HYDRODYNAMIC THEORY OF MULTIPLE PRODUCTION OF PARTICLES IN COLLISIONS BETWEEN FAST NUCLEONS AND NUCLEI

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The three-dimensional hydrodynamic problem of the dispersion of the particles emitted in the collision between a nucleon and a nucleus has been solved by a method which is different from, and more exact than, that applied in references 1 and 2. The distributions of the energy, the angles, and of the transverse momenta are investigated with the aid of the solution obtained. The error in the final formulae is $\sim 20\%$.

1. INTRODUCTION

THE hydrodynamic theory of multiple production of mesons in the collision of high energy particles was developed in a paper by Landau.[1] In view of the great mathematical complexity of this theory, the author confined himself to an approximate theory in which the final formulae have only logarithmic accuracy and describe only collisions between identical particles. Better accuracy is needed for a comparison of theory and experiment. Furthermore it is necessary to generalize the results for the case of collisions between a nucleon and a nucleus, since these are usually observed in experiment.

In the investigation of the collision process between a nucleon and a nucleus one usually uses the tube model,[3] i.e., it is assumed that the nucleon interacts only with the nuclear matter inside a tube which it cuts out of the nucleus during the collision. In the present paper we restrict ourselves to the case where the ratio n of the tube length over the linear dimension of the nucleon, which is approximately equal to the number of nucleons in the tube, does not exceed 3.7.

Belen'kii and the author[4] solved the first part of the problem: the determination of the total number of particles N_0 formed during the collision of the nucleon with the nucleus. They obtained the formula $N_0 = k(n+1)E_0^{1/4}$, where E_0 is the energy of the primary nucleon in the laboratory system, and k is a constant factor.

In the present paper we investigate the distribution of the secondary particles over the angles, the energies, and the transverse momenta. Since the energy, the transverse momentum, and the angle under which it is emitted are determined by the components of the 4-velocity of the particle, our problem is to find the distribution of the particles in velocity space. The symmetry properties of the distribution were discussed in reference 5, where it was shown that the distribution should, in some special coordinate system close to the center of mass system, be symmetric with respect to the plane perpendicular to the direction of motion of the colliding particles. Apart from this it must, of course, have cylindrical symmetry.

According to the hydrodynamic theory, the motion of the system after the collision is governed by the relativistic hydrodynamic equations for the ideal liquid:

$$\partial T_i^k / \partial x^k = 0, \quad T_{ik} = (p+\varepsilon) u_i u_k + p g_{ik}, \qquad (1)$$

where p is the pressure, ϵ the energy density, u^i the 4-velocity of the medium, and g_{ik} the metric tensor with components $-g_{00} = g_{11} = g_{22} = g_{33} = 1$, $g_{ik} = 0$ for $i \neq k$. For the equation of state we take the equation of state of the extremely relativistic medium $p = \epsilon/3$.

Below we shall use a system of units in which the Planck constant $\hbar$, the Boltzmann constant, and the velocity of light are equal to unity.

2. ONE-DIMENSIONAL STAGE OF THE EXPANSION OF THE SYSTEM

Owing to the Lorentz contraction of the colliding particles the first stage of the motion of the system after the collision will be approximately one-dimensional. In this case we can make a special change of variables, proposed by Khalatnikov,[6] which brings the nonlinear equations (1) into the form

$$3\partial^2\chi / \partial\eta^2 - \partial^2\chi / \partial y^2 - 2\partial\chi / \partial y = 0, \qquad (2)$$

where $\eta = \tan^{-1} v$, $y = \ln(T/T_0)$, v and T are the velocity and the temperature of the medium, and T_0 is the initial temperature. The coordinate x and the time t are connected with the potential χ through the relations

$$x = e^{-y}\left(\frac{\partial\chi}{\partial y}\sinh\eta - \frac{\partial\chi}{\partial\eta}\cosh\eta\right); \quad (3)$$

$$t = e^{-y}\left(\frac{\partial\chi}{\partial y}\cosh\eta - \frac{\partial\chi}{\partial\eta}\sinh\eta\right). \quad (4)$$

The stage of one-dimensional motion was discussed in references 2 and 5 for the case $n < 3.7$. It is convenient to use a coordinate system in which the colliding particles have oppositely equal velocities. As a consequence of the discontinuities in the initial conditions the motion of the medium cannot be described by a single analytic formula. We have to do with three regions. Bordering on the vacuum we have the region of traveling waves, whose entropy is very small towards the end of the one-dimensional stage, but which carry a significant part of the energy.[2,7] The main part of the entropy is contained in the region of the so-called nontrivial solution, which is enclosed by traveling waves on both sides. In the following we shall be mainly interested in this region.

