). Thus the calculation of various effects for particles of any spin is reduced to the computation of the traces of certain combinations of matrices. As examples of the application of the method we obtain the general conditions for definiteness of the energy and charge for particles with a single mass and give a simple derivation of the general commutation relations for particles of arbitrary spin.

IN various calculations relating to particles with spin one needs to find the free-field wave functions, which are solutions of the first order equations

$$(\gamma_k \nabla_k + \varkappa)\psi = 0,$$

where $x_4 = it$, $c = h = 1$, and the γ_k are square matrices. For plane waves $\psi \sim e^{ipx}$, corresponding to the four-momentum $p = (p_k)$, these equations take the form

$$(ip_k\gamma_k + \varkappa)\psi = 0. \qquad (1)$$

We use the notation $\hat{p} = ip_k\gamma_k$. As has been shown in reference 1, the minimal equation for the matrix p in the general case has the form

$$P(\hat{p}) = \hat{p}^n(\hat{p}^2+\lambda_1^2p^2)(\hat{p}^2+\lambda_2^2p^2)\ldots(\hat{p}^2+\lambda_q^2p^2) = 0, \qquad (2)$$

where λ_l ($l = 1, 2, \ldots q$) are distinct nonvanishing eigenvalues of the matrix γ_4. To each value $\pm\lambda_l$ there corresponds a set of states of the particle.

Since $p^2 = -m^2$ and $\lambda_l = \kappa/m_l$, where $m_l^2 = -p^{(l)2}$, we can write instead of Eq. (2)

$$P(\hat{p}) = \hat{p}^n \left(\hat{p}^2 - \varkappa^2 \frac{m^2}{m_1^2}\right) \left(\hat{p}^2 - \varkappa^2 \frac{m^2}{m_2^2}\right) \cdots \left(\hat{p}^2 - \varkappa^2 \frac{m^2}{m_q^2}\right) = 0. \quad (3)$$

Let us introduce the operator

$$\sigma = \frac{i}{2|\mathbf{p}|} \delta_{abc} p_a J^{bc}; \quad a, b, c = 1, 2, 3, \quad (4)$$

where J^{bc} are the infinitesimal matrix operators of the representation of the Lorentz group in the space of the functions ψ which correspond to space rotations, and $\mathbf{p}$ is the three-dimensional momentum vector. Using the condition for invariance of Eq. (1)

$$[\gamma_k J^{lm}] = \delta_{kl}\gamma_m - \delta_{km}\gamma_l, \quad (5)$$

it is not hard to verify that the operators $\hat{p}$ and σ commute with each other. Therefore when $p^{(l)}$ and λ_l are prescribed the states of the particle can be characterized also by the different eigenvalues of the operator σ, which is the operator for the component of the spin along the momentum of the particle. Here in the general case we have corresponding to each "mass" state $\pm\lambda_l$ its own maximum value $s^{(l)}$ of the component of the spin along the momentum, i.e., its own value of the spin. Thus for prescribed $p^{(l)}$ and λ_l the particle can still be in $2s^{(l)} + 1$ different spin states. As is well known, the spins for the states of a particle described by the nonseparable equation (1) must be either all integral or all half-integral. Let s be the maximum value of the spin component for all possible states of the particle. Then in the case of half-integral spin the minimal equation for the operator σ must have the form [$s = (2n + 1)/2$, n an integer]

$$Q(\sigma) = \left(\sigma^2 - \left(\frac{1}{2}\right)^2\right)\left(\sigma^2 - \left(\frac{3}{2}\right)^2\right) \cdots \left(\sigma^2 - \left(\frac{2n+1}{2}\right)^2\right) = 0 \quad (6)$$

and in the case of integral spin ($s = n$) it is

$$Q(\sigma) = \sigma(\sigma^2 - 1)(\sigma^2 - 2^2) \cdots (\sigma^2 - n^2) = 0. \quad (7)$$

