COHERENT ELECTRON RADIATION IN A SYNCHROTRON. I

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The potential for the radiation field of a single relativistic electron moving in a circle and simultaneously executing small phase oscillations is considered. A general expression for the radiation generated by a bunch is obtained using the potential for a single phase-modulated electron and a distribution function for the particles in the bunch. The expression applies at low harmonics, in which case the radiation is coherent.

ELECTRONS in high-energy synchrotrons radiate over a wide frequency spectrum: this spectrum extends from the radio-frequency region, corresponding to wave lengths of the order of the orbital radius, to the ultraviolet or x-ray region.

In the region in which the wave lengths are much larger than the characteristic distance between the electrons in a bunch the radiation is partially coherent. The total intensity of the radiation in a given narrow portion of the spectrum is not proportional to the number of accelerated electrons N and is a strong function of the dimensions of the electron bunch and the nature of the particle distribution in a bunch.

The long dimension of a bunch (orbital direction) is determined by the phase oscillations. The transverse dimension, which is characteristic of the betatron oscillations, is small, being of the order of a millimeter in high-energy accelerators. Hence the coherent radiation in a synchrotron is affected primarily by the phase oscillations of electrons and is characteristic of the phase-oscillation distribution of the particles in a bunch.

A comprehensive experimental investigation of coherent radiation has been carried out by Prokhorov,[1] using a 5-Mev synchrotron. This author calculated the coherent radiation for several phase distributions for the particles in a bunch.[2] Somewhat later, Rytov (Report of the Institute of Physics, Academy of Sciences, U.S.S.R., 1950) calculated the coherent radiation assuming that all particles in a bunch execute phase oscillations of the same amplitude. Up to the present time no analysis has been made of the coherent radiation of a bunch with an arbitrary phase distribution for the particles. However, this problem is important because the coherent radiation can be used as a means of investigating various physical processes and particle loss during acceleration.

Before computing the radiation of a bunch we consider the radiation of a single electron which moves in a circle of radius r_0 with an angular velocity ω_0, simultaneously executing phase oscillations at an angular frequency $\Omega \ll \omega_0$.

We introduce a coordinate system with origin at the center of the circle and take the plane of the circle as the xy plane. The electron coordinates as functions of time are:

$$x(t) = (r_0 - \rho_0 \cos \Omega t) \cos (\omega_0 t + \Phi \sin \Omega t),$$
$$y(t) = (r_0 - \rho_0 \cos \Omega t) \sin (\omega_0 t + \Phi \sin \Omega t). \quad (1)$$

Here Φ is the amplitude of the phase oscillations and ρ_0 is the amplitude of the radial oscillations.

We assume that the phase oscillations are harmonic and are given by the relation $\psi = \Phi \sin \Omega t$. The vector potential of the electron field can be expanded in a double Fourier series in the wave zone. Making use of the fact that $\rho_0/r_0 \sim \Omega/\omega_0 = 1/n_0 \ll 1$, we obtain the Fourier amplitudes of the potential of the electron field (to terms of order Ω/ω_0):

$$A_{xnm}(\mathbf{R}_0) = -ie\beta \frac{e^{ik_{nm}R_0}}{R_0} J'_n(n\beta \sin\theta) J_m(n\Phi),$$
$$A_{ynm}(\mathbf{R}_0) = \frac{e}{\sin\theta} \frac{e^{ik_{nm}R_0}}{R_0} J_n(n\beta \sin\theta) J_m(n\Phi). \quad (2)$$

Here $k_{nm} = (n\omega_0 + m\Omega)/c$, $\mathbf{R}_0$ and θ are the radius vector and the polar angle of the point of observation, $\beta = v/c$, and the $J_n(x)$ are Bessel functions.

When $\Phi = 0$, (2) vanishes for $m \neq 0$ and becomes the well-known expression for the Fourier amplitude of the potential of a charge which moves with uniform motion when $m = 0$.[3]

We multiply (2) by $\exp\{-i(n\omega_0 + m\Omega)t\}$ and write the resulting vector in the form:

$$\mathbf{A}_{nm}(\mathbf{R}_0, t) = \mathbf{A}_n(\mathbf{R}_0, t)\exp\{-im\Omega(t - R_0/c)\} J_m(n\Phi), \tag{3}$$

where $\mathbf{A}_n(\mathbf{R}_0, t)$ is the potential of the n-th harmonic in the spectrum of a uniformly rotating electron.

