ON THE PROPAGATION OF ELECTROMAGNETIC WAVES IN A MEDIUM WITH APPRECIABLE SPATIAL DISPERSION

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A method for dealing with electromagnetic waves in a medium with spatial dispersion is presented which is more detailed than that given by Ginzburg. Expansions are obtained for the "direct" and the "inverse" spatial dispersions.

In recent papers by Pekar[1] and Ginzburg[2] treatments have been given of certain effects that appear in the propagation of electromagnetic waves in media with appreciable spatial dispersion of the dielectric constant. In determining the relation between the polarization **P** and the electric field intensity **E**, Pekar has at certain points used a microscopic approach. On the other hand, Ginzburg's treatment was phenomenological. These papers differed in many of their essential conclusions. Thus it is of interest to examine once more the question of including effects of spatial dispersion.

The microscopic approach is not indispensable for the determination of the connection between the macroscopic quantities **P** and **E**. If we confine ourselves hereafter to a range of frequencies close to one of the resonance frequencies,* a sufficiently general relation between **P** and **E** can be established easily. In fact, with accuracy up to terms of order $(a/\lambda)^2$ inclusive,† the energy of the polarized medium, including effects of spatial dispersion, has the form:

$$H = \delta \int\limits_V \left[\dot{P}_i \dot{P}_i + \beta_{ik} P_i P_k + \gamma_{ikl} P_i \frac{\partial P_k}{\partial x_l} + \alpha_{iklm} \frac{\partial P_i}{\partial x_l} \frac{\partial P_k}{\partial x_m} \right] dV + \frac{1}{8\pi} \int (\mathbf{E}\cdot\mathbf{D} + \mathbf{H}\cdot\mathbf{B})\, dV, \qquad (1)$$

where β_{ik}, γ_{ikl}, α_{iklm} are components of tensors characterizing the medium. The tensor γ_{ikl} corresponds to the optical activity, and vanishes if the substance has a center of inversion. If we go over from the quantities **P**, **E**, and **D** to their Fourier components, the relation (1) leads to the following equations of motion:

$$-\omega^2 P_{\mathbf{q}i} + \beta_{ik} P_{\mathbf{q}k} + i\gamma_{ikl} q_l P_{\mathbf{q}k} + \alpha_{iklm} q_l q_m P_{\mathbf{q}k} - (\omega_0^2/4\pi) D_{\mathbf{q}i} = 0, \qquad (2)$$

where **q** is the wave vector, and $\omega_0 = (2\pi/\delta)^{1/2}$ is the plasma frequency corresponding to the dispersion electrons. If there are no free charges in the medium, Maxwell's equations give an additional relation connecting the quantities $P_{\mathbf{q}i}$ and $D_{\mathbf{q}i}$:

$$-\omega^2 \mathbf{D}_\mathbf{q} + c^2 q^2 \mathbf{D}_\mathbf{q} + 4\pi c^2 \{\mathbf{q}(\mathbf{q}\cdot\mathbf{P}_\mathbf{q}) - q^2 \mathbf{P}_\mathbf{q}\} = 0. \qquad (3)$$

Equations (2) and (3) completely solve the problem of the characteristic vibrations of the polarized medium.

For isotropic and, generally speaking, for gyrotropic media $\beta = \beta$, $\alpha = \alpha_{1,2}$, $\gamma = \gamma e$, where e is the completely antisymmetric unit tensor of the third rank. Then the system of equations (2) and (3) takes the following form:

$$(\beta + \alpha_1 q^2 - \omega^2)\mathbf{P}_\mathbf{q} + i\gamma[\mathbf{P}_\mathbf{q}\times\mathbf{q}] + \alpha_2 \mathbf{q}(\mathbf{p}_\mathbf{q}\cdot\mathbf{q}) - (\omega_0^2/4\pi)\mathbf{D}_\mathbf{q} = 0,$$
$$4\pi c^2\{\mathbf{q}(\mathbf{q}\cdot\mathbf{P}_\mathbf{q}) - q^2\mathbf{P}_\mathbf{q}\} + (c^2 q^2 - \omega^2)\mathbf{D}_\mathbf{q} = 0. \qquad (4)$$

For longitudinal waves $\mathbf{D}_\mathbf{q} = 0$, $\omega^2 = \beta + \alpha q^2$, where $\alpha = \alpha_1 + \alpha_2$. If we introduce the index of refraction n, we find that for longitudinal waves

$$n_\parallel^2 = (c^2/\alpha\omega^2)(\omega^2 - \beta). \qquad (5)$$

For the index of refraction of the transverse waves the system (4) gives an equation of the second degree in the variable n^2.