In the solution of various problems about spinning particles and their interactions it is necessary to know the wave functions characterizing a particle that has given energy and momentum and is in a prescribed mass and spin state. Usually these wave functions are found in explicit form by the solution of Eq. (1) with a definite choice of the basis for the representation in the space of the functions ψ. For the classification into the various spin states one has further to solve the equations for the characteristic values of the operator σ. Not only are these calculations very cumbersome, especially in the case of higher spins, but also this method is in principle an unsatisfactory one. The point is that for a single type of particle with prescribed physical properties the equations (1) can be written in an indefinite number of forms that differ from each other in the choice of basis. This is due to the fact that every representation of the Lorentz group (or of any group) is defined only to within the choice of an arbitrary nonsingular basis. It is obvious that all results having physical significance cannot depend on the specific choice of the basis of the representation, i.e., they must be invariant with respect to changes in this choice. Therefore in principle there must exist the possibility of solving problems relating to particles with various spins altogether without reference to any choice of the basis of the representation, i.e., in a consistently invariant way. On the other hand the usual methods of calculation, based on the use of a concrete basis, introduce into the treatment accidental features that have no relation to the essential problem.

As is shown below, the use of the minimal polynomials (3), (6), and (7), which are usually known, make it possible in the general case to reduce the calculation of the main physical quantities to the computation of the traces of certain combinations of matrices, which in many cases can be done in an invariant way.

Let us define the diminished minimal polynomial $P_{\pm l}(\hat{p}^{(l)})$ of the matrix $\hat{p}^{(l)}$, corresponding to the mass state $\pm\lambda_l = \pm\kappa/m_l$, by the following relation [cf. Eq. (3) for $m = m_l$]:

$$(\hat{p}^{(l)} \mp \varkappa) P_{\pm l}(\hat{p}^{(l)}) = P(\hat{p}^{(l)}) = 0. \quad (8)$$

As has been shown in reference 1, the columns of the matrix $P_{\pm l}$ contain all the linearly independent eigenvectors of the matrix $\hat{p}^{(l)}$ that correspond to the eigenvalue $\pm\lambda_l m_l = \pm\kappa$. Let us further introduce a matrix $\alpha_{\pm l}$ differing from $P_{\pm l}$ by a numerical factor

$$\alpha_{\pm l} = P_{\pm l}(\hat{p}^{(l)}) / P_{\pm l}(\pm\varkappa). \quad (9)$$

Since by the results of reference 1 all nonvanishing eigenvalues of the matrix $\hat{p}$ must be single, $P_{\pm l}(\pm\kappa) \neq 0$, and the matrix $\alpha_{\pm l}$ always exists. Since, by Eq. (8), $\hat{p}^{(l)} P_{\pm l} = \pm\kappa P$, multiplication of the matrix $P_{\pm l}(\hat{p}^{(l)})$ by an arbitrary polynomial $R(\hat{p}^{(l)})$ is equivalent to multiplying it by $R(\pm\kappa)$. From this it follows that

$$[\alpha_{\pm l}(\hat{p}^{(l)})]^2 = \alpha_{\pm l}(\hat{p}^{(l)}). \quad (10)$$

Thus the operator $\alpha_{\pm l}$ is a projection operator. This means that by multiplying any state vector ψ by $\alpha_{\pm l}$ we extract from it the part that corresponds to states with the mass m_l.

In just the same way, by using the minimal polynomial $Q(\sigma)$ of the spin operator [cf. Eqs. (6) and (7)], we can construct a projection operator

$$\beta_k = Q_k(\sigma)/Q_k(s_k), \quad (\sigma - s_k)\,Q_k(\sigma) = Q(\sigma) = 0, \tag{11}$$

which extracts from any state the part it contains corresponding to the value s_k for the spin component along the momentum.