According to reference 5, the solution of Eq. (2) in this region is of the form

$$\chi = \frac{\sqrt{3}}{2}\left(l - t_0\frac{\partial}{\partial\eta}\right)e^{y}\int\limits_{\eta/\sqrt{3}}^{y} e^{-2y'}I_0\left(\sqrt{y'^2-\frac{\eta^2}{3}}\right)dy' + \frac{\sqrt{3}}{2}t_0\frac{\partial}{\partial\eta}\int\limits_{\eta/\sqrt{3}}^{y} e^{-y'}I_0\left(\sqrt{y'^2-\frac{\eta^2}{3}}\right)dy', \quad (5)$$

where $l = \frac{1}{4}(n+1)d$, $t_0 = \frac{3}{4}(n-1)d$, and d is the "length" of the Lorentz contracted nucleon. Replacing the Bessel function I_0 by its asymptotic expression, we may write this solution, by (3) and (4), in the form

$$3y = -\ln\frac{t^2-x^2}{\Delta^2} + \left(\ln\frac{t+x}{\Delta}\ln\frac{t-x}{\Delta}\right)^{1/2}; \quad (6)$$

$$\eta = \frac{1}{2}\ln\frac{t+x}{t-x} \quad \text{or } v = \frac{x}{t}, \quad (7)$$

where $\Delta = l/\sqrt{6\pi|y|}$.

We find the distribution function of the entropy with respect to the velocities $dS/d\eta$ in the one-dimensional stage, using (5). Here we must remember that the different volume elements of the nuclear matter decay into the various particles at different time instants. We therefore require the distribution function $dS/d\eta$ not for a given time instant $t = \text{const}$, but on some surface $t = t(x)$. We assume that this surface is space-like.* We can then choose a coordinate system t', x' such that the derivative dt'/dx' is zero at an arbitrary given point of this surface. In this coordinate system the usual formula $dS = su^{0\prime}dx'$ is valid in the neighborhood of the given point, where s is the density of the entropy. Transforming back to the original coordinates, we obtain

$$dS = su^0dx - su^1dt. \quad (8)$$

Going over, using (3) and (4), to the independent variables η, y, we write (8) in the form:

$$dS = -s_0e^{2y}\left(\frac{\partial\psi}{\partial\eta}dy + \frac{1}{3}\frac{\partial\psi}{\partial y}d\eta\right), \quad (9)$$

where $\psi = \partial\chi/\partial y - \chi$, and s_0 is the initial density of the entropy. The derivatives appearing in this expression are equal to

$$\frac{\partial\psi}{\partial y} = \frac{\sqrt{3}}{2}e^{-y}\left(l\frac{\partial}{\partial y} - t_0\frac{\partial}{\partial\eta} - l\right)I_0\left(\sqrt{y^2-\frac{\eta^2}{3}}\right), \quad (10)$$

$$\frac{\partial\psi}{\partial\eta} = \frac{\sqrt{3}}{2}e^{-y}\left(l\frac{\partial}{\partial\eta} - \frac{1}{3}t_0\frac{\partial}{\partial y} - \frac{1}{3}t_0\right)I_0\left(\sqrt{y^2-\frac{\eta^2}{3}}\right). \quad (11)$$

If the temperature of the medium drops to the critical value T_k already in the one-dimensional phase, the distribution of the entropy over the velocity is given by the formula

$$\frac{dS}{d\eta} = -\frac{1}{3}s_0e^{2y_k}\frac{\partial\psi(\eta, y_k)}{\partial y_k} \approx \frac{S_0}{\sqrt{6\pi|y_k|}}\exp\left\{\frac{\eta^2}{6y_k}\right\}, \quad (12)$$

where $y_k = \ln(T_k/T_0)$, and S_0 is the total entropy. This case may arise in the collision of two heavy nuclei with low energies, when the number of secondary particles is large (so that hydrodynamics applies), but the initial temperature is small; then the three-dimensional stage cannot develop.

Thus the so-called quasi-one-dimensional approximation[8] amounts to the assumption that formula (12) gives the final solution to the problem even for high energies of the colliding particles. The distribution of the secondary particles over the transverse momenta is in this approximation completely determined by the thermal motion of decay.[9]

The present paper is mainly devoted to the investigation of the three-dimensional stage of the motion of the medium. We use an essentially different approach from that of references 1 and 2.

3. THREE-DIMENSIONAL STAGE

As our system has boundaries in the transverse direction, the matter flow will not be strictly one-dimensional, and the deviation from unidimensionality will increase as time goes on. At $t \sim a$ ($a = 1/\mu$ is the initial radius of the system, μ is the mass of the π meson) the matter flow will be essentially three-dimensional. The general pic-

*The following calculations will justify this assumption.

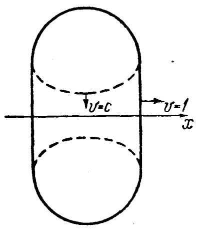

FIG. 1. Illustration of the three-dimensional expansion at the time instant $t \sim a$. The dotted line represents the boundary between the regions of one-dimensional and three-dimensional flow, x is the direction of motion of the colliding particles.

ture of the three-dimensional motion is the following. At the initial moment the temperature at $r = a$ drops suddenly from T_0 to zero, i.e., a sharp discontinuity arises. This discontinuity subsequently splits into two weak discontinuities bounding the region of three-dimensional matter flow. At the borderline with the vacuum the velocity of the medium is equal to the velocity of light, and the boundary between the regions of one-dimensional and three-dimensional flow, like every weak perturbation, moves with the velocity of sound $c = 1/\sqrt{3}$ with respect to the medium. For those elements of the medium which were in the one-dimensional stage, the region of three-dimensional flow is enclosed between the limits $r = a - ct$ and $r = a + t$. For the elements moving along the x axis (in the case of nucleon-nucleon collisions everywhere, with $x \neq 0$), this region has smaller dimensions (Fig. 1), owing to the Lorentz transformation of the transverse components of the velocity (or the time). The instant of time when the region of three-dimensional flow reaches the axis of the system can be called by convention the end of the one-dimensional stage. According to what has been said above, the end of the one-dimensional stage occurs sooner for the slow elements of the medium than for the fast ones.

