RADIATION CORRECTIONS TO BREMSSTRAHLUNG

P. I. FOMIN

Khar' kov State University

Submitted to JETP editor April 2, 1958

J. Exptl. Theoret. Phys. (U.S.S.R.) **35**, 707-718 (September, 1958)

Radiation corrections to the differential cross section for bremsstrahlung are computed using the mass operator method. A general formula is obtained and some limiting cases are considered.

FOR the calculation of radiation corrections to cross sections one ordinarily makes use of Feynman techniques which are described in greatest detail by Brown and Feynman[1] as applied to the Compton effect. By comparison with the Compton effect an additional parameter appears in bremsstrahlung which is associated with the momentum transferred to the nucleus, and as a result of this the calculation of integrals by the Feynman method and also the evaluation of the traces of matrices becomes considerably more complicated. In the present article radiation corrections to bremsstrahlung are calculated by means of the mass operator method.[2,3] The main advantage of this method is the fact that in it one can make use of the expression for the mass operator obtained by Newton[3] in which the integration over the momenta of the virtual particles is carried out in a general way. This, in turn, facilitates the calculation of the traces of matrices. Moreover, it was also possible to develop a quite convenient technique for the evaluation of integrals which remain in the mass operator after the integrations over the momenta have been carried out.

1. GENERAL FORMULAS AND THE CROSS SECTION FOR THE FUNDAMENTAL PROCESS

As is well known,[2,3] single electron processes may be described taking radiation corrections into account, by means of the modified Dirac equation*

$$(\gamma p + m + V)\psi = 0 \tag{1}$$

where $V = e\gamma A + \Delta M$; ΔM is the regularized mass operator; A includes the radiation field A^r, the external field A^e and the regularized vacuum polarization field A':

$$A = A^r + A^e + A';$$

$$\gamma a = \gamma_\mu a_\mu = \gamma_1 a_1 + \gamma_2 a_2 + \gamma_3 a_3 + \gamma_4 a_4,$$

$$\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = -2\delta_{\mu\nu},\ (\mu,\ \nu = 1,\ 2,\ 3,\ 4).$$

*We employ units in which $\hbar = c = 1$, $e^2/4\pi = 1/137$.

On the basis of Eq. (1) it is possible to obtain the following formula for the bremsstrahlung cross section averaged over the electron spins

$$d\sigma = \frac{\pi^2}{4}\frac{d\mathbf{p}_2 d\mathbf{k}}{\omega\varepsilon_2|\mathbf{p}_1|} \times \mathrm{Sp}\,(p_2|H|p_1)(\gamma p_1 - m)(p_1|\gamma_4 H^+ \gamma_4|p_2)(m-\gamma p_2), \tag{2}$$

where

$$H = (1+VG)^{-1}V = V - VGV + \cdots, \tag{3}$$

Here $G = (\gamma p + m)^{-1}$ is the Green's function of the Dirac equation; $(p_2|H|p_1)$ is the matrix element of H between the eigenfunctions of the momentum operator normalized in such a way that $(p_2|p_1) = \delta(p_2 - p_1)$; $p_1 = (\mathbf{p}_1, i\epsilon_1)$ and $p_2 = (\mathbf{p}_2, i\epsilon_2)$ are the initial and final electron momenta; $k = (\mathbf{k}, i\omega)$ is the photon momentum. It is assumed in formula (2) that one of the δ-functions with respect to energy arising here should be replaced by $(2\pi)^{-1}$ (cf., for example, Akhiezer and Berestetskii,[4] § 28).

The radiation field in (2) should be written in the form

$$A_\mu^r(x) = e_\mu e^{-ikx},\ e_\mu^2 = 1. \tag{4}$$

The external field (the field of the nucleus at rest) satisfies the equation

$$\Box A_\mu^e(x) = -Zev_\mu\delta(\mathbf{x}),\ v_\mu = (0,\ 0,\ 0,\ i),$$

from which we obtain

$$A_\mu^e(k_1) = (2\pi)^{-2}\int A_\mu^e(x)\,e^{-ik_1x}dx = a(k_1)\,v_\mu, \tag{5}$$

where

$$a(k_1) = (Ze/2\pi)\,\delta(k_{10})/k_1^2.$$

The bremsstrahlung is described by the terms in H quadratic in the field A. By separating out in ΔM the terms ΔM_1 and ΔM_2 which are linear and quadratic in the field respectively, we can write on the basis of (3)

$$H = -e^2\gamma AG\gamma A + \Delta M_2 + e(\gamma AG\Delta M_1 + \Delta M_1 G\gamma A).$$

Newton[3] has shown [formula (2.19)] that the term $e(\gamma AG\Delta M_1 + \Delta M_1 G\gamma A)$ cancels with a part of ΔM_2 which is defined by formula (2.16). The remaining part of ΔM_2 which we shall denote by H_M is broken up into two terms [cf. Newton[3] (2.14) and (2.17)]:

$$H_M = \Delta M_2^1 + \Delta M_2^3.$$

In H one should make the substitution $A = A^r + A^e + A'$. However, the process which is of interest to us is described only by the cross terms:

$$\begin{aligned} H &= H_0 + H_M + H_{A'};\\ H_0 &= -e^2(\gamma A^r G\gamma A^e + \gamma A^e G\gamma A^r), \\ H_{A'} &= -e^2(\gamma A^r G\gamma A' + \gamma A' G\gamma A^r), \end{aligned} \tag{6}$$

while in H_M one should also retain only the cross terms in A^r and A^e. H_0 describes the fundamental bremmstrahlung process:

$$d\sigma_0 = \frac{\pi^2}{4}\frac{d\mathbf{p}_2 dk}{\omega\varepsilon_2|\mathbf{p}_1|} \times \mathrm{Sp}(p_2|H_0|p_1)(\gamma p_1 - m)(p_1|\gamma_4 H_0^+\gamma_4|p_2)(m-\gamma p_2). \tag{7}$$

