ON THE INTERACTION BETWEEN SMALL DISTURBANCES AND DISCONTINUITIES IN MAGNETOHYDRODYNAMICS AND ON THE STABILITY OF SHOCK WAVES

V. M. KONTOROVICH

Institute of Radiophysics and Electronics, Academy of Sciences, Ukrainian S.S.R.

Submitted to JETP editor May 28, 1958; resubmitted June 30, 1958

J. Exptl. Theoret. Phys. (U.S.S.R.) **35**, 1216-1225 (November, 1958)

A simple geometrical method is presented for the construction of waves (reflected and refracted) diverging from a surface of discontinuity and produced by the incidence of a plane monochromatic wave on a plane stationary surface of discontinuity in a medium describable by the equations of magnetohydrodynamics. The case of a shock wave is considered.

On the basis of the results obtained, the stability of shock waves with respect to splitting up is investigated as regards obliquely incident disturbances. The change in frequency resulting from the interaction of small disturbances with shock waves is considered.

1. INTRODUCTION

WITHIN a stationary homogeneous flow of a fluid of infinite conductivity situated in a magnetic field, there exist several types of small disturbances with different dispersion laws $\omega = \omega(\mathbf{k})$: magnetohydrodynamic, magnetoacoustic (slow and fast), and entropy waves. In the presence of discontinuities, these disturbances can be transformed one into another; when some one disturbance (of frequency ω and propagation vector $\mathbf{k}$) falls on a discontinuity, the waves (reflected and refracted) diverging from the discontinuity will contain, generally speaking, disturbances of all types.

Because of the relatively complicated dispersion law in the case of magnetoacoustic waves, the analytic solution for the propagation vectors (phase velocities) of the divergent waves (the laws of reflection and refraction) becomes inconvenient, since the solution involves roots of a fourth-degree equation. Nevertheless, the problem admits of a simple geometric solution with respect to which the laws of reflection and refraction can be formulated. The calculation of the amplitudes of the divergent waves in terms of the amplitude of the incident disturbance is not given in the present article.

Recently Akhiezer, Liubarskii, and Polovin[1] have investigated the stability of shock waves with respect to splitting, under disturbances that depend only on the distance to the shock wave and on the time (in our treatment this corresponds to normal incidence of the disturbance on the discontinuity).

The problem considered here allows us to investigate the stability of shock waves with respect to splitting under obliquely-incident disturbances as well. Of considerable interest is the fact that within a definite angular interval the components of the phase and of the group velocities along the normal to the discontinuity have opposite signs, and to obtain the correct solution of the problem of stability we must therefore perform the division into incident and divergent waves by considering not their phase velocities but their group velocities.

In ordinary hydrodynamics the interaction of sound with shock waves has been investigated (for the case of an ideal gas) by Blokhintsev,[2] Burgers,[2] and Brillouin,[3] and (in the case of an arbitrary nonviscous fluid) by the author.[4]*

2. SMALL DISTURBANCES

We denote by δA the amplitudes of disturbances proportional to $\exp i(\mathbf{k}\cdot\mathbf{r} - \omega_0 t)$ in the coordinate system in which the fluid is at rest.

*In D'iakov's articles[5] the interactions between stationary discontinuities of low intensity, and also between stationary weak discontinuities and shock waves, have been studied within the framework of ordinary hydrodynamics. We note that owing to the absence of dispersion, the laws of reflection and refraction, and also the amplitudes of the divergent discontinuities, agree with the corresponding quantities for plane monochromatic waves. In magnetohydrodynamics, since the phase velocity depends only on the direction $\mathbf{k}/k$ and not on the frequency ω, the results of the present investigation must also be applicable to discontinuities interacting with shock waves. Naturally, it is not possible to derive from this the shape of the perturbed shock wave, nor the results with respect to weak discontinuities (discontinuities of the derivatives). On the other hand, an investigation of monochromatic waves enables us to determine such quantities as frequency shifts produced by reflection or refraction at a discontinuity (cf. Sec. 7).

These disturbances are therefore proportional to $\exp i(\mathbf{k}\cdot\mathbf{r} - \omega t)$ in the system in which the surface of the unperturbed discontinuity is at rest. We state here only those characteristics of small disturbances[6,7] needed for subsequent discussion, i.e., the dispersion laws $\omega_0(\mathbf{k})$.

