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ON THE THEORY OF THE OPTICAL PROPERTIES OF CONDUCTORS IN THE CASE OF OBLIQUE INCIDENCE OF THE RADIATION

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A macroscopic theory of the optical properties of conductors is given for the case in which the mean free path of the electrons is not small in comparison with the characteristic distance for damping of the electromagnetic field. The difference from the usual optics of conductors comes from the inclusion of the surface current caused by the scattering of the electrons at the surface of the specimen. The case considered is that of incidence of radiation at an arbitrary angle with the surface of a massive conductor. It is shown that the effective complex indices of refraction depend on the angle of incidence, and also that the indices for different polarizations of the radiation differ from each other by a quantity of the order of the ratio of the speed of an electron to the speed of light.

THE theory of the optical properties of conductors with neglect of the anomalous skin effect is a problem of macroscopic electrodynamics. For an isotropic substance one can use as the constitutive equation of the material the relation

$$\mathbf{D} = \varepsilon(\omega)\mathbf{E}, \tag{1}$$

which connects the amplitude of the electric displacement with that of the electric field strength in the case in which the time dependence has the form $e^{i\omega t}$. In Eq. (1) $\epsilon(\omega) = \epsilon_1(\omega) + i\epsilon_2(\omega)$ is the complex dielectric permeability (our $\mathbf{D}$ is often written $\mathbf{D} - i(4\pi/\omega)\mathbf{j}$). On the other hand, it is often necessary to take into account the anomalous character of the skin effect, which is caused by the fact that in the optical region the depth of penetration of the electromagnetic field into a conductor is often comparable with, or even much smaller than, the mean free path of the electrons. Ordinarily the treatment of the optical properties of conductors in the region of the anomalous skin effect is based on a microscopic theory.[1-7] In the present note we shall give a macroscopic theory of the optics of conductors which is valid for the case in which the mean free path of the electrons is not small in comparison with the characteristic distance for damping of the electromagnetic field.

For the applicability of the macroscopic description it is essentially sufficient that during a single period of the field an electron traverses a distance much smaller than the skin depth, and in the optical region this condition is always fulfilled. The distance traversed by an electron during one vibration of the field is of the order of magnitude v/ω, where v is the speed of the electrons. Therefore we assume that

$$v/\omega \ll \delta \sim c/\sqrt{|\varepsilon(\omega)|}\omega \quad \text{or} \quad v/c \ll 1/\sqrt{|\varepsilon(\omega)|}. \tag{2}$$

For ordinary metals in the optical region this is equivalent to the condition $|\epsilon(\omega)| \ll 10^5$. Owing to this we can assume that during a period of the field a conduction electron is not displaced; therefore the connection of the current and the field is a local one, and in particular we can use the relation (1). At the surface of the conductor, however, owing to the diffuse scattering of the electrons, an additional effect arises which produces a surface resistance. Since such an effect of scattering at the surface is important only for electrons that lie within a depth below the surface of the metal of the order of magnitude of the distance traversed by an electron during one vibration of the field, it is clear that because of the smallness of this distance in comparison with the skin depth in the optical region we can assume that the diffuse scattering at the surface leads to a surface current*

$$\mathbf{i} = \gamma(\omega)\{\mathbf{E} - \mathbf{n}(\mathbf{n}\,\mathbf{E})\}. \tag{3}$$

Here $\mathbf{n}$ is the normal to the surface of the conductor and $\gamma(\omega)$ is the surface conductivity. An elementary estimate based on the theory of free

*It follows Onsager's general principle of the symmetry of the kinetic coefficients[8] that for an anisotropic crystal γ is a symmetric surface tensor, $\gamma_{ik} = \gamma_{ki}$. Furthermore in the case of no losses γ_{ik} is pure imaginary, and losses are due to the real part of γ_{ik}.

electrons gives in the optical region $\gamma \sim (e^2n_0/m)v(2\pi/\omega)^2$, where n_0 is the density of the conduction electrons, m is the mass and v the speed of an electron, and ω is the frequency of the alternating field. It is assumed moreover that the fraction of the electrons that undergoes diffuse scattering at the surface of the conductor is not small compared with unity. This value of γ agrees in order of magnitude with the value that can be deduced from the papers devoted to the microscopic optics of conductors.[1-6]

By means of Maxwell's equations and the material equations (1) and (3), one can determine the complex reflection coefficients. For a plane semi-infinite conductor the ratio of the amplitudes of the reflected wave (**R**) and the incident wave (**A**), in the case in which the electric vector is perpendicular to the plane of incidence, is given by

$$r_s = \frac{\cos\theta - \sqrt{\varepsilon(\omega) - \sin^2\theta} - 4\pi\gamma/c}{\cos\theta + \sqrt{\varepsilon(\omega) - \sin^2\theta} + 4\pi\gamma/c} = \frac{\cos\theta - \sqrt{n_s^2 - \sin^2\theta}}{\cos\theta + \sqrt{n_s^2 - \sin^2\theta}}. \tag{4}$$

Here θ is the angle of incidence (the angle between the direction of the wave vector of the incident wave and the normal to the surface of the conductor), and $n_s(\theta)$ is the effective complex index of refraction, given by the expression (cf. reference 7)

$$n_s^2(\theta) = \varepsilon(\omega) + (8\pi\gamma/c)\sqrt{\varepsilon(\omega) - \sin^2\theta} + (4\pi\gamma/c)^2 \tag{5}$$

One takes everywhere the value of the square root that has a negative imaginary part. The effective index of refraction is introduced in such a way that Eq. (4) agrees in form with the usual equation, the only difference being that n_s is now a function of the angle of incidence θ.