The radiation corrections $\sim e^2$ are determined by the interference of H_M and $H_{A'}$ with H_0:

$$d\sigma_R = \frac{\pi^2}{4}\frac{d\mathbf{p}_2 dk}{\omega\varepsilon_2|\mathbf{p}_1|} 2\,\mathrm{Re}\,\mathrm{Sp}(p_2|H_M + H_{A'}|p_1) \times(\gamma p_1 - m)(p_1|\gamma_4 H_0^+\gamma_4|p_2)(m-\gamma p_2) \equiv d\sigma_M + d\sigma_{A'}. \tag{8}$$

Taking (4) and (5) into account we can easily obtain

$$(p_2|H|p_1) = -(e/2\pi)^2 a(q) \times[\gamma e(\gamma p_3 + m)^{-1}\gamma v + \gamma v(\gamma p_4 + m)^{-1}\gamma e], \tag{9}$$

where

$$q = p_2 + k - p_1,\quad p_3 = p_2 + k,\quad p_4 = p_1 - k. \tag{10}$$

From this it follows that

$$(p_1|\gamma_4 H_0^+\gamma_4|p_2) = \gamma_4(p_2|H_0|p_1)^+\gamma_4 = (e/2\pi)^2 a(q)\,Q, \tag{11}$$

where

$$Q = \gamma v(\gamma p_3 + m)^{-1}\gamma e + \gamma e(\gamma p_4 + m)^{-1}\gamma v. \tag{12}$$

The evaluation of the traces of matrices in (7) is easily carried out after summing over the photon polarizations which, as is well known, consists of replacing $e_\mu \to \delta_{\mu\lambda}$ (and at the same time $\gamma e\ldots\gamma e \to \gamma_\lambda\ldots\gamma_\lambda$). As a result of this we obtain the Bethe-Heitler formula which may be conveniently written for future use in the following form [cf. Akhiezer and Berestetskii[4] (31.12)]*

$$d\sigma_0 = \frac{Z^2}{\pi^2}\left(\frac{e^2}{4\pi}\right)^3 \frac{d\mathbf{p}_2 dk\,\delta(\varepsilon_1 - \varepsilon_2 - \omega)}{\omega\varepsilon_2|\mathbf{p}_1|\,\mathbf{q}^4} U_0, \tag{13}$$

*We shall take m = 1 in all specific calculations and results.

where

$$U_0 = T^2 - \rho S^2 - R^2/2\varkappa\tau, \tag{14}$$

$$\mathbf{R} = \mathbf{R}_1 - \mathbf{R}_2,\quad \mathbf{S} = (\mathbf{R}_1/\tau + \mathbf{R}_2/\varkappa)/2\omega, \quad \mathbf{T} = (\varepsilon_2\mathbf{R}_1/\tau + \varepsilon_1\mathbf{R}_2/\varkappa)/\omega,\quad \mathbf{R}_1 = 2[\mathbf{k}\times\mathbf{p}_1],\quad \mathbf{R}_2 = 2[\mathbf{k}\times\mathbf{p}_2], \tag{15}$$

$$\varkappa = p_3^2 + 1,\quad \tau = p_4^2 + 1,\quad \rho = \mathbf{q}^2, \tag{16}$$

with

$$R_1^2 = 4\omega(\varepsilon_1\tau - \omega) - \tau^2,\quad R_2^2 = -4\omega(\varepsilon_2\varkappa + \omega) - \varkappa^2; \quad R_1R_2 = 2\omega(\varepsilon_1\tau - \varepsilon_2\varkappa - \omega\rho - 2\omega) + \varkappa\tau. \tag{17}$$

2. GENERAL EXPRESSION FOR THE RADIATION CORRECTIONS

Let us examine first the contribution to the cross section made by $H_{A'}$. It is determined by the second term in formula (8) and is denoted by $d\sigma_{A'}$. We note that $H_{A'}$ differs from H_0 merely by replacing A^e by A'. The Fourier components $\delta a(q)$ and $a(q)$ of the quantities $A_4'(x)$ and $A_4^e(x)$ are related in the static case by the following expression (Akhiezer and Berestetskii,[4] § 43):

$$\delta a(q) = -(e/2\pi)^2 a(q)\,W(x), \tag{18}$$

where

$$W(x) = (1 - x\coth x)\left(1 - \frac{1}{3}\coth^2 x\right) - \frac{1}{9}, \tag{19}$$

$$4\sinh^2 x = \mathbf{q}^2,\quad \mathbf{q} = \mathbf{p}_2 + \mathbf{k} - \mathbf{p}_1.$$

From this, after taking into account (6), (7), and (8), it follows that

$$d\sigma_{A'} = d\sigma_0[-(e/2\pi)^2 2W(x)]. \tag{20}$$

Let us now write the sum $d\sigma = d\sigma_0 + d\sigma_R$ in the form

$$d\sigma = d\sigma_0[1 - (e/2\pi)^2(2W(x) + U/U_0)], \tag{21}$$

where

$$U = \left(\frac{2\pi}{e}\right)^4 \frac{1}{4a(q)}\mathrm{Re}\,\mathrm{Sp}(p_2|H_M|p_1)(\gamma p_1 - m)\,Q\,(\gamma p_2 - m). \tag{22}$$

According to Newton[3] [formula (2.10) and further],

$$H_M = \Delta M_2^1 + \Delta M_2^3 = \left(\frac{e^2}{4\pi}\right)^2 \int_0^\infty i\,ds \int_0^1 du\, \exp\{-is(u + \lambda^2/u)\} \times \int \frac{dk_1}{(2\pi)^2}\frac{dk_2}{(2\pi)^2} e^{ik_1x}[M_2^1(p, k_1, k_2) + M_2^3(p, k_1, k_2)]\,e^{ik_2x}, \tag{23}$$

where M_2^1 and M_2^3 are given by Newton's formulas[3] (2.14) and (2.17). The factor $\exp(-is\lambda^2/u)$ added by us (λ is the fictitious photon mass) guarantees the convergence of the integrals over u at the lower limit. This factor arises if in the mass operator the photon Green's function is taken to be of the form $(k^2 + \lambda^2)^{-1}$, instead of k^{-2}, in order to eliminate the infrared divergence.