Let us consider an arbitrary polynomial $f(x)$ of degree n which has all its roots x_k distinct. Defining $f_k(x) = f(x)/(x - x_k)$, we construct a new polynomial of degree $n - 1$:

$$\Phi(x) = 1 - \sum_{k=1}^{n} f_k(x)/f_k(x_k).$$

Since $f_k(x_l) \neq 0$ for $k \neq l$, we have $\Phi(x_k) = 0$ for $k = 1, 2, \ldots n$. Therefore the polynomial $\Phi(x)$ of degree $n - 1$ has n distinct roots, which is possible only when it is identically equal to zero. Consequently, $\Sigma f_k(x)/f_k(x_k) \equiv 1$. Since the minimal polynomial of the operator σ has no multiple roots, the identity that has been obtained can be applied to it and we get

$$\sum_k \beta_k \equiv I, \tag{12}$$

where I denotes the unit matrix, and the sum is taken over all values of the spin component. Therefore for an arbitrary state ψ we can write

$$\psi = \sum_k \psi_k, \quad \psi_k = \beta_k\psi, \quad \sigma\psi_k = s_k\psi_k. \tag{13}$$

This relation gives the resolution of an arbitrary state into states corresponding to all the possible values s_k of the spin component. If the state ψ contains no component corresponding to some values s_k of the spin component, then for these k's we have $\beta_k\psi = 0$.

In the general case an analogous resolution into the mass states cannot be carried out, because for this one would have to take into account all the eigenvalues of $\hat{p}$. The zero eigenvalues of $\hat{p}$ operators, however, do not correspond to real states, and furthermore these zero eigenvalues can be multiple (cf. reference 1).

Knowing the minimal polynomials $P(\hat{p})$ and $Q(\sigma)$, we can obtain without any calculations any eigenstate vector $\psi_{lk}(\hat{p}^{(l)})$, corresponding to prescribed energy and momentum $p^{(l)}$, mass m_l, and spin component s_k. For this purpose we have only to multiply together the corresponding projection operators (9) and (11):

$$\tau_{lk} = \alpha_l\beta_k. \tag{14}$$

Since all the α_l and β_k commute with each other, the product (14) is also a projection operator. The matrix τ_{lk} has the property that each of its columns is an eigenvector of the operators $\hat{p}^{(l)}$ and σ for the eigenvalues $\lambda_l m_l = \kappa$ and s_k, respectively. Since such a state is unique, all the columns of the matrix τ_{lk} are proportional to each other. Therefore the matrix τ_{lk} is a simple dyad. By a dyad we mean a matrix in which each element is the product of the corresponding components of two vectors. Thus if A is a dyad formed from the vectors ψ and φ

$$A_{ik} = \psi_i\varphi_k. \tag{15}$$

We shall write the expression for a dyad as a whole in terms of the components of its vectors in the form

$$A = \psi\cdot\varphi. \tag{16}$$

Obviously

$$\tilde{A} = \varphi\cdot\psi, \quad A^+ = \varphi^*\cdot\psi^*, \tag{17}$$

where $\sim$ is the sign of transposition, $+$ is that of the Hermitian adjoint, and $*$ that of the complex conjugate.

Thus we can write

$$\tau_{lk}(\hat{p}^{(l)}) = \alpha_l\beta_k = \psi_{lk}\cdot\varphi. \tag{18}$$

Here the antecedent vector of the dyad ψ_{lk} is the vector for the state in question. We as yet do not know the consequent φ, but it is not hard to determine it. For this it is enough to note that the matrix $\hat{p}$ and the matrix of the invariant bilinear form η satisfy the relation[2]

$$\hat{p}\eta = \eta\hat{p}^+. \tag{19}$$

At the same time the Hermitian spin-component operator σ of Eq. (4) commutes with η[3]:

$$\eta\sigma = \sigma\eta. \tag{20}$$

Therefore the matrix τ_{lk} satisfies the same relation (19) as $\hat{p}$: $\tau_{lk}\eta = \eta\tau_{lk}^+$. Substituting in this the right member of Eq. (18) we get

$$\psi\cdot\varphi\eta = \eta\varphi^*\cdot\psi^*. \tag{21}$$

Multiplying this relation on the left by an arbitrary vector χ and on the right by η, and using the properties

$$\eta = \eta^* = \tilde{\eta} = \eta^+ = \eta^{-1}, \tag{22}$$

we get

$$\varphi = \frac{\chi\eta\varphi^*}{\chi\varphi}\cdot\psi^*\eta = C\psi^*\eta. \tag{23}$$