It is apparent from Eq. (3) that each harmonic in the spectrum of the uniformly rotating electron is to be associated with a band of frequencies in the spectrum of the phase-modulated electron. The amplitude distribution in the n-th band is given by the factor $J_m(n\Phi)$, as in the case of an ordinary phase-modulated signal. $J_m(n\Phi)$ falls off rapidly when $|m| > |n|\Phi$. Hence the number of lines in each branch of the n-th band is $|m| \leq |n|$ (usually $\Phi \leq 1$ in an accelerator). When $n > n_0$, adjacent bands start to overlap and the interpretation of the spectrum in terms of bands is no longer meaningful.

We now consider the phase-oscillation distribution for the particles.

We introduce a normalized particle distribution function in the phase plane: $w(\psi, \dot{\psi}, t)$ which gives the number of particles at a time t in an element $d\Gamma$ of the phase plane: $dn = Nw(\psi, \dot{\psi}, t)\,d\Gamma$. The function w is normalized to unity and the integration extends over the entire region of Γ which is enclosed by the stability boundaries. The nonlinearity of the phase oscillations cannot be neglected near the stability boundaries. We shall analyze the problem in the linear approximation, assuming that the amplitude of the oscillations is small compared with the region of stability; for this reason the exact form of the stability boundaries is not important. We shall assume that they are ellipses: $\psi^2 + \dot{\psi}^2/\Omega^2 = \Phi_{max}^2$; Φ_{max} is the amplitude of the largest nonlinear oscillation. The linear approximation is valid under the assumption that the mean amplitude in a bunch satisfies the condition $\Phi_0 \ll \Phi_{max}$.

Starting from Liouville's theorem and the oscillatory nature of the particle motion, it is a simple matter to find the distribution function at any instant of time once it is known for the entire plane at some earlier time:

$$w(\psi, \dot{\psi}, t) = w(\Phi\sin(u-\alpha),\ \Omega\Phi\cos(u-\alpha),\ t_0),$$

$$\Phi = \sqrt{\psi^2 + \dot{\psi}^2/\Omega^2};\quad u = \arcsin(\psi/\Phi);\quad \alpha = \Omega(t - t_0).$$

In the simplest case the particle distribution is stationary, i.e., the distribution density at each point is independent of time, depending only on the amplitude of the phase oscillations: $w(\psi, \dot{\psi}, t) = w(\Phi)$. Frequently one uses a normalized distribution function (normalized to unity between 0 and Φ_{max}) for the amplitudes $f(\Phi)$; this function gives the number of particles with amplitudes in the range $d\Phi: dn = Nf(\Phi)\,d\Phi$. For a stationary distribution we have $f(\Phi) = 2\pi\Omega\Phi d(\Phi)$. We introduce another normalized (normalized to unity in the interval $-\Phi_{max}$ to $+\Phi_{max}$) particle phase distribution function $\varphi(\psi, t)$, which gives the number of particles at a time t in the angular interval $d\psi$: $dn = N\varphi(\psi, t)\,d\psi$.

For a stationary distribution:

$$\varphi(\psi) = \frac{1}{\pi}\int_{|\psi|}^{\Phi_{max}} \frac{f(\Phi)}{\sqrt{\Phi^2 - \psi^2}}\,d\Phi.$$

For simplicity, we usually assume in the calculations a particle phase distribution of the form:

$$\varphi(\psi) = \begin{cases} 1/4\,\Phi_0; & |\psi| \leqslant 2\Phi_0 \\ 0\ ; & |\psi| > 2\Phi_0 \end{cases}$$

$$f(\Phi) = \Phi/2\Phi_0\sqrt{(2\Phi_0)^2 - \Phi^2}, \tag{4}$$

or else assume that all particles execute phase oscillations of the same amplitude, i.e.,

$$f(\Phi) = \delta(\Phi - \Phi_0),\quad \varphi(\psi) = 1/\pi\sqrt{\Phi_0^2 - \psi^2}. \tag{5}$$

In Eq. (4) $2\Phi_0$ is a singularity and in Eq. (5) Φ_0 is a singularity, so that distributions of this type cannot apply to actual physical cases.