We shall illustrate the method of evaluating U on the example of one of the terms from M_2^3 which we shall denote by $B(p, k_1, k_2)$:

$$B(p, k_1, k_2) = \int_0^1 dv_1 \int_0^{v_1} dv_2 \Phi(k_1, k_2) [f_B(v_1, v_2) i\gamma_\mu J_\mu(k_1) G(p) \times E(p^2) k_{1\nu} F_{\nu\lambda}(k_2) \gamma_\lambda \quad (24)$$
$$- f_B(1 - v_2, 1 - v_1) i\gamma_\lambda F_{\lambda\nu}(k_1) k_{2\nu} G(p) E(p^2) \gamma_\mu J_\mu(k_2)],$$

where

$$f_B(v_1, v_2) = -2isu(1 - v_1)[2 - u + u^2(1 - 2v_1)V],$$
$$V = v_1 - v_2,$$
$$\Phi(k_1, k_2) = \exp\{-isu[v_1(1 - v_1)k_1^2 + v_2(1 - v_2)k_2^2 + v_2(1 - v_1)2k_1k_2]\},$$
$$E(p^2) = \exp\{-is(1 - u)V(p^2 + m^2)\};$$
$$J_\mu = J_\mu^r + J_\mu^e, \quad F_{\mu\nu} = F_{\mu\nu}^r + F_{\mu\nu}^e,$$

in which, in accordance with (4) and (5),

$$\begin{aligned} F_{\mu\nu}^r(k_1) &= (2\pi)^{-2} \int F_{\mu\nu}^r(x) e^{-ik_1x} dx \\ &= i(2\pi)^2 \delta(k_1 + k)(e_\mu k_\nu - e_\nu k_\mu), \\ k_1^{-2} J_\mu^r(k_1) &= (2\pi)^2 \delta(k_1 + k) e_\mu, \quad J_\mu^r(k_1) = 0, \\ F_{\mu\nu}^e(k_1) &= -ia(k_1)(v_\mu k_{1\nu} - v_\nu k_{1\mu}), \quad J_\mu^e(k_1) = a(k_1) k_1^2 v_\mu. \end{aligned} \quad (25)$$

A contribution to bremsstrahlung will be made only by the terms

$$F^r(k_1)\ldots J^e(k_2) \text{ and } J^e(k_1)\ldots F^r(k_2).$$

Corresponding to this $B(p, k_1, k_2)$ will be broken up into two terms:

$$B = B^{re} + B^{er}. \quad (26)$$

We consider

$$H_B \equiv \left(p_2 \left| \int dk_1 dk_2 e^{ik_1x} B(p, k_1, k_2) e^{ik_2x} \right| p_1\right)$$
$$= \int dk_1 dk_2 dp \delta(p - p_1 - k_2) \delta(p - p_2 + k_1) B(p, k_1, k_2).$$

In accordance with (26) and (24) we can easily obtain

$$H_B = H_B^{re} + H_B^{er}, \quad (27)$$

where

$$H_B^{er} = (2\pi)^2 a(q) q^2 \int_0^1 dv_1 \int_0^{v_1} dv_2 \Phi(q, -k)$$
$$\times E(p_4^2) f_B(v_1, v_2) \gamma v(\gamma p_4 + 1)^{-1} \times [\gamma e(kq) - \gamma k(eq)],$$

$$H_B^{re} = (2\pi)^2 a(q) q^2 \int_0^1 dv_1 \int_0^{v_1} dv_2 \Phi(-k, q) E(p_3^2)$$
$$\times f_B(1 - v_2, 1 - v_1)[\gamma e(kq) - \gamma k(eq)](\gamma p_3 + 1)^{-1}\gamma v.$$

We denote the contributions of H_B, H_B^{re} and H_B^{er} to U respectively by U_B, U_B^{re} and U_B^{er}. After the evaluation of traces U_B will be expressed only in terms of the quantities κ, τ, ρ, ϵ_1, and ϵ_2. Therefore

$$U_B = U_B^{re}(\varkappa, \tau, \rho, \varepsilon_1, \varepsilon_2) + U_B^{er}(\varkappa, \tau, \rho, \varepsilon_1, \varepsilon_2).$$

We shall now show that

$$U_B^{er}(\varkappa, \tau, \rho, \varepsilon_1, \varepsilon_2) = U_B^{re}(\tau, \varkappa, \rho, \varepsilon_2, \varepsilon_1). \quad (28)$$

Indeed, if in U_B^{er} we replace $k \to -k$, $q \to -q$, $p_1 \rightleftarrows p_2$ (and at the same time $p_3 \rightleftarrows p_4$), and if in the integrals we introduce new variables $v_1 \to 1 - v_2$, $v_2 \to 1 - v_1$ [with $\Phi(q, -k) \to \Phi(-k, q)$], we shall obtain under the trace sign in U_B^{er} the same factors as in U_B^{re}, but written out in inverted order. After the evaluation of the trace the two expressions will become the same. The replacement of the momenta indicated above is equivalent to the replacement $\kappa \rightleftarrows \tau$, $\epsilon_1 \rightleftarrows \epsilon_2$ in the final result, and this completes the proof of assertion (28). This property is common to all the terms in ΔM_2^3 and ΔM_2^1 and may be easily checked for each term. Thus

$$U = U^{re}(\varkappa, \tau, \rho, \varepsilon_1, \varepsilon_2) + U^{re}(\tau, \varkappa, \rho, \varepsilon_2, \varepsilon_1), \quad (29)$$

and therefore it is sufficient to evaluate only $U^{re}(\kappa, \tau, \rho, \epsilon_1, \epsilon_2)$.