Entropy wave. Just as in the case of hydrodynamics, the disturbance in the entropy $\delta\sigma$ and the disturbance in the density $\delta\rho$ associated with it are at rest with respect to the fluid. All the other quantities are undisturbed:

$$\omega_0(\mathbf{k}) = 0. \tag{2.1}$$

Magnetohydrodynamic waves or Alfven waves. The thermodynamic quantities are undisturbed. The velocity and the related magnetic field both oscillate:

$$\omega_0(\mathbf{k}) = \pm\mathbf{k}\cdot\mathbf{u}, \quad \mathbf{u} \equiv \mathbf{H}/\sqrt{4\pi\rho}. \tag{2.2}$$

Magnetoacoustic waves. This is an adiabatic motion $(\delta\sigma = 0)$ in which the pressure and the velocity, density, and magnetic field associated with it all oscillate:

$$\omega_0^2(\mathbf{k}) = \frac{k^2}{2}[s^2 + u^2 \pm \sqrt{(s^2+u^2)^2 - 4s^2u^2\cos^2\theta}], \tag{2.3}$$

$s^2 = (\partial p/\partial\rho)_\sigma$ is the square of the velocity of sound.

In the system of coordinates in which the discontinuity is at rest while the fluid moves with the velocity $\mathbf{v}$, the frequency ω of the disturbance has the form

$$\omega_0 = \omega - \mathbf{k}\cdot\mathbf{v}. \tag{2.4}$$

By introducing the phase velocity of the disturbance with respect to the fluid at rest, which depends in our case only on the direction of propagation,

$$\mathbf{V}(\varkappa) = \frac{\omega_0(\mathbf{k})}{k}\varkappa, \quad \varkappa \equiv \frac{\mathbf{k}}{k}, \tag{2.5}$$

we can write the relation between the frequency and the propagation vector in the form

$$\omega - \mathbf{k}\cdot\mathbf{v} = \pm kV(\varkappa), \tag{2.6}$$

where we take V to denote the absolute value of the phase velocity. For the Alfven waves we have

$$V_A = u|\cos\theta|. \tag{2.7}$$

Magnetoacoustic waves can be separated into two branches (fast and slow waves):

$$V_\pm = \frac{1}{2}[s^2 + u^2 \pm \sqrt{(s^2+u^2)^2 - 4s^2u^2\cos^2\theta}]. \tag{2.8}$$

The plus sign corresponds to the fast waves which go over into ordinary sound when the magnetic field

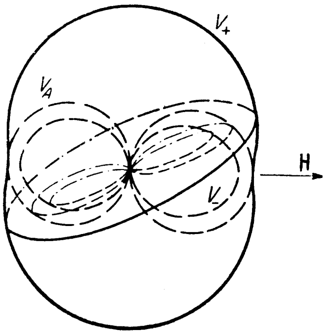

FIG. 1. The surfaces $V(\varkappa)$ and the interaction of these surfaces with the plane of incidence.

is removed, the minus sign corresponds to the second branch, the slow sound, which together with the Alfven waves degenerates, in going over to ordinary hydrodynamics, into a disturbance of the curl of the velocity (rotational wave) which is at rest with respect to the motionless fluid. In a moving fluid all the disturbances considered above are, in addition, carried along by the stream.

We shall make use hereinafter of velocity vector diagrams which specify V_A, V_+, V_-, in accordance with (2.7) and (2.8), as functions of the direction κ.[7] The speed of propagation of a disturbance of a given type is determined by the length of the vector drawn from the origin in the direction $\mathbf{k}$ to its intersection with the corresponding surface $V(\kappa)$ (Fig. 1). $V(\kappa)$ represent surfaces of revolution with respect to the direction of the magnetic field $\mathbf{H}$ and possess central symmetry with respect to the origin.

