ENERGY LOSSES OF AN ELECTRON IN A MEDIUM WITH SPATIAL DISPERSION

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A general formula is obtained for the energy losses of an electron moving in an arbitrary anisotropic medium characterized by spatial dispersion. It is shown that in the nonrelativistic case the total energy losses in an isotropic medium do not change when spatial dispersion of the medium is taken into account; however, the losses due to excitation of longitudinal waves do change in the general case. It is also shown that when spatial dispersion is taken into consideration the Cerenkov radiation at a frequency ω is distributed over several cones. The intensity of this radiation is calculated.

1. Ginzburg[1] has recently analyzed the propagation of electromagnetic waves in media with spatial dispersion. It is especially important to take account of this dispersion at frequencies close to resonances in the medium. The optical properties of a medium in the neighborhood of resonances has been considered in greater detail by Pekar[2] and the present authors.[3] It is also of interest to study effects that arise if one considers the influence of spatial dispersion on the energy losses of an electron which moves in a medium.

Suppose that an electron with velocity $\mathbf{v}$ moves in an anisotropic medium characterized by an arbitrary spatial dispersion. Maxwell's equations for this case are written as follows:

$$\operatorname{div}\mathbf{H}=0,\quad \operatorname{curl}\mathbf{E}=-\frac{1}{c}\frac{\partial\mathbf{H}}{\partial t},$$
$$\operatorname{div}\mathbf{D}=4\pi e\delta(\mathbf{r}-\mathbf{v}t),\ \operatorname{curl}\mathbf{H}=\frac{1}{c}\frac{\partial\mathbf{D}}{\partial t}+\frac{4\pi}{c}e\mathbf{v}\delta(\mathbf{r}-\mathbf{v}t),$$
$$\mathbf{D}=\int\hat{\varepsilon}(\omega,\mathbf{q})\,\mathbf{E}_q\,e^{i(\mathbf{q}\cdot\mathbf{r}-\omega t)}\,d\mathbf{q}. \tag{1}$$

Taking Fourier components of **E**, **D**, and **H** we have

$$\mathbf{E}(\mathbf{r},t)=\frac{ie^2}{2\pi^2c^2}\int(\mathbf{q}\cdot\mathbf{v})\left\{q^2\hat{\eta}-\frac{(\mathbf{q}\cdot\mathbf{v})^2}{c^2}\hat{\varepsilon}(\mathbf{q}\cdot\mathbf{v},\mathbf{q})\right\}^{-1}\mathbf{v}e^{i\mathbf{q}\cdot(\mathbf{r}-\mathbf{v}t)}d\mathbf{q}, \tag{2}$$

where $\hat{\eta}$ is a projection tensor in which $\eta_{ik}=\delta_{ik}-q_iq_k/q^2$.

The energy losses of an electron moving through a medium are determined by the work of the retarding force that acts on the electron; these forces arise by virtue of the field set up by the electron itself. Substituting the field at the electron, i.e., at the point $\mathbf{r}=\mathbf{v}t$, and using the identity

$$\left\{q^2\hat{\eta}-\frac{(\mathbf{q}\cdot\mathbf{v})^2}{c^2}\hat{\varepsilon}(\mathbf{q}\cdot\mathbf{v},\mathbf{q})\right\}^{-1}\mathbf{v}\equiv\hat{b}^{-1}\mathbf{v}+\frac{(\mathbf{q},\hat{b}^{-1})^2\,\hat{b}^{-1}\mathbf{q}}{1-(\mathbf{q},\hat{b}^{-1}\mathbf{q})},$$

where

$$\hat{b}=q^2-(\mathbf{q}\cdot\mathbf{v})^2/c^2\hat{\varepsilon}(\mathbf{q}\cdot\mathbf{v},\mathbf{q}),$$

we find the following expression for the loss per unit path length:

$$F=-\frac{ie^2}{2\pi^2vc^2}\int(\mathbf{q}\cdot\mathbf{v})\frac{|(\mathbf{v},\hat{b}^{-1}\mathbf{q})|^2-(\mathbf{v},\hat{b}^{-1}\mathbf{v})(\mathbf{q},\hat{b}^{-1}\mathbf{q})+(\mathbf{v},\hat{b}^{-1}\mathbf{v})}{1-(\mathbf{q},\hat{b}^{-1}\mathbf{q})}d\mathbf{q}. \tag{3}$$

This is the energy loss of an electron which moves in an anisotropic medium with arbitrary spatial dispersion and is a generalization of the corresponding formulas given, for example, in the review paper by Bolotovskii.[4]

