SHOCK WAVES IN A CONDUCTING ULTRA-RELATIVISTIC GAS

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WHEN a stream of conducting gas passes through the front of a normal shock wave the following quantities are conserved:[1,2]

$$\frac{1}{\theta^2}\left[p + \rho a^2 + \frac{\mu H^2}{8\pi}\left(1 + \frac{a^2}{c^2}\right)\right] = J; \tag{1}$$

$$\frac{1}{\theta}\left[pV + \rho V c^2 + \frac{\mu H^2 V}{4\pi}\right] = \frac{w^*}{\theta} = w_0^*; \tag{2}$$

$$a / V\theta = m. \tag{3}$$

At the same time the following condition is also satisfied.

$$HV = b = \text{const.} \tag{4}$$

Here p is the pressure, ρ the density, V the specific volume, a the velocity, $\theta = \sqrt{1 - a^2/c^2}$, and H the intensity of the magnetic field ($H \perp a$).

On eliminating the quantity H from Eqs. (1) and (2) with the aid of (4), we arrive at the following equations:

$$\left[\frac{1}{\theta^2}\left(p + \rho a^2 + \frac{b_0 c^2}{2V^2}\right)\right] = 0; \tag{5}$$

$$\left[\frac{1}{\theta}\left(pV + \rho V c^2 + \frac{b_0 c^2}{V}\right)\right] = 0, \tag{6}$$

where

$$b_0 = \mu b^2 / 4\pi c^2.$$

For an ultrarelativistic gas we have

$$p = (k-1)\rho c^2; \quad \rho V^k = \sigma^k, \tag{7}$$

where σ is the entropy function.

On eliminating from Eq. (6) the pressure and the specific volume we shall find that

$$\left[\frac{2\rho}{\theta^2}\left(k - 1 + \frac{a^2}{c^2}\right) + \frac{b_0 m^2}{a^2}\right] = 0;$$
$$\left[a\left(\frac{k\rho}{\theta^2} + \frac{b^0 m^2}{a^2}\right)\right] = 0. \tag{8}$$

Denoting the parameters of the gas ahead of the front by the subscript 1 and behind the front of the shock wave by the subscript 2, we obtain, on the basis of (8), the following equations:

$$2\frac{\rho_1}{\theta_1^2}\left(k - 1 + \frac{a_1^2}{c^2}\right) + \frac{b_0 m^2}{a_1^2} = 2\frac{\rho_2}{\theta_2^2}\left(k - 1 + \frac{a_2^2}{c^2}\right) + \frac{b_0 m^2}{a_2^2};$$
$$k\rho_1 a_1 / \theta_1^2 + b_0 m^2 / a_1 = k\rho_2 a_2 / \theta_2^2 + b_0 m^2 / a_2. \tag{9}$$

If we eliminate ρ_2 from the system (9) we obtain a cubic equation for a_2. This equation has, first, the trivial solution $a_1 = a_2 = c$ which has a meaning in the case of a photon gas whose equation of state is the same as for the case of the ultrarelativistic gas. As is well known, no shock wave arises in this case.

The resulting equations lead to the solution

$$\beta_2 = \frac{1}{4\beta_1}\{k - 1 + \alpha^2 + ((k - 1 + \alpha^2)^2 + 8\beta_1^2[\alpha^2 - (k-1)])^{1/2}\}, \tag{10}$$

where

$$\beta_1 = \frac{a_1}{c}; \quad \beta_2 = \frac{a_2}{c};$$

$$\alpha = \frac{\omega^*}{c} = \left(\frac{k - 1 + b_0 / k\rho_1 V_1^2}{1 + b_0 / k\rho_1 V_1^2}\right)^{1/2}.$$

(Evidently the only meaningful solution is one with a plus sign in front of the square root). Analysis of this solution shows that if $\beta_1 = \alpha$, then also $\beta_2 = \alpha$.

Further, it is evident that the condition $\beta_2 < \beta_1$ yields the result: $\beta_1 > \alpha$; $\beta_2 < \alpha$. For $\beta_1 < \alpha$, no shock wave is formed.

If the gas pressure is negligibly small in comparison with the field pressure (there is no gas), then $\alpha = 1$, and since the condition $1 \geq \beta_1 \geq \alpha$ must be satisfied, $\beta_1 = \beta_2 = 1$ and, consequently, the shock wave can not be formed.

If there is no field ($H = 0$; $b = 0$), $\alpha = (k-1)^{1/2}$ and (10) leads to

$$\beta_1 \beta_2 = k - 1. \tag{11}$$

If $\beta_1 = (k-1)^{1/2}$, then $\beta_2 = (k-1)^{1/2}$ and there is no shock wave. As β_1 increases, the amplitude of the shock wave also increases.

In the system of reference in which the gas ahead of the wave front is at rest, and the speed of the wave front is $D = a_1$, the velocity of the gas behind the front of the shock wave will be given by

$$a_H = \frac{a_1 - a_2}{1 - a_1 a_2 / c^2} = c\frac{\beta_1^2 - (k-1)}{\beta_1(2-k)}. \tag{12}$$

When $\beta_1 = 1$, then $\beta_2 = k - 1$, but $a_H/c = 1$.

It is evident that a shock wave is impossible when the speed of gas flow behind its front is equal to the velocity of light for particles with a rest mass different from zero, because such a speed is not attainable.

It may turn out that for a given $a_H = a_H^*$ the amplitude of the shock wave will be a maximum. In the case of a further increase in the speed a_H, a part of the energy of the particles will be converted into energy of radiation, pairs will begin to be created in the photon gas, and the amplitude of the shock wave will be reduced.