Because of the cylindrical symmetry of the problem and because of the relation $u^i u_i = -1$, only two of the components u^i are independent. It is convenient to introduce new variables η, ξ, which automatically satisfy the above-mentioned conditions:

$$u^0 = \cosh\eta\,\cosh\xi,\quad u^1 = \sinh\eta\,\cosh\xi,$$

$$u^2 = \sinh\xi\cos\varphi,\quad u^3 = \sinh\xi\sin\varphi.$$

Here $\tanh\xi$ is the radial velocity in the coordinate system moving with the medium along the direction of the x axis.

We make use of the following circumstance in solving the three-dimensional problem. According to (7), $v = x/t$ in the one-dimensional stage, i.e., each element of the medium moves with almost constant velocity, as a quasi-inertial system. It is natural to expect the radial dispersion of the medium not to change the quasi-inertial character of the motion along the x axis, i.e., $v_x \approx x/t$ also in the region of three-dimensional flow. The discussion of the three-dimensional stage is therefore greatly simplified by eliminating the velocity $v_x = x/t$ through a transition to a suitable (curvilinear) coordinate system. This leads to the following coordinates:

$$\omega = \tanh^{-1}(x/t),\quad t' = \sqrt{t^2 - x^2}.$$

The coordinate lines $\omega = \text{const}$ in the x, t plane are straight lines $x/t = \tanh\omega$. Hence the trajectories of the points with coordinates $\omega = \text{const}$ coincide with the trajectories of those elements of the medium moving as inertial systems. The variable t' is the proper time of such elements.

We introduce polar coordinates r, φ in the plane perpendicular to the x axis.

We can use the formulae of general relativity to write down the equations of motion in the new coordinates. The generalization of equation (1) to the case of curvilinear coordinates is:[10]

$$\frac{1}{\sqrt{-g}}\frac{\partial(\sqrt{-g}T_i^k)}{\partial x^k} = \frac{1}{2}\frac{\partial g_{kl}}{\partial x^i}T^{kl}. \tag{13}$$

In the transition from the coordinates x^i to the new coordinates $x^{i'}$, the tensor quantities transform according to the law $u^{i'} = (\partial x^{i'}/\partial x^k)u^k$ etc. With our choice of coordinates $x^{i'}$ ($x^{0'} = t'$, $x^{1'} = \omega$, $x^{2'} = r$, $x^{3'} = \varphi$) the metric tensor and the 4-velocity have the form

$$g'_{00} = -1,\ g'_{11} = t'^2,\ g'_{22} = 1,\ g'_{33} = r^2;\ g_{ik} = 0 \text{ when } i \neq k$$

$$u^{0'} = \cosh\eta'\cosh\xi,\quad u^{1'} = \frac{1}{t'}\sinh\eta'\cosh\xi,$$

$$u^{2'} = \sinh\xi,\quad u^{3'} = 0,$$

where $\eta' = \eta - \omega$.

It is hardly possible to solve (13) exactly. We use an approximate method, whose main features will be explained in the example of one-dimensional flow. It is convenient for this purpose to rewrite Eq. (13) in a form which is solved for the derivatives $\partial y/\partial x^i$:

$$\frac{\partial y}{\partial x^i} = \frac{1}{3}\frac{u_i}{\sqrt{-g}}\frac{\partial(\sqrt{-g}u^k)}{\partial x^k} - u^k\left(\frac{\partial u_i}{\partial x^k} - \frac{1}{2}u^l\frac{\partial g_{kl}}{\partial x^i}\right). \tag{14}$$

We convince ourselves of the equivalence of Eqs. (13) and (14) by "projecting" them on the direction of u^i and on the direction perpendicular to u^i. In the one-dimensional case the equations (14) have the form

$$\frac{\partial y}{\partial\omega} = -\frac{1}{3}\sinh 2\eta'\left(1 + \frac{\partial\eta'}{\partial\omega}\right) - \frac{1}{3}(\cosh 2\eta' + 2)\,t'\frac{\partial\eta'}{\partial t'}; \tag{15}$$

$$t'\frac{\partial y}{\partial t'} = \frac{1}{3}(\cosh 2\eta' - 2)\left(1 + \frac{\partial\eta'}{\partial\omega}\right) + \frac{1}{3}\sinh 2\eta'\cdot t'\frac{\partial\eta'}{\partial t'}. \tag{16}$$

The following assumption, which will be justified below, is essential for the whole approximate method of solution:*

$$t'\partial\eta'/\partial t',\ \partial\eta'/\partial\omega \ll 1.$$

With this assumption, (15) and (16) take the form

$$\partial y/\partial\omega = -{}^1/_3 \sinh 2\eta'; \qquad (17)$$

$$t'\partial y/\partial t' = {}^1/_3 (\cosh 2\eta' - 2). \qquad (18)$$

Eliminating η' from (17) and (18), we obtain

$$(3t'\partial y/\partial t' + 2)^2 - 9(\partial y/\partial\omega)^2 = 1. \qquad (19)$$

The general solution of this equation is, as is easily seen,

$$y = {}^1/_3 [(\cosh 2\alpha - 2)\ln(t'/\beta) - \omega \sinh 2\alpha], \qquad (20)$$

where α and β are arbitrary constants. We note that the general solution (20), together with the relation $\eta' = \alpha$, is at the same time the exact solution of (15) and (16). A special case of this solution is the traveling wave which is obtained by setting $\tanh \alpha = 1/\sqrt{3}$. In our coordinate system the velocity of the medium $\tanh \eta'$ changes thus from zero in the center of the system to the velocity of sound in the region of traveling waves.