3. BOUNDARY CONDITIONS AT THE SURFACE OF DISCONTINUITY

At the surface of discontinuity in the coordinate system I, in which the surface of discontinuity is at rest, the system of boundary conditions which follows from the continuity of the fluxes of mass $\rho\mathbf{v}$, of energy $\mathbf{g}$, of momentum π_{ik}, of the tangential components of the electric field $-[\mathbf{v}\times\mathbf{H}]/c$, and of the normal component of the magnetic field[6,7] has the form

$$\{\rho v_n\} = 0, \quad \{g_n\} = 0, \quad \{\pi_{in}\} = 0,$$
$$\{[\mathbf{v}\times\mathbf{H}]_t\} = 0, \quad \{H_n\} = 0. \tag{3.1}$$

Here n and t denote components normal or tangential to the discontinuity. $\{A\}$ denotes the abrupt change in the value of the quantity A at the discontinuity. In the coordinate system I we shall choose the x axis to be directed along the normal to the discontinuity in the direction of flow (for shock waves). The quantities belonging to the semi-infinite space $\overline{\text{II}}$, for which $x < 0$ (ahead of the shock wave), will be denoted by $\overline{A}$ to distinguish them from the quantities A belonging to the semi-

infinite space Π (behind the shock wave);
$\{A\} \equiv A - \bar{A}$.

When a small disturbance falls on the discontinuity it is necessary, in the approximation linear in terms of the amplitude of the incident disturbance $\delta A^{(i)}$ (or $\overline{\delta A}^{(i)}$), to take into account the deformation and the additional velocity of the surface of discontinuity. In this approximation the boundary conditions for the disturbance can be written[8] with reference to the unperturbed surface of discontinuity at $x = 0$ (cf. the cases of ordinary hydrodynamics[9] and relativistic hydrodynamics[10]).

For disturbances of the type $\sim \exp i(\mathbf{k}\cdot\mathbf{r} - \omega t)$ we must guarantee the "continuity" at the surface of discontinuity ($x = 0$) of the frequency ω and of the component $\mathbf{q}$ of the propagation vectors $\mathbf{k}^{(i)}$ and $\mathbf{k}^{(d)}$, i.e., the divergent waves will have the same frequency and component of the propagation vector as the incident wave. We note that this is not associated with the specific form of the boundary conditions (3.1), but only with the smallness of the disturbance, which allows us to reduce the perturbed boundary condition to the plane $x = 0$.[11] As a consequence of linearity, the equation of the disturbed discontinuity has the form

$$x = \eta \exp i(\mathbf{q}\cdot\mathbf{r} - \omega t) \qquad (3.2)$$

and is characterized by the amplitude η. By expanding the incident and the divergent disturbances in terms of eigenwaves, we obtain from (3.1) a system of equations for the amplitudes of the divergent waves $\delta A^{(d)}$ and of the amplitude of the disturbance of the discontinuity η. When $\delta A \sim \exp i(\mathbf{k}\cdot\mathbf{r} - \omega t)$, the last perturbed equation in (3.1) is a consequence of the two preceding ones. Finally we obtain a system of seven inhomogeneous equations enabling us to find $\delta A^{(d)}$ and η in terms of given values of $\delta A^{(i)}$:

$$\sum T_{ik}\delta A_k^{(d)} + \sum T_{il}\delta \bar{A}_l^{(d)} + T_i\eta = \sum U_{im}\delta A_m^{(i)} + \sum U_{in}\delta \bar{A}_n^{(i)},$$
$$i = 1, 2, \ldots, 7 \qquad (3.3)$$

Here we take into account the fact that both incident and divergent waves, generally speaking, may appear in Π and in $\bar{\Pi}$. The specific form of the coefficients will be needed only for the calculation of the amplitudes, where it will be given in due course.

4. THE LAWS OF REFLECTION AND REFRACTION

The relation between the angles of incidence and reflection or refraction may be obtained by equating the frequency ω and the component $\mathbf{q}$ of the propagation vector along the surface of discontinuity for the incident and the divergent waves.