From this we find for τ_{lk}

$$\tau_{lk} = C_{lk}\psi_{lk}\cdot\bar{\psi}_{lk}, \quad \bar{\psi}_{lk} = \psi_{lk}^*\eta. \tag{24}$$

Since the matrix τ_{lk} is a projection operator ($\tau_{lk}^2 = \tau_{lk}$),

$$C_{lk}\bar{\psi}_{lk}\psi_{lk} = (\tau_{lk})_c = 1. \quad (25)$$

Here $(\tau_{lk})_c$ denotes the trace of the matrix τ_{lk}. Since η is a Hermitian matrix, $\bar{\psi}\psi = \psi^*\eta\psi$ is a real number, and therefore C_{lk} is also always real. Replacing ψ_{lk} by $C'\psi_{lk}$, we can normalize C_{lk} to $+1$ for $\bar{\psi}_{lk}\psi_{lk} > 0$ and to -1 for $\bar{\psi}_{lk}\psi_{lk} < 0$. Consequently

$$\tau_{lk} = \alpha_l\beta_k = \pm\psi_{lk}\cdot\bar{\psi}_{lk},$$

and the normalization conditions are, respectively,

$$\bar{\psi}_{lk}\psi_{lk} = \psi^*_{lk}\gamma_l\psi_{lk} = \pm 1. \quad (27)$$

Instead of the invariant normalization (27) the charge normalization by

$$\bar{\psi}\gamma_4\psi = \pm 1. \quad (28)$$

is frequently used. The change from one normalization to the other is accomplished very simply, if we use the fact that, by Eq. (1),

$$\bar{\psi}^{(l)}\hat{p}^{(l)}\psi^{(l)} = -\varkappa\bar{\psi}^{(l)}\psi^{(l)}, \quad (\psi^{(l)} = \psi(p^{(l)})). \quad (29)$$

On the other hand, $\bar{\psi}^{(l)}\hat{p}^{(l)}\psi^{(l)} = ip_k^{(l)}\bar{\psi}^{(l)}\gamma_k\psi^{(l)}$. Since the last expression is an invariant, we must have

$$\bar{\psi}^{(l)}\gamma_k\psi^{(l)} = Cp_k^{(l)}. \quad (30)$$

Multiplying Eq. (30) by $ip_k^{(l)}$ and comparing with Eq. (29), we get

$$iCp^{(l)^2} = -iCm_l^2 = -\varkappa\bar{\psi}^{(l)}\psi^{(l)} = -\lambda_l m_l\bar{\psi}^{(l)}\psi^{(l)}$$

Consequently, $C = (\lambda_l/im_l)\bar{\psi}^{(l)}\psi^{(l)}$ and, by Eq. (30),

$$\bar{\psi}^{(l)}\gamma_4\psi^{(l)} = \frac{\lambda_l p_4^{(l)}}{im_l}\bar{\psi}^{(l)}\psi^{(l)} = \frac{\lambda_l p_0^{(l)}}{m_l}\bar{\psi}^{(l)}\psi^{(l)}, \quad (p_4 = ip_0). \quad (31)$$

Thus with the normalization (27)

$$\bar{\psi}^{(l)}\gamma_4\psi^{(l)} = \pm(\lambda_l/m_l)p_0^{(l)} = \pm\lambda_l^2 p_0^{(l)}/\varkappa. \quad (32)$$

If, on the other hand, we take the normalization (28), then

$$\bar{\psi}^{(l)}\psi^{(l)} = \pm\varkappa/\lambda_l^2 p_0^{(l)}. \quad (33)$$

Corresponding to this normalization we get instead of Eq. (26)

$$\psi_{lk}\cdot\bar{\psi}_{lk} = \pm(\varkappa/\lambda_l^2 p_0^{(l)})\alpha_l\beta_k. \quad (34)$$