It is customary to use the boundary conditions to obtain the solution of the problem from the general integral. However, we cannot do this, as our approximation is, as is seen from the following, an asymptotic one, which is valid only for large t'. We therefore use the asymptotic formula (6), which in our variables has the form

$$y = {}^1/_3 (\sqrt{\tau^2 - \omega^2} - 2\tau), \qquad (21)$$

where $\tau = \ln(t'/\Delta)$. This expression satisfies Eq. (19) and is therefore the required solution. It is obtained from (20) by setting

$$\beta = \Delta,\ \tanh 2\alpha = \omega/\tau. \qquad (22)$$

Substituting (21) in (17) or (18), we obtain $\eta' = \alpha$. The derivatives $\partial\eta'/\partial\omega$, $\partial\eta'/\partial\tau$ are of order $1/\tau$; they can thus indeed be neglected for $\tau \gg 1$. Hence the velocity of the medium $\tanh \eta'$ is a slowly varying function of ω and τ, and its magnitude does not exceed the velocity of sound. These properties are the mathematical expression of the fact that the flow along the x axis is quasi-inertial.

The boundary between the regions of one-dimensional and three-dimensional flow is given by a surface on which the derivatives of the hydrodynamic quantities are discontinuous, i.e., by the characteristic surface of Eq. (13). We can find the differential equation determining the characteristic surface $\Sigma(x^i) = 0$ by the usual method[11] from the requirement that it be impossible to determine all derivatives of first order from the equations of motion (13) by fixing the initial values on this surface. Some cumbersome calculations, which we shall not repeat here, lead to

$$2\left(u^k \frac{\partial\Sigma}{\partial x^k}\right)^2 - g^{ik}\frac{\partial\Sigma}{\partial x^i}\frac{\partial\Sigma}{\partial x^k} = 0. \qquad (23)$$

The physical meaning of this equation is that the characteristic surface always moves with the velocity of sound with respect to the medium.

With the help of (23) we determine the equation for the boundary between the regions of one-dimensional and three-dimensional flow. Noting that $\xi = 0$, $\eta' = \alpha$ on this boundary, we obtain

$$2\left(\cosh\alpha \frac{\partial\Sigma}{\partial t'} + \frac{\sinh\alpha}{t'}\frac{\partial\Sigma}{\partial\omega}\right)^2 + \left(\frac{\partial\Sigma}{\partial t'}\right)^2 - \left(\frac{1}{t'}\frac{\partial\Sigma}{\partial\omega}\right)^2 - \left(\frac{\partial\Sigma}{\partial r}\right)^2 = 0. \qquad (24)$$

Dividing (24) by $(\partial\Sigma/\partial r)^2$, we obtain an equation for $r = r(\omega, t')$:

$$2\left(\cosh\alpha \frac{\partial r}{\partial t'} + \frac{\sinh\alpha}{t'}\frac{\partial r}{\partial\omega}\right)^2 + \left(\frac{\partial r}{\partial t'}\right)^2 - \left(\frac{1}{t'}\frac{\partial r}{\partial\omega}\right)^2 = 1. \qquad (25)$$

We seek a solution of this equation in the form $r = r(t')$. Then (25) takes the form

$$dr/dt' = \pm(\cosh 2\alpha + 2)^{-1/2}. \qquad (26)$$

With neglect of the derivatives of α we obtain a solution which satisfies the initial conditions:

$$r = a - t'/\sqrt{\cosh 2\alpha + 2}. \qquad (27)$$

We turn to the discussion of the region of three-dimensional motion of the medium. We make the substitution

$$y = y_1 + y_2, \qquad (28)$$

where y_1 is the value of y in the absence of transverse dispersion, i.e., the solution of the one-dimensional problem obtained earlier. The variable y_2 is therefore responsible for an additional lowering of the temperature as a consequence of the radial motion of the medium. Corresponding to (28) we also transform the components of the energy-momentum tensor: $T_{ik} = e^{4y_1} I_{ik}$. Then (13) takes the form

$$\frac{\partial(\sqrt{-g}I_i^k)}{\partial x^k} + \left(4g_{il}\frac{\partial y_1}{\partial x^k} - {}^1/_2\frac{\partial g_{kl}}{\partial x^i}\right)\sqrt{-g}I^{kl} = 0. \qquad (29)$$

For the derivatives of y_1 we have to substitute the expressions (17) and (18), in which we replace η' by the known function α. The boundary conditions for Eq. (29) have the form

*This corresponds to neglecting the derivatives of the slowly varying function f in reference 1.

on the surface $r = a - t' / \sqrt{\cosh 2\alpha + 2}$:

$$y_2 = 0, \ \xi = 0, \ \eta' = \alpha. \quad (30)$$

After the substitution (28) the variable ω clearly enters into the equations of motion (29) and into the boundary conditions (30) only through the slowly varying function α, whose derivatives can be neglected, as was shown earlier. We can therefore regard α as a parameter, and look for the solution in the form

$$y_2 = y_2(t', r, \alpha), \quad \xi = \xi(t', r, \alpha), \quad \eta' = \eta'(t', r, \alpha).$$

We thus obtain the important result that we have separated the variables in the problem of the three-dimensional motion of the medium in the asymptotic approximation, which, as a consequence, lowers the number of independent variables from three to two. We note that this result is exact in the region of traveling waves, since the derivatives of α are strictly zero in this region.