2. In the nonrelativistic approximation Eq. (3) is simplified and assumes the following form:

$$F=-\frac{ie^2}{2\pi^2v}\int\frac{(\mathbf{q}\cdot\mathbf{v})}{(\mathbf{q},\hat{\varepsilon}\mathbf{q})}d\mathbf{q}. \tag{4}$$

As an example we may consider the case of an isotropic non-gyrotropic medium with arbitrary spatial dispersion. The dielectric-constant tensor is then

$$\varepsilon_{ik}(\mathbf{q}\cdot\mathbf{v},\mathbf{q})=\varepsilon_0(\mathbf{q}\cdot\mathbf{v},\mathbf{q})\,\delta_{ik}+\varepsilon_1(\mathbf{q}\cdot\mathbf{v},\mathbf{q})\,q_iq_k/q^2. \tag{5}$$

We take the z axis in the direction of $\mathbf{v}$ and introduce the notation $k=\sqrt{q_x^2+q_y^2}$, $\omega=q_y^2v$. Equation (4) becomes*

$$F=\frac{ie^2}{\pi}\int_{-\infty}^{+\infty}\int_0^{k_0}\frac{\omega d\omega\,kdk}{\bar{\varepsilon}(\omega,k^2+\omega^2/v^2)(v^2k^2+\omega^2)}, \tag{6}$$

where

$$\bar{\varepsilon}\equiv(\mathbf{q},\hat{\varepsilon}\mathbf{q})/q^2=\varepsilon_0+\varepsilon_1. \tag{7}$$

Assuming that the complex dielectric constant

*The upper limit k_0 in the integration over k is determined by the condition of applicability of the macroscopic analysis (cf. reference 5, §84).

$\bar{\epsilon}(\omega, k^2+\omega^2/v^2)$ has no poles in the upper half-plane, we have from Eq. (6)

$$F = \frac{e^2}{v^2}\int_0^{ik_0 v} \omega d\omega [1 - 1/\bar{\varepsilon}(\omega, 0)]. \quad (8)$$

Since $\epsilon_1(\omega, 0) = 0$ (cf. reference 1) and $\bar{\epsilon}(\omega, 0) = \epsilon_0(\omega, 0)$ is the dielectric susceptibility when spatial dispersion is not taken into account, we may conclude that in the nonrelativistic approximation spatial dispersion does not effect the total energy loss of an electron which moves in an isotropic medium. Equation (8) has been investigated in detail in reference 5. It should be noted, however, that while the total energy loss is not changed when spatial dispersion is considered, there can be a change in the energy lost by excitation of longitudinal waves. As follows from Eq. (6), the losses due to excitation of the longitudinal waves are determined by the zeroes of the quantity $\bar{\epsilon}(\omega, k^2+\omega^2/v^2)$ which, when spatial dispersion is introduced, lead to poles at frequencies $\omega_n = \pm\omega_n(k^2)$. Hence, the loss due to excitation of longitudinal waves (with spatial dispersion) is given by the following expression, which also applies in the relativistic case:

$$F_{\text{long}} = \frac{e^2}{2}\sum_n \int_0^{k_0^2} \omega_n(x)\, dx / [v^2 x + \omega_n^2(x)] \frac{d}{d\omega_n}\bar{\varepsilon} \times \left[\omega_n(x), x + \frac{\omega_n^2(x)}{v^2}\right]. \quad (9)$$

According to reference 3, the dielectric-constant tensor associated with one of the resonant absorption frequencies of the medium (assuming spatial dispersion in an isotropic, non-gyrotropic medium) is:

$$\varepsilon_{ik}(\omega, \mathbf{q}) = \frac{\omega^2 - \alpha q^2 - \beta}{\omega^2 - \alpha q^2 - \beta + \omega_0^2}\delta_{ik} - \frac{\alpha_1 q_i q_k}{\omega^2 - \alpha q^2 - \beta + \omega_0^2}, \quad (10)$$

where the quantities α and α_1 characterize the spatial dispersion of the medium. In this case the zeroes of the quantity $\bar{\epsilon}(\omega, k^2+\omega^2/v^2)$ determine two poles at frequencies:

$$\omega_{1,2}(k^2) = \pm\left[\frac{(\alpha+\alpha_1)k^2+\beta}{1-(\alpha+\alpha_1)v^{-2}}\right]^{1/2}. \quad (11)$$

Using Eqs. (7) and (10), we obtain from Eq. (9) the loss due to excitation of longitudinal waves, on the basis of the model considered in reference 3

$$F_{\text{long}} = \frac{\omega_0^2}{v^2}e^2 \ln\frac{k_0 v}{\sqrt{\beta}} - \frac{\alpha_1 e^2 k_0^2}{2(v^2-\alpha-\alpha_1)}. \quad (12)$$

This expression differs by the presence of the second term from the energy loss in a medium in which spatial dispersion is neglected.