The quantity U_B^{re} may be written in the form

$$U_B^{re} = J_B h_B / 16, \quad (30)$$

where

$$J_B = \mathrm{Re} \int_0^\infty ids \int_0^1 du e^{-isu} \int_0^1 dv_1 \int_0^{v_1} dv_2 \Phi(-k, q) \times E(p_3^2) f_B(1 - v_2, 1 - v_1), \quad (31)$$

$$h_B = \mathrm{Sp}\, q^2 [\gamma_\lambda(kq) - q_\lambda(\gamma k)](\gamma p_3 + 1)^{-1}\gamma v(\gamma p_1 - 1) Q_\lambda(\gamma p_2 - 1), \quad (32)$$
$$Q_\lambda = \gamma v(\gamma p_3 + 1)^{-1}\gamma_\lambda + \gamma_\lambda(\gamma p_4 + 1)^{-1}\gamma v.$$

The evaluation of the trace of h_B leads to the result

$$\begin{aligned} h_B &= (2\rho/\varkappa)(2\rho b/\varkappa\tau + \tau\alpha) \\ &- 4\rho\{\rho\omega(\varepsilon_1/\tau + \varepsilon_2/\varkappa) + (2\rho/\varkappa)(\varepsilon_1^2 + \varepsilon_2^2) \\ &+ (\varepsilon_1/\varkappa + \varepsilon_2/\tau)[\varepsilon_1(1 + 4\alpha/\varkappa) - \varepsilon_2\tau]\}, \end{aligned} \quad (33)$$

where

$$\alpha = \varkappa + \tau, \quad b = \alpha^2 + \varkappa\tau(\rho - \alpha).$$

With the aid of (15) and (17) this may be written in the form

$$h_B = 2\rho[2\tau U_0 + T(R_1 + R_2) - 2\omega RS]. \quad (34)$$

We introduce the notation

$$J[f(v_1, v_2)] = \int_0^\infty ids \int_0^1 du \exp\{-is(u+\lambda^2/u)\} \int_0^1 dv_1 \int_0^{v_1} dv_2 \times \Phi(q, -k) E(p_3^2) f(v_1, v_2), \qquad (35)$$

where

$$\Phi(q, -k) = \exp\{-is[v_1(1-v_1)\rho - v_2(1-v_1)\alpha]u\},$$

$$E(p_3^2) = \exp\{-is(1-u)V\varkappa\}.$$

After introducing in (31) new variables $v_1 \to 1 - v_2$, $v_2 \to 1 - v_1$, (with $\Phi(-k, q) \to \Phi(q, -k)$, while $E(p_3^2)$ remains unchanged) we obtain:

$$J_B = J[(f_B(v_1, v_2)]$$
$$= J\{-2isu(1-v_1)[2-u+u^2V(1-2v_1)]\}.$$

Making use of obvious properties of the integrals $J(f)$ we can easily rewrite J_B in the form

$$J_B = 2J[isu(u-2)(1-v_1)]$$
$$+ 2\frac{\partial}{\partial\alpha} J(u^2) + 2\frac{\partial}{\partial\rho}[J(u^2) - 2J(u^2V)].$$

It is shown in the Appendix [cf. (A.6) and (A.5)] that

$$J[isu^2(1-v_1)] = -\varkappa\frac{\partial}{\partial\varkappa} J[isu(1-v_1)],$$

$$2J(u^2) = \left(1 - \varkappa\frac{\partial}{\partial\varkappa}\right) J(u).$$

Thus it now remains to calculate $J(isu)$, $J(isuv_1)$, $J(u)$ and $J(u^2V)$. The method of carrying out these calculations is given in the Appendix.

It turns out that all the integrals appearing in ΔM_2^3 with the exception of $J(ise^{-is\lambda^2/u})$ can be expressed in a similar manner in terms of these four integrals and their derivatives with respect to the parameters κ, α and ρ. The integrals appearing in ΔM_2^1, and likewise $J(ise^{-is\lambda^2/u})$, may be evaluated directly; the results are given in the Appendix.

We have described the method of calculating the contribution to U^{re} of the term B. The contribution made by the remaining terms may be evaluated in an analogous manner. As a result of all these calculations the following result has been obtained for U^{re}

$$\begin{aligned} 2U^{re}(\varkappa, \tau, \rho, \varepsilon_1, \varepsilon_2) = {} & T^2(2y/\sinh 2y) + U_0\{2(1-2y\coth 2y)\ln\lambda + 4y\coth 2y[h(2y) - h(y)] + 2 - y\coth y \\ & - (4+\rho)y/\sinh 2y + 2\tau M + \text{antisymm. terms} - (\tau U_0 + S_1\partial/\partial\varkappa)J[u^2(1-V)] + (S_2\partial/\partial\varkappa + S_3\partial/\partial\alpha \\ & + S_4\partial/\partial\rho)J(u) + (S_5\partial/\partial\varkappa + S_6\partial/\partial\alpha + S_7\partial/\partial\rho)J(u^2) + (S_8\partial/\partial\alpha + S_9\partial/\partial\rho)J(u^2V) + (S_{10} - S_{11}\varkappa\partial/\partial\varkappa \\ & + {}^1/_2 S_{12}\varkappa^2\partial^2/\partial\varkappa^2)J(isu) + (S_{13} - S_{14}\varkappa\partial/\partial\varkappa + {}^1/_2 S_{15}\varkappa^2\partial^2/\partial\varkappa^2)J(isuv_1), \end{aligned} \qquad (36)$$

where

$$2bJ(u) = [\varkappa(\alpha-\rho) - 2\alpha]L - \varkappa(\rho-\alpha)(\rho-\alpha+4)M + \varkappa[(\rho-\alpha)(\rho-\varkappa+4) + 2\alpha]N; \qquad (37)$$

$$2bJ(isu) = \alpha L + (\varkappa-\tau)(\rho-\alpha)M + [(\varkappa-\tau)(\alpha-\rho) + \alpha\tau]N, \qquad (38)$$

$$2bJ(isuv_1) = \tau L - [\tau(\rho-\alpha) + 2\alpha]M + [\tau(\rho-\varkappa) + 2\alpha]N, \qquad (39)$$

$$\begin{aligned} \alpha J(u^2V) = {} & (\rho-\alpha)\left[\varkappa\left(\frac{\partial}{\partial\alpha} + \frac{\partial}{\partial\rho}\right)J(u) - J(u^2)\right] - \frac{\varkappa^2}{4\rho}\left[(\rho-\varkappa)\frac{\partial N}{\partial\varkappa} - N + \frac{\ln|\varkappa|}{\varkappa-1}\right] - \frac{1}{\varkappa}[F(\varkappa-1) - F(-1)] \\ & - \frac{(\varkappa-2)\ln|\varkappa|}{2(\varkappa-1)} + x\coth x + \frac{2(\rho-\alpha)}{\alpha}(x\coth x - y\coth y); \end{aligned} \qquad (40)$$