We expand the required functions in a series in powers of α. Since α is an odd function of x, the expansions of y_2 and ξ contain only even powers of α, and the expansion of η', only odd powers of α. The main part of the entropy is concentrated in the region where $\omega \sim 1$. In this region, according to (22), $\alpha \sim 1/\tau$, i.e., α is an asymptotically small quantity. We therefore confine ourselves to the linear approximation in α, i.e., we require the solution in the form

$$y_2 = y_2^{\cdot}(r, t'), \ \xi = \xi(r, t'), \ \eta' = \alpha f(r, t').$$

4. AVERAGING OF THE HYDRODYNAMIC QUANTITIES OVER THE RADIUS

A consequence of the equations of motion is the conservation law for the entropy, which, in curvilinear coordinates, has the form

$$\frac{1}{\sqrt{-g}} \frac{\partial(\sqrt{-g}\, su^k)}{\partial x^k} = 0. \quad (31)$$

After the substitution $s_2 e^{3y_1}$ according to (28), we can write Eq. (31) in our approximation as

$$\frac{\partial(rs_2u^{0'})}{\partial t'} + \frac{\partial(rs_2u^{2'})}{\partial r} = 0. \quad (32)$$

The conservation law for the quantity $\int s_2 u^{0'} r\, dr$ follows from this equation. It implies that the amount of entropy enclosed in the interval $d\omega$ is the same as that in the absence of the transverse dispersion, which means it is given by the formula

$$\frac{dS}{d\omega} = -s_0 e^{2y_1}\left(\frac{\partial\psi}{\partial\eta_1}\frac{dy_1}{d\omega} + {}^1/_3 \frac{\partial\psi}{\partial y_1}\frac{d\eta_1}{d\omega}\right), \quad (33)$$

where η_1 is the value of η in the absence of the transverse dispersion. Since the derivative $d\eta/d\omega$ is asymptotically equal to zero, the same formula describes the distribution $dS/d\eta$ in the three-dimensional stage. Formula (33) can approximately be written in a form analogous to (12):

$$\frac{1}{S_0}\frac{dS}{d\omega} = \frac{\exp(\eta^2/6y_1)}{\sqrt{6\pi|y_1|}}. \quad (34)$$

The temperature of the elements in the interval $d\omega$ with different coordinates r reaches the critical value T_k at different time instants. The values of $y_1 = y_k - y_2$ which have to be substituted in (34) in order to find the distribution of the produced particles with respect to η are therefore different for different elements. However, this difference is small. Estimates show that the decay of the elements of the medium occurs at $t' \sim a$. At that time the system is still quite uniform in the transverse direction, and $y_2 \sim 1$. In view of the weak dependence of (34) on y_1 we can therefore neglect this difference in the calculation of the distribution function $dS/d\eta$. We shall assume that all elements of the medium with the same coordinate ω decay simultaneously when their average temperature reaches the critical value. We restrict ourselves to the computation of this average value of the temperature.

In the calculation of the distribution function for the produced particles with respect to the transverse components of the velocity we must take account of the thermal motion of the particles, which is superposed on the hydrodynamic motion in a given volume element. The thermal motion of the particles plays indeed a fundamental role, so that the radial component of the hydrodynamic 4-velocity is small at the instant $t' \sim a$, and $u^2 \sim 1$. For each plane ω = const we can therefore take a value for the hydrodynamic radial velocity which is averaged over the radius.

Below we shall estimate the accuracy of the results obtained in this approximation.

To find the average values of the hydrodynamic quantities, we make use of the conservation laws in the integral form, which refer to finite, instead of infinitesimal, elements of the medium. These laws were also used in reference 1 in the investigation of the three-dimensional stage of the motion of the medium. However, our separation of variables gives the possibility to obtain more accurate results. We integrate Eq. (29) over the radius in the region of three-dimensional flow, i.e., from $r = a - ct'$ to $r = a + t'$ for $t' < \sqrt{3}\, a$, and from $r = 0$ to $r = a + t'$ for $t' > \sqrt{3}\, a$. Replacing all hydrodynamic quantities under the integral sign by their average values and taking these outside the integral, we obtain a system of ordinary differential equations for the mean values of the hydrodynamic quantities. These equations were solved numerically. The results of the numerical calculation of the hydrodynamic quantities averaged over the area of the transverse cross section are shown in Fig. 2.