Since α and α_1 are approximately $(d/\lambda_0)^2 v_{el}^2$ where v_{el} is the velocity of an electron in an atom ($v_{el} \sim 10^8$ cm/sec), $d \sim 10^{-8}$ cm is the lattice constant, $k_0 \sim 1/d$, when $v > v_{el}$ the ratio of the second term to the first term in Eq. (12) is a small quantity $(d/\lambda_0)^2 \sim 10^{-6}$. This estimate indicates that if we take account of the spatial dispersion, there is practically no change in the amount of energy lost by a moving electron in the excitation of longitudinal waves. This result is to be expected because spatial dispersion is important only near resonant frequencies of the medium $[\epsilon_0(\omega, 0) \to \infty]$ while the frequency region associated with longitudinal waves is very far from these resonance frequencies.

3. For an isotropic non-gyrotropic medium, we have from Eq. (5)*

$$\hat{b}^{-1} = \frac{1}{q^2 - c^{-2}(\mathbf{q}\cdot\mathbf{v})^2\bar{\varepsilon}}\,\frac{q^2 - c^{-2}(\mathbf{q}\cdot\mathbf{v})^2\varepsilon_0 - c^{-2}(\mathbf{q}\cdot\mathbf{v})^2\varepsilon_1\hat{\eta}}{q^2 - c^{-2}(\mathbf{q}\cdot\mathbf{v})^2\varepsilon_0}. \quad (13)$$

Hence, in the relativistic case, using Eq. (3) we find the following expression for the total energy loss:

$$F = -\frac{ie^2}{\pi}\int_{-\infty}^{+\infty}\omega d\omega \int_0^{k_0} k dk \times \frac{[v^{-2} - c^{-2}\bar{\varepsilon}(\omega, k^2+\omega^2/v^2) + \omega^2\varepsilon_1(\omega, k^2+\omega^2/c^2)/(v^2k^2+\omega^2)c^2]}{\bar{\varepsilon}(\omega, k^2+\omega^2/v^2)\{k^2+\omega^2[v^{-2} - c^{-2}\varepsilon_0(\omega, k^2+\omega^2/v^2)]\}}. \quad (14)$$

If spatial dispersion is neglected, the quantity ϵ_1 vanishes and $\bar{\epsilon}$ becomes $\epsilon_0(\omega, 0)$, so that Eq. (14) becomes the well-known expression for loss in an isotropic medium in which spatial dispersion is neglected (cf. references 4 and 5).

Since Cerenkov radiation losses are determined by the zeroes of the expression in the curly brackets in the denominator of Eq. (14), we may conclude that these losses are centered in frequency regions in which $v > c/n_i(\omega)$, where the $n_i(\omega)$ are defined by the equation

$$n^2 = \varepsilon_0(\omega, n^2\omega^2/c^2). \quad (15)$$

If spatial dispersion is neglected, Eq. (15) has a single root $n^2 = \epsilon_0(\omega, 0)$. When spatial dispersion is taken into account, in general, Eq. (15) has several roots $n_i = n_i(\omega)$. If θ is the angle between the direction of motion of the electron and the radiation direction, since $\cos\theta_i = c/vn_i(\omega)$, we find that the Cerenkov radiation at frequency ω is distributed over several cones with aperture angles θ_i. It is easy to show that the intensity of the Cerenkov radiation in the frequency interval $\omega, \omega + d\omega$ is given by the following expression:

*To verify the identity in (13) we recall that $\hat{\eta}^2 = \hat{\eta}$.

$$dF = \frac{e^2}{c^2} \sum_i \frac{1 - c^2/v^2 n_i^2(\omega)}{|\, 1 - (d/dn_i^2)\, \varepsilon_0(\omega, n_i^2\omega^2/c^2)\, |}\, \omega d\omega. \qquad (16)$$

Since spatial dispersion is important close to the absorption poles of the medium, the new Cerenkov radiation cones will, in general, correspond to a frequency region close to a resonance frequency of the medium. Thus the transmission of Cerenkov radiation is difficult because of absorption; to observe this effect experimentally, it would be necessary to use very thin film. The situation is simplified in optically-active media in which, according to reference 1, the frequency regions in which the new solutions appear are rather broad, so that absorption effects are not important.

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