For the case of a wave with the electric vector lying in the plane of incidence we have the following expression for the ratio of the components of the respective amplitudes along the normal to the surface of the conductor:

$$r_p = \frac{\cos\theta\,[\sqrt{\varepsilon - \sin^2\theta} + 4\pi\gamma/c + \sin^2\theta/\sqrt{\varepsilon - \sin^2\theta}] - 1}{\cos\theta\,[\sqrt{\varepsilon - \sin^2\theta} + 4\pi\gamma/c + \sin^2\theta/\sqrt{\varepsilon - \sin^2\theta}] + 1} = \frac{n_p^2\cos\theta - \sqrt{n_s^2 - \sin^2\theta}}{n_p^2\cos\theta + \sqrt{n_s^2 - \sin^2\theta}}, \tag{6}$$

where $n_p(\theta)$ is defined by the relation

$$n_p^2(\theta) = \varepsilon(\omega) + \frac{8\pi\gamma}{c}\sqrt{\varepsilon - \sin^2\theta} + \left(\frac{4\pi\gamma}{c}\right)^2 + \frac{4\pi\gamma}{c}\,\frac{\sin^2\theta}{\sqrt{\varepsilon(\omega) - \sin^2\theta}}. \tag{7}$$

From a comparison of the formulas (5) and (7) we see that $n_s(\theta)$ and $n_p(\theta)$ are the same, and we can speak of the existence of a single complex index of refraction, only under the condition that $\epsilon(\omega) \gg 1$, so that we can neglect the last term of the right member of Eq. (7), or else under the condition that we can neglect $\gamma(\omega)$. The latter condition corresponds to the normal skin effect.

In the limiting case $\epsilon(\omega) \gg 1$ the complex indices of refraction are not only equal, but also are independent of the angle of incidence. Therefore under these conditions the existence of the surface current (3) can be taken into account by the introduction of the effective dielectric permittivity

$$\varepsilon_{\text{eff}}(\omega) = \varepsilon(\omega) + (8\pi\gamma/c)\sqrt{\varepsilon(\omega)} + (4\pi\gamma/c)^2.$$

In the short-wave region $\epsilon(\omega)$ is not large in comparison with unity, so that in taking the anomalous skin effect into account in this region it is necessary to use different complex indices of refraction for waves with different polarizations. Such an assertion is already made in the paper of Collins[7] on the microscopic theory of the anomalous skin effect for oblique incidence of the radiation on the surface of the metal. Quantitatively, however, there is an important difference here. Namely, in our treatment it turns out that n_p and n_s differ by a quantity $\sim 4\pi\gamma/c$, which is of the order v/c in the region $\epsilon(\omega) \sim 1$. In Collins' work, on the other hand, the two indices of refraction differ in this region by a quantity of the order of unity. This difference is due to the fact that in reference 7 the equation div **E** = 0 is assumed for the microscopic field, and this is approximately legitimate only in the region $|\epsilon| \gg 1$.

We write out below the formulas that determine the principal angle of incidence Θ and the principal azimuth Ψ; the former is given by the equation

$$(\operatorname{Re}\Xi)^2 + (\operatorname{Im}\Xi)^2 = 1, \tag{8}$$

and for the latter we have

$$\cot 2\Psi = -\operatorname{Re}\Xi/\operatorname{Im}\Xi, \tag{9}$$

where

$$\Xi = \frac{\cos\Theta}{\sin^2\Theta}\left\{\sqrt{\varepsilon - \sin^2\Theta} - \frac{4\pi\gamma}{c}\left[\frac{1}{1 - \varepsilon - (4\pi\gamma/c)\sqrt{\varepsilon - \sin^2\Theta}} - 1\right]\right\}. \tag{10}$$

In the particular case in which we can neglect the imaginary part of $\epsilon(\omega)$ and the real part of $\epsilon(\omega)$ is negative, we have in place of Eqs. (8) and (9):

$$\sin^4\theta - \sin^2\theta\cos^2\theta + \varepsilon(\omega)\cos^2\theta \approx 0, \qquad (8')$$

$$\cot 2\Psi = \frac{4\pi\gamma'}{c\sqrt{|\varepsilon - \sin^2\theta|}}\,\frac{\varepsilon(\omega)}{\varepsilon(\omega)-1}. \qquad (9')$$

In this same case we have for the reflection coefficients:

$$R_s = |r_s|^2 = 1 - \frac{16\pi\gamma\cos\theta}{c(1-\varepsilon(\omega))}, \qquad (11)$$

$$R_p = |r_p|^2 = 1 - \frac{16\pi\gamma\cos\theta}{c(1-\varepsilon(\omega))}\,\frac{\sin^2\theta - \varepsilon(\omega)}{\sin^2\theta - \cos^2\theta\,\varepsilon(\omega)}. \qquad (12)$$

According to Eq. (10) the terms containing $\gamma(\omega)$ can also play a very important part in the case in which the imaginary part of $\epsilon(\omega)$ is small and the real part is positive and close to unity.

We note finally that our formula (6) has been obtained under the condition that the inequality $|\epsilon(\omega)| \ll 1$ does not hold. In the opposite case it is necessary to consider the excitation of plasma waves, and to do this the material equations (1) must be generalized to include spatial derivatives of the electric field.

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