If the medium is inactive, the equation has the following form:

$$(n_\perp^2 \alpha_1 \omega^2/c^2 - \omega^2 + \beta)(n_\perp^2 - 1) - n_\perp^2 \omega_0^2 = 0. \qquad (6)$$

If we set $\alpha_1 = 0$, then

$$n_0^2 = (\omega^2 - \beta)/(\omega^2 - \beta + \omega_0^2) \equiv \varepsilon_0(\omega), \qquad (7)$$

where n_0 is the index of refraction of the medium with neglect of spatial dispersion. From the relations (5) and (7) it follows that at resonance, where

*Just this frequency region is sensitive for the effects of spatial dispersion.

†For crystals a is of the order of the lattice constant, for a plasma it is the Debye screening radius, and so on.

$\epsilon_0 \to \infty$, $n_{||} \neq 0$. This conclusion agrees with Ginzburg's work and contradicts that of Pekar. In the general case of an anisotropic medium one gets from Eqs. (2) and (3) a complicated equation for the determination of $n(\omega)$:

$$\left\| \left(\omega^2 + \frac{n^2\omega_0^2}{n^2-1}\right)\delta_{ik} - \beta_{ik} - in\frac{\omega}{c}\gamma_{ikl}s_l - n^2\frac{\omega^2}{c^2}\alpha_{iklm}s_l s_m - \frac{n^2\omega_0^2}{n^2-1}s_i s_k \right\| = 0 \quad (8)$$

$(\mathbf{q} = n\omega\mathbf{s}/c)$. In particular there follows from Eq. (8) the conclusion of Ginzburg, that in cubic crystals inclusion of the spatial dispersion leads to a weak anisotropy of the index of refraction.*

If in Eq. (2) we introduce instead of the quantities $\mathbf{P_q}$ quantities $\mathbf{E_q}$ and $\mathbf{D_q} = \mathbf{E_q} + 4\pi\mathbf{P_q}$, we get the relation

$$(\hat{A} + \omega_0^2)\mathbf{D_q} = \hat{A}\mathbf{E_q}, \quad (9)$$

where

$$A_{ik} = \omega^2\delta_{ik} - \beta_{ik} - i\gamma_{ikl}q_l - \alpha_{iklm}q_l q_m.$$

We introduce two matrices $\hat{a}$ and $\hat{b}$ $(\hat{a} + \hat{b} = A)$:

$$a_{ik} = \omega^2\delta_{ik} - \beta_{ik}, \quad b_{ik} = -i\gamma_{ikl}q_l - \alpha_{iklm}q_l q_m.$$

Let us suppose that the coordinate axes coincide with the principal axes of the tensor β_{ik} $(\beta_{ik} = \beta_i\delta_{ik})$. Then the expansions of the direct and inverse dispersion give as expressions for the quantities $\mathbf{D_q}$ and $\mathbf{E_q}$ as power series in the small matrix $\hat{b}$ the following:

$$\mathbf{D_q} = \mathbf{E_q} - \omega_0^2[(\hat{a}+\omega_0^2)^{-1} - (\hat{a}+\omega_0^2)^{-1}\hat{b}(\hat{a}+\omega_0^2)^{-1} + (\hat{a}+\omega_0^2)^{-1}\hat{b}(\hat{a}+\omega_0^2)^{-1}\hat{b}(\hat{a}+\omega_0^2)^{-1} - \ldots]\mathbf{E_q}, \quad (10)$$

$$\mathbf{E_q} = \mathbf{D_q} + \omega_0^2[\hat{a}^{-1} - \hat{a}^{-1}\hat{b}\hat{a}^{-1} + \hat{a}^{-1}\hat{b}\hat{a}^{-1}\hat{b}\hat{a}^{-1} - \ldots]\mathbf{D_q}. \quad (11)$$

The expansion (10) is valid in the range of frequencies in which $|\omega^2 - \beta_i + \omega_0^2| \gg |b_{ik}|$, and the expansion (11) in the range in which $|\omega^2 - \beta_i| \gg |b_{ik}|$. From the expansion (10) one can determine the "gyration vector" $\mathbf{f}$ (cf. reference 3). For isotropic media and cubic crystals $\mathbf{f} = f\mathbf{s}$, where

$$f = (\omega/c)\,\omega_0^2\gamma/(\omega^2 + \omega_0^2 - \beta)^2. \quad (12)$$

Since the amount of rotation of the plane of polarization is proportional to ωf, far from resonance the relation (12) leads to Chandrasekhar's empirical formula,[4] which describes the dispersion of the optical activity in certain crystals consisting of inactive molecules. For molecular crystals of arbitrary symmetry the Chandrasekhar formula has been obtained previously by one of the present writers.[5] It is easy to show that the results of reference 5 also follow from the treatment that has been carried through above.

In conclusion the writers express their sincere gratitude to V. L. Ginzburg for valuable discussions, which were of very considerable help in the carrying out of this work.

*This anisotropy is absent in Pekar's work.

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