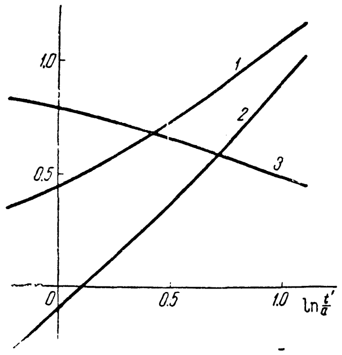

FIG. 2. Curve 1) y_2; curve 2) $\ln \overline{\sinh \xi}$; curve 3) $f = \overline{\eta'}/\alpha$.

5. DISTRIBUTION OF THE PARTICLES IN VELOCITY SPACE

As mentioned earlier, the distribution of the particles with respect to the transverse components of the velocity depends essentially on the thermal motion of the particles. We retain the notation ξ for the hydrodynamic velocity, and denote the radial component of the actual 4-velocity of the particles by $\sinh \zeta$. We write the required distribution function $F(\eta, \zeta)$ in the form

$$F(\eta, \zeta) = N_0 F_1(\eta) F_2(\eta, \zeta). \quad (35)$$

The quantity $N_0 F_1(\eta)\,d\eta$ is the total number of particles in the interval $d\eta$, and the function $F_2(\eta, \zeta)$ gives the distribution of the particles with respect to the variable ζ for a given η. The functions $F_1(\eta) = N_0^{-1} dN/d\eta$ and $F_2(\eta, \zeta)$ must, of course, be normalized:

$$\int_{-\infty}^{\infty} F_1(\eta)\,d\eta = 1; \quad \int_{0}^{\infty} F_2(\eta, \zeta)\,d\zeta = 1. \quad ($$

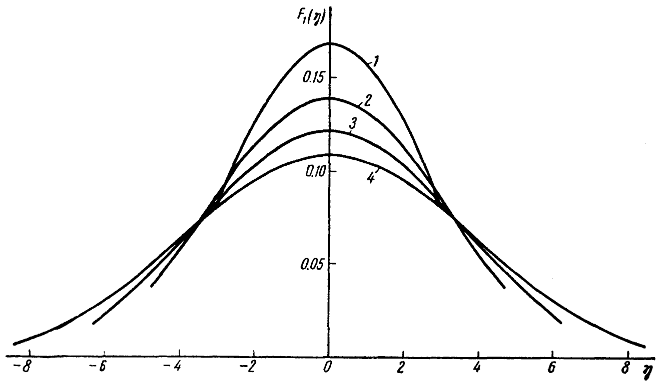

FIG. 3. Function $F_1(\eta)$ for $n = 1$. The initial energy E_0 is equal to: 1) 10^{12} ev, 2) 10^{14} ev, 3) 10^{16} ev, 4) 10^{18} ev.

The function $F_1(\eta)$ is computed with the help of formula (33) and the curves in Fig. 2. Figures 3 and 4 show the results of the calculation for $T_k = \mu$. According to (34), $F_1(\eta)$ can approximately be represented by a Gauss function. We therefore approximate the function $F_1(\eta)$ in the center of mass system by a Gauss function with parameter L:

$$F_1(\eta) = (2\pi L)^{-1/2} \exp(-\eta^2/2L); \quad (37)$$

$$L = 0.56 \ln \frac{E_0}{M} + 1.6 \ln \frac{2}{n+1} + 1.6, \quad (38)$$

where M is the mass of the nucleon.

Formulae (37) and (38) give an approximation to the curves in Figs. 3 and 4, with an error not exceeding 10%.

We turn to the computation of the function $F_2(\eta, \zeta)$. The thermal motion of the particles at the moment of breakup of the element into separate particles may be described by the formulae for the ideal gas. In the rest system of the element of the medium the distribution of the particles in momentum space is known to have the form

$$dN = A \frac{dp'_x\,dp'_y\,dp'_z}{\exp(E'/T) \pm 1}, \quad (39)$$

where the plus sign refers to Fermi statistics, and the minus sign to Bose statistics. The factor A is determined from the normalization condition on the function F_2. We introduce a coordinate system in which the element moves along the z axis with the velocity $v_z = \tanh \xi$. Energy and momentum are then transformed in the following way:

$$E' = E \cosh \xi - p_z \sinh \xi;$$

$$p'_z = p_z \cosh \xi - E \sinh \xi; \quad p'_x = p_x; \quad p'_y = p_y.$$

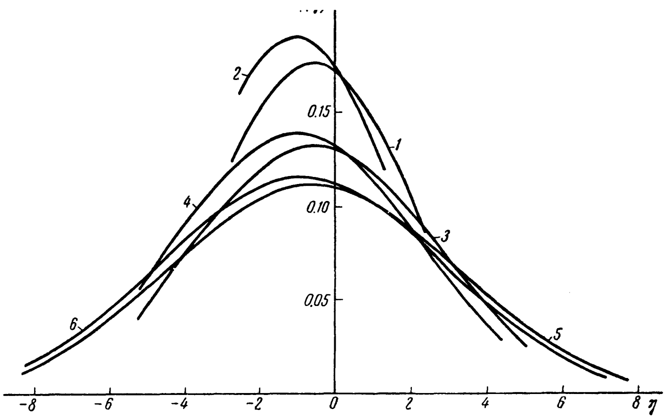

FIG. 4. Function $F_1(\eta)$ for the collision of a nucleon with the tube in the system of oppositely equal velocities of the colliding particles. $n = 2$ for curves 1, 3, and 5; $n = 3.7$ for curves 2, 4, and 6. E_0 is equal to: 1, 2) 10^{12} ev, 3, 4) 10^{15} ev, and 5, 6) 10^{18} ev.