[1] F. Hoffman and E. Teller, Phys. Rev. 80, 692 (1950).

[2] K. P. Staniukovich, Неустановившиеся движения сплошной среды (Unsteady Motion of Continuous Media), Gostekhizdat, 1955, Ch. 15, § 87.

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THE ENERGY OF EXCITONS FOR VERY SMALL QUASI-MOMENTA

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LET us divide the crystal into cubic regions, let the edge of the cube L contain a large number of lattice constants, and let the exciton quasi-momentum $\mathbf{k}$ have an absolute magnitude much less than 1/L. We construct the wave function of the crystal in the form of a linear combination of antisymmetrized products of wave functions of the separate regions:

$$\Psi_{\mathbf{k}} = A \sum_{\mathbf{m}l\lambda} e^{i\mathbf{k}\cdot\mathbf{m}}(-1)^{\lambda} \prod_{\substack{\mathbf{n}=1\\ \mathbf{n}\neq\mathbf{m}}}^{N} \psi_{\mathbf{n}}(\lambda)\, c_l \varphi_{\mathbf{m}}\ (\lambda). \quad (1)$$

Here $\mathbf{n}$ (or $\mathbf{m}$) is a vector determining the position of the center of the cube and at the same time an index, numbering the cube. $\psi_{\mathbf{n}}$ is the (antisymmetrized) ground state of the separate cube, $\varphi_{\mathbf{m}l}$ an excited state of the cube, l numbers the degenerate excited states of the cube, and λ is the number giving the distribution of the electrons over the cube. The distribution is numbered arbitrarily but in such a way that each subsequent distribution is obtained from the previous one by the interchange of two electrons belonging to different cubes.

The exciton energy is evaluated from the wave function (1) in the same way as in the Heitler-London-Heisenberg method (see reference 1, § 2, case b). The only difference is that the Heitler-London-Heisenberg method involves the wave functions of the elementary cells of the crystal, while here we have instead the wave functions of the above-mentioned cubes. We can neglect the exchange interaction energy for electrons belonging to different cubes. The energy is then equal to

$$\mathscr{E}(\mathbf{k}) = \mathscr{E}_0 + u(\mathbf{k}), \quad u(\mathbf{k}) = \sum_{\mathbf{m}} L^{(\mathbf{m})} e^{i\mathbf{k}\mathbf{m}}, \quad (2)$$

where $\mathscr{E}_0$ is a constant that does not depend on the quasi-momentum $\mathbf{k}$, and

$$L^{(\mathbf{m})} = \sum_{ll'} c_l^* c_{l'} \int \varphi_{0l}^*(r_0)\, \psi_{\mathbf{m}}^*(r_{\mathbf{m}})\, V_{\mathbf{m}0}(r_{\mathbf{m}} r_0)\, \varphi_{\mathbf{m}l'}(r_{\mathbf{m}})\, \psi_0(r_0)\, dr_0 dr_{\mathbf{m}}. \quad (3)$$

$V_{\mathbf{m}0}$ is the potential energy of the Coulomb interaction between charged particles belonging to cubes at the points $\mathbf{m}$ and 0; $r_{\mathbf{m}}$ and r_0 indicate the totality of the coordinates of these particles (for an arbitrary distribution λ). Since the cubes are electrically neutral, $V_{\mathbf{m}0}$ can be reduced to their dipole-dipole interaction. Let $\mathbf{P}(\mathbf{m})$ and $\mathbf{P}(0)$ be the dipole moments of the cube. We introduce the notation

$$\mathbf{P}_l = \int \varphi_{0l}^*(r_0)\, \mathbf{P}(0)\, \psi_0(r_0)\, dr_0, \quad \mathbf{P} = \sum_l c_l^* \mathbf{P}_l. \quad (4)$$

One can then easily show by expressing $V_{\mathbf{m}0}$ explicitly in terms of $\mathbf{P}(\mathbf{m})$ and $\mathbf{P}(0)$, substituting the result into (3) and using the notation (4) that the quantity $u(\mathbf{k})$ can be interpreted in the following way: u is equal to the electrostatic energy of the interaction of the dipole $\mathbf{P}$ at the origin with all dipoles $\mathbf{P}^* e^{i(\mathbf{k}\cdot\mathbf{m})}$ at the centers of all the cubes. Since $kL \ll 1$ one can replace the collection of dipoles placed at the centers of the cubes by a polarized continuum with a specific dipole moment $L^{-3}\mathbf{P}^* e^{i(\mathbf{k}\cdot\mathbf{r})}$. The latter produces, as is well-known, a fictitious dielectric polarization field which is equal to

$$\mathbf{E}'(\mathbf{r}) = -4\pi L^{-3}\mathbf{s}(\mathbf{P}^*, \mathbf{s})\, e^{i\mathbf{k}\mathbf{r}}, \quad \mathbf{s} = \mathbf{k}/k. \quad (5)$$

In this way u can be evaluated as the energy of the interaction between the dipole $\mathbf{P}$ and the field $\mathbf{E}'(0)$:

$$u = -\mathbf{P}\cdot\mathbf{E}'(0) = 4\pi L^{-3} |\mathbf{P}\cdot\mathbf{s}|^2 = 4\pi L^{-3} |\mathbf{P}|^2 \cos^2\alpha, \quad (6)$$

where α is the angle between $\mathbf{P}$ and $\mathbf{s}$. We can show that the quantity (6) does not depend on L, since $\mathbf{P}$ is proportional to $L^{3/2}$. We can thus,