Thus a knowledge of the minimal polynomials of the operators $\hat{p}$ and σ is enough to enable us to get in invariant form the solution of Eq. (1) for any state of the free field, normalized by the condition (27) or (28). The signs in the relations (27) and (28) relate to the definiteness of the energy and the charge, respectively. As is well known (cf., e.g., reference 3), the energy is definite if $\bar{\psi}_0^{(l)}\psi_0^{(l)} > 0$ for all $\psi_0^{(l)}$ that correspond to a particle at rest. Analogously, for definiteness of the charge we must have $\bar{\psi}_0^{(l)}\gamma_4\psi_0^{(l)} > 0$ for all these $\psi_0^{(l)}$. According to Eq. (26) $(\tau_{lk}\eta)_c = \pm\psi_{lk}\psi^*_{lk}$, with the sign here corresponding to that in the relation (26) or (27). Since $\psi_{lk}\psi^*_{lk} > 0$, the sign of the expression $(\tau_{lk}\eta)_c$ will be the sign of the energy. Since for the state at rest all the values of the spin component are on an equal footing (cf. reference 1) and the sign of the energy can change only when there is a change of the mass state, instead of $\tau_{lk} = \alpha_l\beta_k$ we can take $\alpha_l = \sum_k \tau_{lk}$. Consequently, for positive definiteness of the energy it is necessary and sufficient that for all l

$$(\alpha^0_{\pm l}\eta)_c > 0, \quad (35)$$

where $\alpha^0_{\pm l}$ denotes the operator $\alpha_{\pm l}(\hat{p}^{(l)})$ for $p^{(l)} = (0, im_l)$. Similarly, the requirement of definiteness of the charge reduces to the condition

$$(\gamma_4\alpha^0_{\pm l}\eta)_c > 0. \quad (36)$$

Let us examine the application of these general criteria to the case of a particle with a single rest mass. For such a particle the minimal polynomial of the matrix γ_4 can always be written in the form

$$\gamma_4^n(\gamma_4^2 - 1) = 0. \quad (37)$$

From this we get for $\alpha^0_\pm$, according to Eq. (9),

$$\alpha^0_\pm = \frac{\gamma_4^n(\gamma_4 \pm 1)}{(\pm 1)^n(\pm 2)} = \frac{1}{2}(\pm 1)^{n+1}(\gamma_4^{n+1} \pm \gamma_4^n). \quad (38)$$

According to Eq. (35) a necessary and sufficient condition for definiteness of the energy is that the expressions

$$2(\alpha^0_+\eta)_c = (\gamma_4^{n+1}\eta)_c + (\gamma_4^n\eta)_c,$$

$$2(\alpha^0_-\eta)_c = (-1)^{n+1}[(\gamma_4^{n+1}\eta)_c - (\gamma_4^n\eta)_c]$$

have the same sign, i.e., that their product be positive:

$$(-1)^{n+1}[((\gamma_4^{n+1}\eta)_c)^2 - ((\gamma_4^n\eta)_c)^2] > 0. \quad (39)$$

Since $\gamma_4\alpha^0_\pm = \pm\alpha^0_\pm$, the analogous condition for the definiteness of charge (36) becomes

$$(-1)^n[((\gamma_4^{n+1}\eta)_c)^2 - ((\gamma_4^n\eta)_c)^2] > 0. \quad (40)$$

Taking n to be even or odd, we can conclude from this that in both cases a necessary and sufficient condition for definiteness of the energy (charge) is that the larger of the two expressions $|(\gamma_4^{n+1}\eta)_c|$

and $|(\gamma_4^n\eta)_c|$ be the one in which the exponent of γ_4 is even (or odd).

It is not hard to verify that for spin $\frac{1}{2}$ and spin $\frac{3}{2}$ (reference 3) that one of the traces $(\gamma_4^{n+1}\eta)_c$ and $(\gamma_4^n\eta)_c$ in which γ_4 has an even exponent vanishes, and that for the spins 0, 1, and 2 (references 4 and 5) the trace with the odd power of γ_4 vanishes. This is in full agreement with the facts of the definiteness of the energy (charge) of particles with integral (half-integral) spin.