$$L = 2(y^2 - x^2) + F(\varkappa-1) - F(-1),$$

$$M = \int_0^1 \frac{dv}{1+(1-v)v(\rho-\alpha)} \ln\left|\frac{1+(1-v)v\rho}{\varkappa v}\right| = 2(\sinh 2y)^{-1}\{(y+x)h(y+x) + (y-x)h(y-x) - yh(y) - y\ln|\varkappa|\},$$

$$N = \int_0^1 \frac{dv}{1+(1-v)v\rho - \varkappa v} \ln\left|\frac{1+(1-v)v\rho}{\varkappa v}\right|; \quad yh(y) = \int_0^y u\coth u\,du, \quad F(x) = \int_0^x \frac{\ln|1+u|}{u}du, \quad F(-1) = -\pi^2/6,$$

$$4\sinh^2 x = \rho, \quad 4\sinh^2 y = \rho - \alpha; \quad \alpha = \varkappa + \tau, \quad b = \alpha^2 + \varkappa\tau(\rho-\alpha)$$

and

$$S_1 = \varkappa\tau U_0 + R^2 + \varkappa RT + \varkappa\tau\alpha S^2 - 2\rho\omega SR_2, \quad S_2 = (\varkappa-2)\tau U_0 - (\alpha/2\varkappa\tau)R^2 - 2RT - 2\varkappa\tau(T^2 + \varkappa S^2) + 4\rho\varepsilon_1 SR_2,$$

$$S_3 = -(\alpha/\varkappa\tau)R^2 - 6\varkappa\tau\rho S^2 + (\alpha - 4)RT + 2\rho TR_1 + 2\rho(\omega - 4\varepsilon_1)RS,$$

$$S_4 = S_3 + (1 + \alpha/\varkappa\tau)R^2 + 4RT + \varkappa\tau\alpha S^2 - \rho(R_1^2/\tau + R_2^2/\varkappa) - 2P,$$

$$S_5 = -2\tau U_0 + \varkappa\tau[T^2 + (\rho + \varkappa)S^2] - (1/\varkappa + \alpha/2\varkappa\tau)R^2 - 2RT - \rho[TR_1 + 2\varepsilon_1 S(R_2 - R)],$$

$$S_6 = \tau\rho U_0 - (1 + \alpha/\varkappa\tau)R^2 + \varkappa\tau(4\rho - \alpha)S^2 - (4 + \alpha)RT + \rho[2(\varepsilon_1 + \varepsilon_2)RS + R_1^2/\tau + R_2^2/\varkappa],$$

$$S_7 = S_6 + \alpha R^2/\varkappa\tau + 4RT - 2P, \qquad S_8 = 2\rho\omega RS + (\varkappa - \tau)RT, \qquad S_9 = -2\tau\rho U_0 + 2\rho[2\omega RS - T(R_1 + R_2)],$$

$$S_{10} = (3\varkappa - 2\rho - 2)\tau U_0 + \alpha R^2/2\varkappa\tau + 2RT, \qquad S_{11} = (\rho - \varkappa)\tau U_0 - R^2/\varkappa - (\rho/\varkappa\tau)RR_1 - 3TR - 4\tau\rho S^2 - P,$$

$$S_{12} = 2\tau U_0 + \frac{\alpha}{2\varkappa\tau}R^2 + 3RT + \frac{1}{\varkappa}\left(R^2 + \frac{\rho}{\tau}RR_1 + 4\varkappa\tau\rho S^2\right) - P,$$

$$S_{13} = 2\tau\rho U_0, \qquad S_{14} = -\tau\rho U_0 - S_{15} + 2P, \qquad S_{15} = -\rho\left(\frac{2}{\varkappa\tau}RR_1 + 4\tau S^2\right) + P,$$

$$P = \frac{\rho}{\varkappa}\left[\frac{1}{2}\left(\frac{1}{\tau} - \frac{1}{\varkappa}\right)R^2 - 2T(R_1 + R_2) - 2\tau U_0\right]. \tag{41}$$

The cross section for the fundamental process $d\sigma_0$ vanishes for zero angles when $\mathbf{k} \times \mathbf{p}_1 = \mathbf{k} \times \mathbf{p}_2 = 0$. At the same time the radiation corrections found by us also vanish, as should have been expected, since they are determined by interference with the matrix element of the fundamental process.

If the accuracy with which the energy is measured does not exceed a certain value $\Delta E \ll m$ then in actual fact the quantity which will be measured will be the total cross section of the ordinary bremsstrahlung of a single photon of momentum $\mathbf{k}$ and of processes in which additional soft photons are emitted whose total energy does not exceed ΔE. To the order in e^2 which we are at present considering we should also add to $d\sigma$ the cross section $d\sigma_D$ of double bremsstrahlung when the additional radiation consists of the emission of a single photon of low energy $\omega_1 < \Delta E$. According to the general rule[5]

$$d\sigma_D = d\sigma_0(e/2\pi)^2 D, \tag{42}$$

where

$$D = \frac{1}{4\pi}\int_0^{\Delta E}\frac{d\mathbf{k}_1}{\sqrt{\mathbf{k}_1^2 + \lambda^2}}\left(\frac{p_2}{p_2\cdot k_1} - \frac{p_1}{p_1\cdot k_1}\right)^2. \tag{43}$$

The expression for D may be written in the following form (Akhiezer and Berestetskii,[4] § 32.4):

$$D = 2(1 - 2y\coth 2y)\ln\frac{\lambda}{2\Delta E} \tag{44}$$

$$+\frac{1}{2}\left[\frac{1}{v_1}\ln\frac{1+v_1}{1-v_1} + \frac{1}{v_2}\ln\frac{1+v_2}{1-v_2} - \coth 2y\cdot Y(y, \varepsilon_1, \varepsilon_2)\right],$$

where

$$v_1\varepsilon_1 = |p_1|, \quad v_2\varepsilon_2 = |p_2|,$$

$$Y(y, \varepsilon_1, \varepsilon_2) = \int_1^1 \frac{dz}{\coth^2 y - z^2\sinh^2 y}\frac{\varepsilon_z}{p_z}\ln\frac{\varepsilon_z + p_z}{\varepsilon_z - p_z}, \tag{45}$$

$$2\varepsilon_z = \varepsilon_1 + \varepsilon_2 + \omega z, \quad p_z^2 = \varepsilon_z^2 + z^2\sinh^2 y - \coth^2 y.$$

It may be easily seen that the term in $\ln\lambda$ cancels out in the total cross section $d\sigma + d\sigma_D$.