From this it follows that the propagation vectors

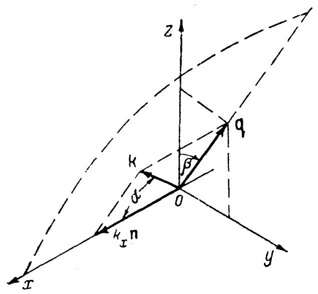

FIG. 2. The plane of incidence contains the normal to the discontinuity (0x) and the propagation vector $\mathbf{k}$. The surface of discontinuity (zy) contains the component of the propagation vector along the discontinuity $\mathbf{q}$.

of all the divergent waves $\mathbf{k}^{(d)}$ lie in the plane of incidence formed by $\mathbf{k}^{(i)}$ and by the normal to the undisturbed discontinuity $\mathbf{n}$ (the x axis). In future we shall consider the intersections of the surfaces $V(\kappa)$ (Fig. 1) with the plane of incidence. We introduce the angles α and β defining the direction of κ:

$$k_i(q \cot\alpha, \quad q\sin\beta, \quad q\cos\beta). \qquad (4.1)$$

The angle α between the vector $\mathbf{k}$ and the normal $\mathbf{n}$ lies in the plane of incidence. The angle β (Fig. 2) defines the position of the plane of incidence. The intersection gives the curves $V(\alpha)$ [more accurately $V(\alpha, \beta)$ for $\beta = \text{const}$]. The equations of these curves will be obtained from (2.7) and (2.8) by the substitution

$$\cos\theta = \cos\alpha \frac{H_x}{H} + \sin\alpha \frac{\mathbf{q}\cdot\mathbf{H}}{qH}. \qquad (4.2)$$

We introduce the angle ψ, which plays a fundamental role in subsequent development:

$$\cot\psi \equiv (\omega - \mathbf{q}\cdot\mathbf{v})/qv_x. \qquad (4.3)$$

By using the relation (2.6) we shall obtain a formula for ψ in terms of the velocity vectors

$$\cot\psi = (v_x\cos\alpha \pm V(\alpha))/v_x\sin\alpha. \qquad (4.4)$$

A simple geometric meaning of the angle ψ (Fig. 2) follows from (4.4): in the system of coordinates in which the velocity of the fluid $\mathbf{v}$ has only a normal component, it is the angle between the total velocity of the disturbance $\pm V\kappa + v_x\mathbf{n}$ (taking drift into account) and the velocity of this disturbance $\pm V\kappa$ with respect to the fluid at rest.

The solution with the minus sign corresponds to the wave for which $\omega_0 < 0$ and for which the propagation vector and the phase velocity are therefore oppositely directed [$\delta A \sim \exp(\mathbf{k}\cdot\mathbf{r} + |\omega_0|t)$]. Such disturbances result from the fact that the flow is "supersonic" for the given disturbance.

It follows directly from (4.3) that the angle ψ is not changed on reflection. This is the content of the law of reflection

$$\psi^{(i)} = \psi^{(r)}. \qquad (4.5)$$

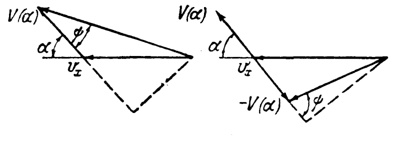

FIG. 3. The angles α and ψ for $\omega_0 > 0$ (on the left) and for $\omega_0 < 0$.

It follows from (4.3) that the law of refraction has the form

$$\{v_x \cot \psi\} + \frac{q}{q}\{v\} = 0. \tag{4.6}$$

In the case when the tangential components do not undergo a discontinuity (ordinary hydrodynamics, parallel and perpendicular shock waves in magnetohydrodynamics), $q\{v\} = 0$, and the law of refraction takes on the simple form of the "law of tangents":[4]