We see that although the projection matrix τ_{lk} gives not the wave function ψ_{lk} itself, but the dyad (26), nevertheless such fundamental quantities as the charge and energy of the field are expressed directly in terms of τ_{lk}. This fact is of general significance, and all the basic bilinear expressions in ψ that have direct physical meaning, and the calculation of which is the main problem of the theory, can be expressed directly in terms of τ_{lk}.

In particular, let us consider the transition probability for a certain particle to go from the state ψ_1 to the state ψ_2 as the result of an interaction. As is well known, the general expression for this probability w is proportional to the square of the absolute value of the corresponding matrix element for the transition:

$$w = C\,|\,\psi_2 R\psi_1\,|^2 = C\,(\psi_2 R\psi_1)^*\bar{\psi}_2 R\psi_1, \tag{41}$$

where R is an operator of a form determined by the nature of the interaction. Since

$$(\bar{\psi}_2 R\psi_1)^* = \bar{\psi}_2^* R^* \psi_1^* = \psi_1^* R^+ \eta\psi_2 = \bar{\psi}_1 \eta R^+ \eta\psi_2,$$

Eq. (41) can be written in the form

$$w = C\bar{\psi}_1\eta R^+\eta\,(\psi_2\cdot\bar{\psi}_2)\,R\psi_1 = C\,[\bar{R}\,(\psi_2\cdot\bar{\psi}_2)R\,(\psi_1\cdot\bar{\psi}_1)]_c, \tag{42}$$

where $\bar{R} = \eta R^+\eta$. According to Eqs. (18) and (26) the dyads $\psi_1\cdot\bar{\psi}_1$ and $\psi_2\cdot\bar{\psi}_2$ can be represented in the form

$$\begin{aligned}\psi_1\cdot\bar{\psi}_1 &= \alpha\,(\hat{p}_1)\,\beta\,(\sigma_1) = \alpha_1\beta_1,\\ \psi_2\cdot\bar{\psi}_2 &= \alpha\,(\hat{p}_2)\,\beta\,(\sigma_2) = \alpha_2\beta_2,\end{aligned} \tag{43}$$

where $\hat{p}_1$, $\hat{p}_2$, σ_1, σ_2 are the operators for the states ψ_1 and ψ_2 respectively. Thus in the general case the calculation of transition probabilities or effective cross-sections for various processes reduces to the computation of traces of products of operators:

$$w = C\,[\bar{R}\alpha_1\beta_1 R\alpha_2\beta_2]_c. \tag{44}$$

Since computations of traces can be carried out in an invariant way, the expression (44) can be obtained without the use of the explicit form of the matrices γ_k in any particular basis.

In many calculations of various interaction effects it is required to find the corresponding probabilities summed over the spins in the final states and averaged over the spins in the initial states. In the expression (44) the summation over the spins is obtained remarkably simply. Since according to Eq. (12) $\sum_k \beta_k \equiv 1$, for a particle with the spin s the expression (44), averaged over the spin of the initial state and summed over the spin of the final state, will have the form

$$\frac{1}{2s_1+1}\sum_{s_1,s_2} w = \frac{C}{2s_1+1}\,(\bar{R}\alpha_1 R\alpha_2)_c. \tag{45}$$

Here it is assumed that the operator R does not depend on the spin of the particle.

Expressions of the type (45) are used for the calculation of various effects in quantum electrodynamics.[6] It follows from the above developments, however, that such expressions have an extremely general significance and are valid for the calculation of all sorts of interaction effects for particles of arbitrary spin. It must be noted that in all its generality the method used here to derive these basic relations is considerably simpler than those used in the standard literature[6,7] to obtain results that relate to just the special case of the interaction of electrons with the electromagnetic field.