3. LIMITING CASES

On the basis of the general formulas (36) and (42) we can easily obtain certain limiting expressions for the radiation corrections to bremsstrahlung. Five independent parameters: ϵ_1, ω, $\theta_1 = \widehat{\mathbf{k}\mathbf{p}_1}$, $\theta_2 = \widehat{\mathbf{k}\mathbf{p}_2}$, and $\theta = \widehat{\mathbf{p}_1\mathbf{p}_2}$ appear in the cross section. All the other parameters may be expressed in terms of them:

$$\varkappa = -2\omega(\varepsilon_2 - p_2\cos\theta_2), \quad \varepsilon_2 = \varepsilon_1 - \omega,$$

$$\tau = 2\omega(\varepsilon_1 - p_1\cos\theta_1), \quad p_1 = \sqrt{\varepsilon_1^2 - 1},$$

$$\rho = -2 + \varkappa + \tau + 2(\varepsilon_1\varepsilon_2 - p_1 p_2\cos\theta), \quad p_2 = \sqrt{\varepsilon_2^2 - 1}.$$

We consider four particularly interesting limiting cases in which the general formula can be considerably simplified. We write the total cross section $d\bar{\sigma} = d\sigma + d\sigma_D$ in the form

$$d\sigma = d\sigma_0[1 - (e/2\pi)^2\delta], \tag{46}$$

where

$$\delta = U/U_0 + 2W(x) - D. \tag{47}$$

(1) The case of low frequencies ($\omega\epsilon_1 \ll 1$). In the limiting case of almost elastic scattering, i.e., for radiation of low frequencies, we should expect a simple relation between the radiation corrections to bremsstrahlung and the radiation corrections to elastic scattering.[6] Indeed, in the limit $\omega \to 0$ ($p_2 \to p_1 = p$, $\epsilon_2 \to \epsilon_1 = \epsilon$) the bremsstrahlung cross section may be represented in the form of the product

$$d\sigma_0 = d\sigma_y^0 \cdot w_\omega^0 \tag{48}$$

of the elastic scattering cross section

$$d\sigma_y^0 = \left(\frac{Ze^2}{8\pi pv}\right)^2\frac{1 - v^2\sin^2(\theta/2)}{\sin^4(\theta/2)}dO_2 \tag{49}$$

by the probability of radiation occurring during scattering

$$w_\omega^0 = \frac{e^2}{2(2\pi)^3}\cdot\frac{v^2}{p^2}\,\frac{d\omega dO}{\omega V_1 V_2}\left(4p^2\sin^2\frac{\theta}{2} + 2 - \frac{V_1}{V_2} - \frac{V_2}{V_1}\right), \tag{50}$$

where

$$v = p/\varepsilon, \quad V_1 = 1 - v\cos\theta_1, \quad V_2 = 1 - v\cos\theta_2.$$

On taking radiation corrections into account

$$d\sigma_y^0 \to d\sigma_y = d\sigma_y^0\,[1 - (e/2\pi)^2\delta_y],$$

$$w_\omega^0 \to w_\omega = w_\omega^0\,[1 - (e/2\pi)^2\delta_\omega],$$

so that our formula (46) should in the limit $\omega \to 0$ reduce to

$$d\bar{\sigma} = d\sigma_y^0\cdot w_\omega^0\,[1 - (e/2\pi)^2(\delta_y + \delta_\omega)], \tag{51}$$

where δ_y is determined by Schwinger's formula[6] (cf. also Akhiezer and Berestetskii,[4] § 45).

If we assume $\omega\epsilon_1 \ll 1$ [and at the same time $|\kappa|$, $\tau \ll 1$, $\rho = 4p^2\sin^2(\theta/2)$], we obtain the following value* for δ

$$\delta = 2(1 - 2x\coth 2x)\ln 2\Delta E + x\tanh x + 4x\coth 2x\,[h(2x) - h(x)] + 2W(x) + \frac{2x}{\sinh 2x}\,\frac{\rho}{4\varepsilon^2 - \rho}$$
$$+ \frac{1}{v}\ln\frac{1+v}{1-v} + \frac{(1-v^2)\coth 2x}{v\sin\theta/2}\,G(v, \theta) \tag{52}$$
$$+ O(\omega\varepsilon_1\ln\omega, \omega\varepsilon_1) = \delta_y + O(\omega\varepsilon_1\ln\omega, \omega\varepsilon_1).$$

Here

$$G(v, \theta) = \int\limits_{\cos\theta/2}^{1}\frac{du}{(1 - v^2u^2)(u^2 - \cos^2(\theta/2))^{1/2}}\ln\frac{1+vu}{1-vu}.$$

Thus, $\delta_\omega = 0(\omega\epsilon, \ln\omega)$, i.e., it behaves like $\omega\ln\omega$ as $\omega \to 0$. In the non-relativistic limit (cf. below)

$$\delta_\omega = {}^4/_3(\mathbf{p}_1 - \mathbf{p}_2)\,\mathbf{k}\ln\omega. \tag{53}$$

(2) Relativistic case (ϵ_1, $\epsilon_2 \gg 1$). At high energies, small angles (θ, θ_1, $\theta_2 \sim 1/\epsilon$) play the principal role in the cross section $d\sigma_0$. For these values of the angles the formula for the corrections is considerably simplified only if we assume $\omega \ll \epsilon_1$, in which case we may impose on the angles the less restrictive condition $\theta_1^2 \ll 1/\omega\epsilon_1$, $\theta_2^2 \ll 1/\omega\epsilon_2$. Under these conditions $|\kappa|$, $\tau \ll 1$, $\rho = \epsilon_1\epsilon_2\theta^2 \ll \epsilon_1/\omega$, and

$$\delta = 2(1 - 2x\coth 2x)\left(\ln\frac{\Delta E}{\varepsilon_1} + 1\right) + x\tanh x + 2W(x) + O\left(\frac{\omega}{\varepsilon_1}\ln\varepsilon_1, \frac{\omega}{\varepsilon_1}\right). \tag{54}$$