Formula (39) takes the form

$$dN = A\left(\cosh\xi - \frac{p_z}{E}\sinh\xi\right) \times\left[\exp\left(\frac{E\cosh\xi - p_z\sinh\xi}{T}\right) \pm 1\right]^{-1} dp_x dp_y dp_z. \quad (40)$$

Changing to cylindrical coordinates in momentum space and integrating over the azimuthal angle and the component p_x, we obtain

$$F_2(\eta,\zeta) = 4\pi A m^3 \sinh\zeta\cosh\zeta \sum_{n=1}^{\infty} (\mp)^{n-1}\Big\{\cosh\xi\cosh\zeta K_1\left(n\frac{m}{T}\cosh\xi\cosh\zeta\right) I_0\left(n\frac{m}{T}\sinh\xi\sinh\zeta\right) - \sinh\xi\sinh\zeta\, K_0\left(n\frac{m}{T}\cosh\xi\cosh\zeta\right) \times I_1\left(n\frac{m}{T}\sinh\xi\sinh\zeta\right)\Big\}. \quad (41)$$

In the integration we expanded the denominator of the expression under the integral sign into a series in powers of $\exp\{-(E\cosh\xi - p_z\sinh\xi)/T\}$ and applied the relations

$$K_m(\lambda) = \int_0^{\infty} e^{-\lambda\cosh t}\cosh(mt)\,dt;\quad I_m(\lambda) = \frac{1}{2\pi}\int_0^{2\pi} e^{\lambda\cos\theta}\cos(m\theta)\,d\theta.$$

Integrating (39) over all momentum space we similarly obtain an expression for A:

$$\frac{1}{A} = 4\pi m^2 T \sum_{n=1}^{\infty} (\mp)^{n-1}\frac{K_2(nm/T)}{n}. \quad (42)$$

Figure 6 shows the distribution functions with respect to the transverse momenta $(\mu/N_0)\,dN/dp_\perp = F_2/\cosh\zeta$ for Bose particles for $T_k = \mu$ and various values of the parameter ξ.

The hydrodynamic transverse velocity ξ at the moment of break-up is with good accuracy (~10%) given by the approximate formula

$$\sinh\xi = 0.53\left(\frac{n+1}{2}\right)^{0.4}\left(\frac{E_0}{M}\right)^{1/14} e^{-\eta^2/6L}. \quad (43)$$

It follows from this formula that the transverse momenta of the fast particles should on the average be smaller than the transverse momenta of the slow particles. At energies of 10^{12} to 10^{13} ev the difference between them is, however, very small. Only at energies of order 10^{15} to 10^{18} ev do the transverse momenta of the fastest particles become significantly smaller (by a factor of $\frac{1}{2}$ or $\frac{1}{3}$) than those of the other particles.

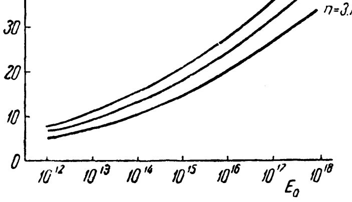

FIG. 5

6. DISTRIBUTION OF THE PARTICLES WITH RESPECT TO THE ANGLES, ENERGIES, AND TRANSVERSE MOMENTA

After the determination of the function $F(\eta, \zeta)$, we find the distribution of the particles with respect to the angles, energies, and transverse momenta with the help of the formula

$$\tan\theta = \frac{\tanh\zeta}{\sinh\eta},\quad E = \mu\cosh\eta\,\cosh\zeta,\quad p_\perp = \mu\sinh\zeta.$$

Figure 6 shows that for the overwhelming majority of the particles $p_\perp > \mu$, i.e., $\tanh\zeta$ is rather close to unity. The angular distribution of the particles is therefore quite accurately given by the relations

$$\frac{1}{N_0}\frac{dN}{d\eta} = F_1(\eta) = \frac{\exp(-\eta^2/2L)}{\sqrt{2\pi L}}, \quad (44)$$

$$\tan\theta = 1/\sinh\eta. \quad (45)$$

In going over to the laboratory system we change η in formula (45) to $\eta + \eta_c$, where $V_c = \tanh\eta_c$ is the velocity of the center of mass system with respect to the laboratory system. Since η_c is a big quantity, we can write

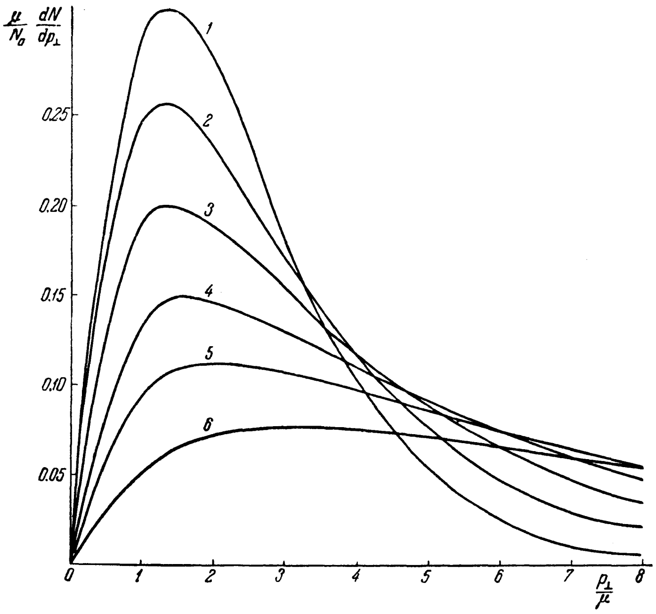

FIG. 6. Values of sinh ξ: 1) 0; 2) 0.5; 3) 1; 4) 1.5; 5) 2; 6) 3.