By means of these same relations, (26) and (34), we can work out in a very simple way the general commutation relations for particles with arbitrary spin. As is well known (cf., e.g., reference 8):

$$\begin{aligned}&[\psi_\alpha(x'),\,\bar{\psi}_\beta(x'')]_\pm\\ &= \sum_{l,k}\frac{1}{(2\pi)^3}\int d\,\mathbf{p}\,\{\psi_\alpha(p^{(l)},\lambda_l,s_k)\,\bar{\psi}_\beta(p^{(l)},\lambda_l,s_k)\,e^{ip^{(l)}x}\\ &\pm\psi_\alpha(p^{(l)},-\lambda_l,s_k)\,\bar{\psi}_\beta(p^{(l)},-\lambda_l,s_k)\,e^{-ip^{(l)}x}\}.\end{aligned} \tag{46}$$

Here $x = x' - x''$; the expressions $\psi_\alpha\bar{\psi}_\beta$ form the dyad $(\psi\cdot\bar{\psi})_{\alpha\beta}$ [cf. Eq. (15)]. We use the normalization (28). Then in the case of integral spin, on account of the definiteness of the energy, by Eq. (27) we must take the positive sign for both the states $\psi_{\pm lk}$ in Eq. (34), that is,

$$\psi_{\pm lk}\cdot\bar{\psi}_{\pm lk} = \tau_{\pm lk} = (\varkappa/\lambda_l^2 p_0^{(l)})\,\alpha_{\pm l}\beta_k. \tag{47}$$

For half-integral spin (indefinite energy), on the other hand, we get

$$\psi_{\pm lk}\cdot\bar{\psi}_{\pm lk} = \pm\tau_{\pm lk} = \pm\,(\varkappa/\lambda_l^2 p_0^{(l)})\,\alpha_{\pm l}\beta_k. \tag{48}$$

According to Eq. (12) the summation over the spins k in Eq. (46) reduces simply to striking out β_k, and we obtain for the two cases of integral and half-integral spins

$$[\psi(x')\cdot\bar{\psi}(x'')]_{\pm}=\sum_{l}(2\pi)^{-3}\frac{\varkappa}{\lambda_l^2}\int\frac{d\mathbf{p}}{p_0^{(l)}}(\alpha_{+l}(\hat{p}^{(l)})\,e^{ip^{(l)}x}$$
$$-\alpha_{-l}(\hat{p}^{(l)})\,e^{-ip^{(l)}x}) \qquad (49)$$
$$=\sum_{l}\frac{1}{(2\pi)^3}\frac{\varkappa}{\lambda_l^2}\int\frac{d\mathbf{p}}{p_0^{(l)}}(\alpha_{+l}(\nabla)\,e^{ip^{(l)}x}-\alpha_{-l}(-\nabla)\,e^{-ip^{(l)}x}).$$

By means of Eqs. (3), (8), (9), setting $m=\pm m_l=\pm\varkappa/\lambda_l$, we get

$$\alpha_{\pm l}(\hat{p}^{(l)})=\frac{(\hat{p}^{(l)}\pm\varkappa)\,(\hat{p}^{(l)})^n}{\pm 2\varkappa\,(\pm\varkappa)^n}\prod_{\substack{l'=1\\ l'\neq l}}^{q}\frac{(\hat{p}^{(l)})^2-\lambda_l^2 m_l^2}{\varkappa^2-\lambda_{l'}^2 m_l^2}. \qquad (50)$$

Therefore $\alpha_{-l}(\hat{p}^{(l)})=\alpha_{+l}(-\hat{p}^{(l)})$, since change of the sign of $\varkappa$ in Eq. (50) gives the same result as change of the sign of $p^{(l)}$. Consequently, $\alpha_{+l}(\nabla)=\alpha_{-l}(-\nabla)$, and it follows from Eq. (49) that

$$[\psi(x')\cdot\bar{\psi}(x'')]_{\pm}=2i\varkappa\sum_{l=1}^{q}\frac{1}{\lambda_l^2}\alpha_l(\nabla)\,D_l(x), \qquad (51)$$
$$D_l(x)=(2\pi)^{-3}\int\frac{d\mathbf{p}}{p_0^{(l)}}e^{ipx}\sin p_0^{(l)}x_0.$$

It has here taken only a few lines to derive this result, which was obtained in the paper of Karpman[9] by rather lengthy arguments. The example that has been given can serve as an illustration of the effectiveness of the method expounded above. Obviously the relations (26) and (34) can find application in all sorts of other calculations relating to spinning particles.

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