THE BEHAVIOR OF A PARTICLE OF ARBITRARY SPIN IN AN EXTERNAL MAGNETIC FIELD

V. S. POPOV

Moscow State University

Submitted to JETP editor May 13, 1958

J. Exptl. Theoret. Phys. (U.S.S.R.) **35**, 985-988 (October, 1958)

A method of disentangling given by Feynman is used to solve the problem of the way the polarization of a particle possessing a magnetic moment changes in an external magnetic field.

In the present paper we treat the problem of the change of the polarization of a particle possessing a magnetic moment under the action of an external magnetic field of arbitrary time dependence. The solution is obtained by means of a method given by Feynman[1] for disentangling operator expressions containing noncommutating operators. This method has hitherto not been widely applied. It has been used to solve only one problem — that of the harmonic oscillator subjected to the action of an arbitrary external force (cf. reference 1, Sec. 5, and reference 4). Therefore it is of interest to solve other quantum-mechanical problems by the Feynman method, in order to elucidate the properties of this method and the difficulties that arise in applying it to concrete problems. One such problem, which can be solved completely by the Feynman method, is considered below. The results themselves are not new, and are for the most part contained in a paper by Majorana,[2] where they were obtained by a different method.

Let us consider a particle with magnetic moment $\mathbf{M} = \gamma\hbar\mathbf{I}$. Here γ is the gyromagnetic ratio and $\mathbf{I}$ is the spin angular momentum of the particle. The transformation of the wave function $\psi(t)$ from the time t to the infinitesimally different time $t + \Delta t$ is accomplished by means of the unitary operator

$$S(t, t+\Delta t) = 1 + i\gamma \mathbf{H}(t)\,\mathbf{I}\,\Delta t = \exp(i\gamma\mathbf{H}(t)\,\mathbf{I}\,\Delta t),$$

where $\mathbf{H}(t)$ is the external magnetic field. From this we get for a finite time interval:

$$\psi(t_2) = S(t_2, t_1)\,\psi(t_1),$$
$$S(t_2, t_1) = \lim_{N\to\infty}\prod_{i=1}^{N}\exp(i\gamma\mathbf{H}(t_i)\,\mathbf{I}\,\Delta t_i) = \exp\left[i\gamma\int_{t_1}^{t_2}\mathbf{H}(t)\,\mathbf{I}\,dt\right], \quad (1)$$

where in this formula t is an ordering parameter and indicates the order of action of the infinitesimal operators $1 + i\gamma\mathbf{H}(t)\mathbf{I}\Delta t$. Because of the fact that the components of the vector $\mathbf{I}$ do not commute with the exponent in Eq. (1) we cannot proceed with the usual rules familiar in analysis.

We shall try to represent $S(t_2, t_1)$ in a "disentangled" form:

$$S(t_2, t_1) = \exp(a\hat{I}_1)\exp(b\hat{I}_0)\exp(c\hat{I}_{-1}),$$
$$\hat{I}_{\pm 1} = (\mp\hat{I}_x - i\hat{I}_y)/\sqrt{2},\ \hat{I}_0 = \hat{I}_z, \quad (2)$$

where a, b, c are for the present unknown functions of the time.

By means of Eq. (2) one can easily find the probability amplitude of a transition from the state $|jm\rangle$ at the time t_1 to the state $|jm'\rangle$ at the time t_2, since on using for the right-hand exponent the series expansion

$$\exp(c\hat{I}_{-1}) = \sum_{k=0}^{\infty}\frac{c^k}{k!}\hat{I}_{-1}^k$$

we see that when it acts on ψ_{jm} only a finite number of terms of this series remain. Similarly, $\psi^*_{jm'}\exp(a\hat{I}_1)$ reduces to a finite sum, and consequently the transition amplitude $\langle jm'|S(t_2, t_1)|jm\rangle$ also contains only a finite number of terms.

To disentangle the operators $\hat{I}_1$, $\hat{I}_0$, $\hat{I}_{-1}$ we use the following artifice: we break up H_1I_1 into a sum of two terms:

$$\mathbf{HI} = xI_1 + (H_1 - x)I_1 + H_0I_0 + H_{-1}I_{-1},$$
$$H_{\pm 1} = (\mp H_x + iH_y)/\sqrt{2},\ H_0 = H_z \quad (3)$$

and apply to the first term the theorem on the disentangling of an exponential factor proved by Feynman (cf. reference 1, Sec. 3):

$$S(t_2, t_1) = \exp\left[i\gamma\hat{I}_1\int_{t_1}^{t_2}x(t')\,dt'\right]$$
$$\times\exp\left\{i\gamma\int_{t_1}^{t_2}[(H_1 - x)I_1' + H_0I_0' + H_{-1}I_{-1}']\,dt'\right\}, \quad (4)$$
$$I_\mu'(t) = \exp\left[-i\gamma\hat{I}_1\int_{t_1}^{t_2}x(t')\,dt'\right]I_\mu\exp\left[i\gamma\hat{I}_1\int_{t_1}^{t_2}x(t')\,dt'\right].$$