In order to estimate the order of magnitude of the corrections we take $\epsilon_1 = 100$, $x = 1$. In this case

$$(e/2\pi)^2\delta \simeq 0.02\,(1 - 0.25\ln 2\,\Delta E). \tag{55}$$

*cf. also Fomin.[7]

(3) Ultrarelativistic case. It is of interest to investigate the behavior of the corrections for angles which are not too small. In this case, to achieve simplification, we take energies so large that

$$\ln(\rho - \alpha),\ \ln\rho,\ \ln|\varkappa|,\ \ln\tau \gg 1,$$

but at the same time

$$\ln\frac{\rho - \alpha}{|\varkappa|},\ \ln\frac{\rho - \alpha}{\tau},\ \ln\frac{\rho}{|\varkappa|},\ \ln\frac{\rho}{\tau} \sim 1.$$

At the same time

$$\delta = 2(1 - 2y)\ln 2\Delta E + 4y^2 - 3y - 2y\ln\frac{\rho - \alpha}{4\varepsilon_1\varepsilon_2} + \frac{4}{3}x$$
$$- \ln\varepsilon_1\varepsilon_2 + O(1), \qquad 2x = \ln\rho,\ 2y = \ln(\rho - \alpha). \tag{56}$$

From this it follows that, roughly speaking, the dependence on the energy is of the form

$$\delta = A\ln^2\varepsilon_1 + B\ln\varepsilon_1 + C.$$

Such behavior is characteristic in general for all the radiation corrections of order e^2 which we are at the moment discussing. At sufficiently high energies the correction may attain values of 10 to 20%.

(4) Nonrelativistic case ($p_1 \ll 1$). At low energies we obtain

$$\delta = \frac{2}{3}\left\{(\mathbf{p}_1 - \mathbf{p}_2)^2\left(\frac{19}{30} - \ln 2\Delta E\right) + 2(\mathbf{p}_1 - \mathbf{p}_2)\,\mathbf{k}\ln\omega\right\} + O(p^3). \tag{57}$$

The fact that the corrections tend to zero as $p_1 \to 0$ is a characteristic feature. We note that in the nonrelativistic limit corrections to the elastic scattering,[6] to the Compton effect,[1] and to the Møller scattering,[8] also vanish. The corrections to the two-photon pair annihilation[9] approach a finite limit as the relative velocity of the electron and the positron tends to zero. However, this limit is not a truly nonrelativistic one, but rather a threshold one, since in the inverse process of pair creation this will correspond to the threshold of the reaction. On the basis of these results we may apparently make a general assertion that the radiation corrections always vanish in the true nonrelativistic limit, while in the case of threshold reactions they approach a finite limit at the threshold.

The author takes this opportunity to express his gratitude to Prof. A. I. Akhiezer for suggesting the subject of the present investigation and for valuable aid, and also to V. G. Bar'iakhtar and S. V. Peletminskii for considerable aid in calculations at the earlier stages of the work, and to V. F. Aleksin and D. V. Volkov for valuable advice.

APPENDIX

We shall now present the method of evaluating the integrals. We shall first give the values of certain auxiliary integrals:

$$\int_0^1 [1 + v(1-v)\cdot 4\sinh^2 y]^{-1} dv = 2y/\sinh 2y; \qquad \int_0^1 dv_1 \int_0^{v_1} dv_2 [1 + (1-v_1)v_2\cdot 4\sinh^2 y]^{-1} = y^2/2\sinh^2 y;$$

$$\int_0^1 \ln v\cdot[1 + (1-v)v\cdot 4\sinh^2 y]^{-1} dv = -2yh(y)(\sinh 2y)^{-1}; \tag{A.1}$$

$$\int_0^1 \frac{\ln|1+(1-v)v\cdot 4\sinh^2 y|}{1+(1-v)v\cdot 4\sinh^2 y} dv = \frac{4}{\sinh 2y}\int_0^y u\tanh u\,du = \frac{4y}{\sinh 2y}[h(2y) - h(y)].$$

We now consider $J(isu)$. First of all, we note that the integral over the proper time s is defined in such a way that

$$\int_0^\infty i ds e^{-isa} = a^{-1} \text{and} \int_0^\infty i ds e^{-isa} is = a^{-2}.$$

In accordance with this, we have

$$J(isu) = \int_0^1 du \int_0^1 dv_1 \int_0^{v_1} dv_2 \cdot u\,\{[1 + (1-v_1)(v_1\rho - v_2\alpha)]u + (1-u)V\varkappa]^{-2}.$$

After introducing the new variable $(1-u)u^{-1} = t$ the integration over v_2 and T may be easily carried out and leads to

$$J(isu) = \int_0^1 \frac{v_1 dv_1}{1+(1-v_1)v_1\rho - v_1\varkappa}\,\frac{1}{1+(1-v_1)v_1(\rho-\alpha)} \ln\left|\frac{1+(1-v_1)v_1\rho}{\varkappa v_1}\right|.$$

With the aid of the auxiliary integrals given above this integral may be easily brought to the final form (38). The evaluation of $J(isuv_1)$ is carried out in a completely analogous manner.