The true distribution in a bunch has not been studied in any detail. From the experimental amplitude-distribution curve for the 250-Mev synchrotron at the Institute of Physics, Academy of Sciences, U.S.S.R.,[4] it may be concluded that the particle phase distribution is essentially Gaussian, i.e.,

$$\varphi(\psi) = \frac{1}{2\Phi_0}\exp\left\{-\frac{\pi}{4}\left(\frac{\psi}{\Phi_0}\right)^2\right\},$$

$$f(\Phi) = \frac{\pi\Phi}{2\Phi_0^2}\exp\left\{-\frac{\pi}{4}\left(\frac{\Phi}{\Phi_0}\right)^2\right\}. \tag{6}$$

Here, in computing the normalized integral we set $\Phi_{max} = \infty$, making use of the fact that $\Phi_{max} \gg \Phi_0$.

We now consider the coherent radiation of a bunch; we shall analyze the low-wavelength region, where the radiation is coherent. In Eq. (3) we introduce the initial phase of the i-th electron $\psi_{0i} = \Phi\sin\Omega t_{0i}$ and integrate over the electrons in a bunch. The potential for a component in the bunch field designated by the numbers n and m is of the form

$$\bar{A}_{nm}(\mathbf{R}_0, t) = NF_{nm}\mathbf{A}_{nm}(\mathbf{R}_0, t), \tag{7}$$

where we have introduced the form factor

$$F_{nm} = \int J_m(n\Phi)\exp\{im\arcsin(\psi/\Phi)\}\,w(\psi, \dot{\psi}, t_0)\,d\Gamma \tag{8}$$

For a stationary distribution, using the variable

Φ and $u = \sin^{-1}(\psi/\Phi)$ in Eq. (8) and integrating with respect to u from 0 to 2π, we have

$$F_{nm} = 0 \text{ for } m \neq 0, \quad F_{n0} = \int_0^{\Phi_{max}} f(\Phi) J_0(nf)\, d\Phi. \tag{9}$$

The bunch spectrum contains only harmonics of the rotation frequency ω_0. The power in the n-th component of the bunch spectrum is $I_n = N^2 |F_{n0}|^2 \times I_{n0}$, where I_n is the power of the n-th component in the spectrum of the unmodulated electron (3).

We compute the integral in (9) for various distributions: for the distribution in (4) we find $F_{n0} = \sin 2n(\Phi_0)/2n\Phi_0$; for the distribution in (5) $F_{n0} = J_0(n\Phi_0)$; finally, for the Gaussian distribution (6) $F_{n0} = \exp(-n^2\Phi_0^2/\pi)$.

In the first two cases the form factor oscillates, falling off slowly as Φ_0 increases; this behavior is due to the singularities in the distributions. In the Gaussian case the function falls off rapidly without oscillating. By an appropriate choice of $f(\Phi)$ it is possible to obtain any intermediate case. Whence it follows that the energy variations in the coherent radiation observed by Prokhorov correspond to a well-defined distribution of particles over phase-oscillation amplitudes. These variations need not necessarily be found in other systems.

We now consider a non-stationary distribution. As a typical example we consider a distribution in which all the particles lie in the upper half of the phase plane at the starting time, i.e.,

$$w(\psi, \dot{\psi}, t_0) = 2w(\Phi) \quad \text{if } \dot{\psi} > 0, \qquad w(\psi, \dot{\psi}, t_0) = 0 \quad \text{if } \dot{\psi} < 0. \tag{10}$$

Substituting (10) in (8) and integrating with respect to u from 0 to π we have

$$F_{n0} = \int_0^{\Phi_{max}} f(\Phi) J_0(n\Phi)\, d\Phi, \qquad F_{nm} = 0; \; m = 2p,$$
$$F_{nm} = \frac{2i}{m\pi} \int_0^{\Phi_{max}} f(\Phi) J_m(n\Phi)\, d\Phi; \quad m = 2p+1. \tag{11}$$

The spectrum now contains the fundamental and the odd harmonics only. In the frequency region in which $n < n_0$ and the bands do not overlap it is meaningful to sum over all m harmonics which form a band. As a result we obtain a signal which is modulated in phase and also amplitude modulated at the frequency of the phase oscillations. This means that the energy radiated by a bunch in each band of the spectrum is modulated by the phase frequency.

The general properties of the spectrum are maintained for an arbitrary non-stationary particle distribution. The only changes are in the distribution of energy in the lines of each band and the form of the oscillations which modulate the radiation energy. This modulation of the energy of the coherent radiation at a frequency close to the calculated frequency of the phase oscillations has recently been observed at $\lambda = 3$ cm $(n = 170)$ by Iu. M. Ado at the synchrotron of the Institute of Physics, Academy ofSciences, U.S.S.R.

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