We set

$$a(t) = i\gamma \int_{t_1}^{t_2} x(t')\,dt' \qquad (5)$$

and determine a(t) in such a way that the operator I_1 in Eq. (4) may be completely disentangled. To do this we first find the explicit form of the operators $I'_\mu(t)$. Let $I_\mu(\alpha) = e^{\alpha I_1} I_\mu e^{-\alpha I_1}$; differentiating with respect to α, we obtain the system of equations

$$dI_1(\alpha)/d\alpha = 0,\ dI_0(\alpha)/d\alpha = -I_1(\alpha),$$
$$dI_{-1}(\alpha)/d\alpha = -I_0(\alpha)$$

with the initial conditions $I_\mu(0) = \hat{I}_\mu$. The solution of these equations is

$$I_1(\alpha) = \hat{I}_1,\ I_0(\alpha) = \hat{I}_0 - \alpha\hat{I}_1,\ \hat{I}_{-1}(\alpha) = \hat{I}_{-1} - \alpha\hat{I}_0 + \frac{\alpha^2}{2}\hat{I}_1. \qquad (6)$$

In an analogous way we get

$$e^{\alpha I_0} I_1 e^{-\alpha I_0} = I_1 e^{\alpha},\ e^{\alpha I_0} I_{-1} e^{-\alpha I_0} = I_{-1} e^{-\alpha}. \qquad (7)$$

Now substituting the operators I'_μ from Eq. (6) into Eq. (4) and equating the coefficient of I_1 to zero, we get the equation for the determination of a(t)

$$da/dt = i\gamma(H_1 + H_0 a + {}^1/_2 H_{-1} a^2),\ a(t_1) = 0. \qquad (8)$$

Hereafter we shall everywhere set $t_1 = 0$.

We now have only to disentagle I_0 by means of a transformation of the type (7), and S(t, 0) is then reduced to the form (2), where a(t) is determined from Eq. (8) and the determination of b(t) and c(t) reduces to quadratures:

$$b(t) = i\gamma \int_0^t [H_0(t') + H_{-1}(t')\,a(t')]\,dt'.$$
$$c(t) = i\gamma \int_0^t H_{-1}(t')\,e^{b(t')}\,dt'. \qquad (9)$$

An essential point is that a, b, c depend on the gyromagnetic ratio γ, but do not depend on the value of the spin j. Therefore the solution is obtained all at once for particles of arbitrary spin with the same value of γ.

Let us consider an example: a constant field, and perpendicular to it a uniformly rotating alternating field

$$H_z = H_0,\ H_x = H_1\cos\omega t,\ H_y = H_1 \sin\omega t. \qquad (10)$$

Such a magnetic field is used in experiments on the measurement of the magnetic moments of atomic nuclei, and in this connection was first considered by Rabi.[3] Equation (8) takes the form:

$$-i\frac{da}{d\tau} = \frac{\sin\theta}{\sqrt{2}}(e^{-i\lambda\tau} - {}^1/_2 e^{i\lambda\tau}a^2) - a\cos\theta,$$

where

$$\tau = \omega_0 t,\ \omega_0 = -\gamma\sqrt{H_0^2 + H_1^2},\ \tan\theta = H_1/H_0,\ \lambda = \omega/\omega_0.$$

Solving the equation with the boundary condition a(0) = 0, we find

$$a(\tau) = 2(e^{-i\lambda\tau} - e^{-i\mu\tau})/(n_1 e^{i(\lambda-\mu)\tau} - n_2),$$
$$e^{b(\tau)} = (n_1 - n_2)^2 e^{-i\mu\tau}/(n_1 e^{i(\lambda-\mu)\tau} - n_2)^2,$$
$$c(\tau) = 2(e^{i(\lambda-\mu)\tau} - 1)/(n_1 e^{i(\lambda-\mu)\tau} - n_2), \qquad (11)$$
$$\mu = \cos\theta + n_1\sin\theta/\sqrt{2},$$

where n_1, n_2 are the roots of the equation

$$x^2 + 2\sqrt{2}x(\cos\theta - \lambda)/\sin\theta - 2 = 0.$$

Suppose the particle has spin ½ and that at the initial time the z component of the spin is ½. By means of Eq. (2) we find that the probability of reversal of the spin during the time t is given by

$$P_{{}^1/_2\to-{}^1/_2}(t) = {}^1/_2\,|c(t)\,e^{-b(t)/2}|^2$$
$$= \frac{q^2\sin^2\theta}{1+q^2-2q\cos\theta}\sin^2\left(\frac{\omega t}{2}\sqrt{1+q^2-2q\cos\theta}\right), \qquad (12)$$
$$q = \omega_0/\omega = 1/\lambda,$$

which agrees with the result of Rabi.[3]