After integration over s and after the introduction of the new variable $(1-u)/u = t$ the integral $J(u)$ takes on the form

$$J(u) = \int_0^\infty \frac{dt}{(1+t)^2}\int_0^1 dv_1 \int_0^{v_1} dv_2 [1 + (1-v_1)(v_1\rho - v_2\alpha) + \varkappa Vt]^{-1}. \tag{A.2}$$

Integration by parts with respect to t gives

$$J(u) = 2(x^2 - y^2)/\alpha - \varkappa J(isuV). \tag{A.3}$$

By introducing the new variables $v_1 \to v_1$, $v_2 \to v_1 - V$ and by changing the order of integration over V and v_1 the integral $J(isuv)$ may be brought to the form

$$J(isuV) = -\frac{1}{\alpha}\int_0^\infty \frac{dt}{1+t}\int_0^1 dV \int_V^1 dv_1 \frac{[V\alpha - (1-2v_1)(\rho-\alpha)] + (1-2v_1)(\rho-\alpha)}{\{1+(1-v_1)[v_1(\rho-\alpha)+V\alpha] + \varkappa Vt\}^2}. \tag{A.4}$$

The part of the integral which corresponds to the last term in the numerator is evidently equal to $\frac{\rho-\alpha}{\alpha} J[isu(1-2v_1)]$ and may be found directly with the aid of (38) and (39). The remaining part of the integral assumes the following form after integration over v_1 and t has been carried out:

$$-\frac{1}{\alpha}\left\{\frac{1}{\varkappa}\cdot[F(\varkappa-1) - F(-1)] - N\right\}.$$

Finally we obtain the result (37) for $J(u)$.

By means of differentiating (A.2) with respect to κ and then by means of integrating by parts with respect to t it can be shown that

$$\varkappa\frac{\partial}{\partial\varkappa} J(u) = J(u) - 2J(u^2),$$

from which it follows that

$$J(u^2) = \tfrac{1}{2}(1 - \varkappa\partial/\partial\varkappa)\, J(u). \tag{A.5}$$

In a similar way it can be shown that

$$\begin{gathered} J(isu^2) = -\varkappa\frac{\partial}{\partial\varkappa}J(isu),\quad J(isu^2v_1) = -\varkappa\frac{\partial}{\partial\varkappa}J(isuv_1),\\ J(isu^3) = \frac{1}{2}\left(1-\varkappa\frac{\partial}{\partial\varkappa}\right)J(isu^2) = \frac{\varkappa^2}{2}\frac{\partial^2}{\partial\varkappa^2}J(isu),\qquad J(isu^3v_1) = \frac{\varkappa^2}{2}\frac{\partial^2}{\partial\varkappa^2}J(isuv_1). \end{gathered} \tag{A.6}$$

We now consider the integral $J(u^2V)$. It can be shown in a manner similar to that used for deriving (A.3) that

$$J(u^2V) = \frac{1}{2}\int_0^1 dv_1\int_0^{v_1} dv_2\frac{V}{1+(1-v_1)(v_1\rho - v_2\alpha)} - \frac{\varkappa}{2}J(isu^2V^2).$$

The first integral can be easily evaluated. By carrying out a transformation similar to (A.4) we readily obtain for $J(isu^2V^2)$

$$\begin{gathered} J(isu^2V^2) = \frac{\rho-\alpha}{\alpha}J[isu^2(1-2v_1)V] + \frac{1}{\varkappa}\left\{\frac{2}{\varkappa}[F(\varkappa-1) - F(-1)] + \frac{\varkappa-2}{\varkappa-1}\ln|\varkappa| - 2\right\}\\ + \frac{\varkappa}{2\rho}\left[(\rho-\varkappa)\frac{\partial N}{\partial\varkappa} - N + \frac{\ln|\varkappa|}{\varkappa-1}\right]. \end{gathered}$$

Further, it can be easily checked that

$$J[isu^2(1-2v_1)V] = -2\left(\frac{\partial}{\partial\alpha} + \frac{\partial}{\partial\rho}\right)J(u) - J(isu^2V).$$

$$J(isu^2V) = J(isuV) - J[isu(1-u)V] = \frac{2}{\varkappa\alpha}(x^2-y^2) - \frac{1}{\varkappa}\left(1-\varkappa\frac{\partial}{\partial\varkappa}\right)J(u).$$

As a result we obtain for $J(u^2V)$ the value (40).

The evaluation of integrals appearing in ΔM_2^1 and also of $J(ise^{-is\lambda^2/u})$ may be easily carried out by means of the procedures outlined above. We now give the results:

$$J(ise^{-is\lambda^2/u}) = \frac{2}{\varkappa\sinh 2y}\left[yh(y) - (y+x)h(y+x) - (y-x)h(y-x) - y\ln\frac{\lambda}{|\varkappa|}\right];$$

$$I\{v(1-2v)[u^2+2isu(1-u)]\} = \frac{1}{\rho-\alpha}\left\{1 - y\coth y + \frac{4y}{\sinh 2y}[h(2y) - h(y)] + \left(1 - \frac{2y}{\sinh 2y}\right)(1+\ln\lambda^2)\right\}; \tag{A.7}$$

$$I[u(u-1)] = -y/\sinh 2y; \qquad I(2-u) = \frac{2y}{\sinh 2y}\{2[h(2y) - h(y)] - 1 - \ln\lambda^2\}.$$

Here we have introduced the notation

$$I[f(s,u,v)] = \int_0^\infty ids\int_0^1 du\int_0^1 dvf(s,u,v)\times\exp\{-isu[1+(1-v)v(\rho-\alpha)] - is\lambda^2/u\} \tag{A.8}$$

and in the course of the calculation we have neglected all those terms which vanish as $\lambda\to 0$.

[1] L. M. Brown and R. P. Feynman, Phys. Rev. **85**, 231 (1952).
[2] J. Schwinger, Proc. Nat. Acad. Sci. **37**, 455 (1951).
[3] R. G. Newton, Phys. Rev. **94**, 1773 (1954).
[4] A. I. Akhiezer and V. B. Berestetskii, **Квантовая электродинамика** (Quantum Electrodynamics), M. 1953, (Engl. transl. publ. by U. S. Dept. of Commerce).
[5] J. M. Jauch and F. Rohrlich, Helv. Phys. Acta. **27**, 613 (1954).
[6] J. Schwinger, Phys. Rev. **76**, 790 (1949).
[7] P. I. Fomin, J. Exptl. Theoret. Phys. (U.S.S.R.) **34**, 227 (1958), Soviet Phys. JETP **7**, 156 (1958).
[8] M. L. G. Redhead, Proc. Roy. Soc. **A220**, 219 (1953).
[9] I. Harris and L. M. Brown, Phys. Rev. **105**, 1656 (1957).

Translated by G. Volkoff
141