$$\vartheta / 2 = \exp(-\eta - \eta_c). \qquad (46)$$

Formulae (44) and (45) give a more anisotropic distribution than the corresponding formulae of Landau.

A very sensitive characteristic of the angular distribution is the quantity $X = x/x_1$ introduced by Kaplon and Ritson,[12] where $x = \vartheta_{3/4}\ \vartheta_{1/4}$, and $\vartheta_{1/4}$ and $\vartheta_{3/4}$ are the angles within which $\frac{1}{4}$ and $\frac{3}{4}$ of all particles are, respectively, emitted; x_1 is the analogous quantity for the case of an isotropic distribution in the center-of-mass system.

To find the distribution of the particles over the transverse momenta, we have to compute the integral

$$\int_{-\infty}^{\infty} F(\eta, \zeta)\, d\eta = \frac{1}{N_0}\frac{dN}{d\zeta}.$$

The factor $\exp(-\eta^2/6L)$ in formula (43) is significant in the region $\eta \gtrsim \sqrt{L}$, where only a small fraction of the particles is to be found. The required distribution can therefore with good accuracy be written in the form

$$\frac{1}{N_0}\frac{dN}{d\zeta} = F_2(0, \zeta), \quad p_\perp = \mu \sinh \zeta. \qquad (47)$$

For the determination of the energy distribution we have to integrate the function $F(\eta, \zeta)$ over the surface $E = \text{const}$ in velocity space. We shall not concern ourselves with this distribution in detail. We remark only that since the transverse momenta of all particles are of the same order, the energy of the particles can approximately be written as

$$E \approx (\overline{p}_\perp^2 + \mu^2)^{1/2} \cosh \eta, \qquad (48)$$

where $\overline{p}_\perp$ is the average value of the transverse momentum of the particles. Formulae (44) and (48) determine the order of manitude of the energy distribution of the particles.

We estimate the accuracy of the formulae obtained. Formula (9), which describes the distribution of the entropy with respect to the velocities in the one-dimensional state, is exact. In the investigation of the three-dimensional stage we made use of the asymptotic approximation, i.e., we neglected quantities of order $1/\tau$. In order to estimate the effect of this neglect on $F_1(\eta)$, it is convenient to consider the conservation law for the entropy (31). The most significant of the terms discarded in equation (32) has the order of magnitude $\partial(\alpha - \eta')/\partial\omega \approx (1-f)/2\tau$. The inclusion of this term leads to a change of 5 to 7% in the entropy enclosed in the interval $d\omega$, which also gives a measure for the error. Replacing all hydrodynamic quantities by their averages over the area of the transverse cross section entails a further approximation. The corresponding error was estimated in the following fashion. The region of three-dimensional flow was subdivided into three sections with respect to the radius, and the averaging was then done over these sections separately. The more exact solution obtained in this way was used to calculate the function $F_1(\eta)$ for the energies 10^{12} ev and 10^{15} ev. The result agrees with that obtained earlier within $\sim 10\%$ of error.

Despite the very approximate character of the treatment of the three-dimensional stage of the motion of the medium, the accuracy of the computed function $F_1(\eta)$ appears to be rather good ($\sim$ 10 to 15%). The physical reason for this is that the elements of the medium move almost inertially along the x axis at the moment of transition to the three-dimensional stage. As a consequence, the distribution of the entropy with respect to the velocities $dS/d\eta$ is mainly determined by the one-dimensional stage of motion.

Analogous estimates yield $\sim$ 20 to 25% as the accuracy for $F_2(\eta, \zeta)$. In this case the effect of the three-dimensional stage is small also on account of the great role played by the thermal motion of the particles. We note that the transverse hydrodynamic velocity increases slowly with increasing energy, whereas the role of the thermal motion becomes less significant. The accuracy of the calculation of $F_2(\eta, \zeta)$ is therefore the better, the lower the initial energy, in contradistinction to the situation in the case of $F_1(\eta)$.

In conclusion we remark on the following interesting point. It is seen from Figs. 3, 4, and 6 that the functions $F_1(\eta)$ and $F_2(\eta, \zeta)$ depend very weakly on the energy of the primary nucleon. According to (38) and (43),

$$L \sim \ln(E_0/M), \quad \sinh \zeta \sim E_0^{1/14}.$$

At the same time the initial width of the disk changes by three orders of magnitude when the energy is increased from 10^{12} ev to 10^{18} ev. This points to a very weak dependence of the results on the initial conditions. We may therefore suppose that, for $n > 3.7$, the distribution over energies and angles is described by formulae which are not very different from those obtained above. One should also expect that the quantum effects,[13] which are strongest at the initial moment, do not change our results significantly.

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