Let us consider another representation of the S matrix, which is more convenient for particles with larger spin j; we try to represent S(t, 0) in the form

$$S(t, 0) = \exp(i\alpha\hat{I}_x)\exp(i\beta\hat{I}_y)\exp(i\gamma\hat{I}_z), \qquad (13)$$

where α, β, γ are real functions of t (the reality follows from the unitary character of the S matrix; for the representation (2) a, b, c do not have to be real). Carrying out calculations analogous to those performed above, we find that the disentangling of $\hat{I}_x$, $\hat{I}_y$, $\hat{I}_z$ is possible and α, β, γ are determined by a system of equations of the type of Eq. (8), but more complicated. An important fact is that these equations involve only the gyromagnetic ratio γ, and not the spin j itself. Consequently, α, β, and γ do not depend on j and can be expressed uniquely in terms of the a, b, c determined from Eqs. (8) and (9). But each factor in Eq. (13) is the operator for a finite rotation around one of the coordinate axes, and therefore

$$S(t, 0) = D^{(j)}(\varphi, \vartheta, \psi), \qquad (14)$$

and furthermore the Eulerian angles φ, ϑ, ψ that define the resultant rotation are uniquely related to a, b, c and do not depend on j. To express φ, ϑ, ψ in terms of a, b c it suffices to examine the S matrix for a particle with spin ½ situated in the same magnetic field. From Eq. (2)

we find the form of the S matrix in the system of functions $|\frac{1}{2}m\rangle$, $m = \pm\frac{1}{2}$:

$$S^{(1/2)}(t, 0) = \begin{pmatrix} e^{b/2} - (ac/2)\, e^{-b/2} & (c/\sqrt{2})\, e^{-b/2} \\ -(a/\sqrt{2})\, e^{-b/2} & e^{-b/2} \end{pmatrix} = D^{(1/2)}(\varphi, \vartheta, \psi), \tag{15}$$

from which we have

$$\sin\frac{\vartheta}{2} = \left(\frac{ac}{2}\, e^{-b}\right)^{1/2}, \quad e^{2i\varphi} = \frac{a}{c\,(e^{b} - ac/2)}, \quad e^{2i\psi} = \frac{c}{a\,(e^{b} - ac/2)}. \tag{16}$$

For the example (10) discussed above we have

$$\sin\frac{\vartheta}{2} = \frac{q \sin\theta}{\sqrt{1 + q^2 - 2q\cos\theta}} \sin\left(\frac{\omega t}{2}\sqrt{1 + q^2 - 2q\cos\theta}\right). \tag{17}$$

The probability of a transition during the time t from the state $|jm\rangle$ into the state $|jm'\rangle$, for a particle with spin j, is given by

$$P_{m\to m'}(t) = |D^{(j)}_{mm'}(\varphi, \vartheta, \psi)|^2$$
$$= [(j+m)!\,(j-m)!\,(j+m')!\,(j-m')!] \cos^{4j}\left(\frac{\vartheta}{2}\right) \tag{18}$$
$$\times \left[\sum_{\nu} (-1)^{\nu} \frac{\left(\tan\frac{\vartheta}{2}\right)^{2\nu - m + m'}}{\nu!\,(\nu - m + m')!\,(j + m - \nu)!\,(j - m' - \nu)!}\right]^2.$$

If we take here $j = \frac{1}{2}$, $m = \frac{1}{2}$, $m' = -\frac{1}{2}$ and substitute the value (17), we get the result (12).

In the general case the polarization state of a particle is characterized by a density matrix ρ. Expanding it in terms of the tensor operators T_{IM}, we find the way the polarization changes with time:

$$\rho = \sum_{0\leqslant I\leqslant 2j} \rho_{IM} T_{IM}; \quad \rho_{IM}(t) = \sum_{M'} \rho_{IM'}(0)\, D^{(I)}_{M'M}(\varphi, \vartheta, \psi). \tag{19}$$

If the particle has not only a magnetic dipole moment but also higher multipole moments, a complete disentangling of the S matrix cannot be carried through. This is due to the fact that for $I > 1$ the commutator $[T_{IM}, T_{IM'}]_-$ of two tensor operators of the same rank I cannot be expressed in terms of tensor operators of this same rank. It is always possible however, to separate out from the S matrix the part corresponding to the magnetic-dipole part of the Hamiltonian, $-\gamma(\mathbf{HI})$. A meaning can be given to this if it is permissible to regard the rest of the Hamiltonian as a perturbation.

I am sincerely grateful to I. S. Shapiro for his interest in this work and a number of valuable suggestions.

[1] R. P. Feynman, Phys. Rev. **84**, 108 (1951).
[2] E. Majorana, Nuovo cimento **9**, 43 (1937).
[3] I. I. Rabi, Phys. Rev. **51**, 652 (1937).
[4] I. Fujiwara, Prog. Theoret. Phys. **7**, 433 (1952).

Translated by W. H. Furry
205