$$\tan \psi / \tan \bar{\psi} = v / \bar{v}. \tag{4.7}$$

We now turn to the construction of the propagation vectors κ restricted by the relation (4.5), i.e., to the construction of $\alpha^{(r)}$ in terms of the given $\alpha^{(i)}$. We see (Fig. 3) from (4.4) that the segment of the normal $-v_x$ subtends the angle ψ at the end of the vector $V = V\kappa$ [in the case of the plus sign in formula (4.4)], or subtends the angle $\pi - \psi$ at the end of the vector $V = -V\kappa$ [in the case of the minus sign in (4.4)]. It follows from the law of reflection (4.5) that if the segment subtends the angle ψ at the end of the vector $V^{(i)}$ of the incident wave, then it must subtend either the same angle ψ or the angle $\pi - \psi$ at the ends of the vectors $V^{(r)}$ for all the reflected waves. This means that the only waves that can be reflected are those for which the ends of the vectors $V^{(r)}$ lie on the circle drawn through the ends of the segment $-v_x$ and the end of the vector $V^{(i)}$. We note that in virtue of the central symmetry of the surfaces $V(\kappa)$, each point of the graph of V corresponds also to a point of the graph of $-V$. Naturally, this property is preserved also for a central section, such as the section defined by the plane of incidence. We can thus construct, with the aid of the velocity vector diagram in the plane of incidence, the phase velocity vectors satisfying the boundary condition of reflection. To do this we must draw a circle through the ends of the segment $-v_x$ and the end of the velocity vector of the incident wave $V^{(i)}$. The points of intersection of this circle with the curves $V(\alpha)$ will give the ends of the required vectors $V = V\kappa$, $V = -V\kappa$ for the intersections with the larger and the smaller circular arcs respectively (for $\psi < \pi/2$) (Fig. 4). The points of intersection of the circle with the curve $V_+(\alpha)$ will give the velocity vector (and the corresponding angle α) for fast sound. The points of intersection of the circle with the curve $V_-(\alpha)$ will

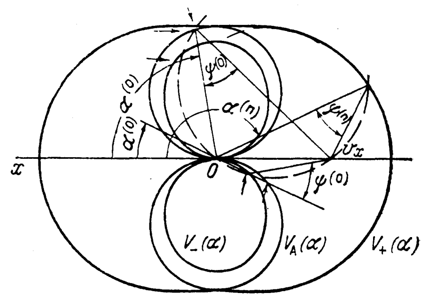

FIG. 4. Arrows indicate points of intersection of the ψ circle circle (dotted line) with the curves $V_+(\alpha)$, $V_A(\alpha)$, $V_-(\alpha)$. The vectors drawn from the center to these points are the velocities of the reflected waves behind the shock wave, which arise when the V_+ wave is incident on the discontinuity. The point at which the ψ circle is tangent to V_+ (dot-dashed line) divides the V_+-waves into incident and divergent ones.

In the drawing the magnetic field lies in the plane of the discontinuity and in the plane of incidence. This case is not in any way special for making the construction.

give the velocity vector (and the corresponding angle) for slow sound. The points of intersection of the circle with the curve $V_A(\alpha)$ will determine the velocity vector for the Alfven wave.

For a given value of v_x, each angle ψ corresponds to its own circle (ψ circle). To construct the refracted waves we must, in accordance with the law of refraction (4.6), find the angle $\bar{\psi}$ in terms of the angle ψ, and then construct the $\bar{\psi}$ circle on the segment $-\bar{v}_x$. The points of intersection with $V(\alpha)$ and $\bar{V}(\alpha)$ respectively determine the types and velocities of the disturbances which are restricted by the laws (4.5) and (4.6). For the Alfven waves, the angles α may be easily found in terms of the given angle ψ also in analytic form, by making use of formulas (4.4), (2.7), and (4.2). However, in the case of magnetoacoustic waves the analytic solution of the problem of refraction, i.e., of determining the angles α in terms of the given angle ψ, requires the solution of the complete fourth-degree equation, obtained from (4.4), (2.8), and (4.2). The graphical method given above is nothing but a graphical solution of this equation.

In the case of normal incidence, the circle degenerates into a straight line and the obvious solution is given by the points of intersection of the normal to the discontinuity with the curves $V(\alpha)$ of the diagram.

5. INCIDENT AND DIVERGENT WAVES

In accordance with the results of the preceding section, the points where the circle drawn through the ends of the vector $-\mathbf{n}v_x$ intersect the curves of the velocity diagram determine the phase velocity vectors of all the waves that are interrelated by the reflection condition.

Among these waves there are divergent ones as well as those incident on the discontinuity, and we must know how to classify them by making use of this property.

However, we have no justification for using phase velocities to separate waves into incident and divergent ones, since the phase and the group velocities may have opposite signs for certain angles of incidence. In such cases, as has already been pointed out by Mandel'shtam,[12] the physical requirement, that the energy should flow away from the boundary of the discontinuity for the reflected or for the refracted wave, must lead to the classification of waves in terms of the group velocity $\mathbf{V}^{gr} = \partial\omega/\partial\mathbf{k}$. Taking into account the fact that $V(\boldsymbol{\kappa})$ is a function of $\cos\theta$ only, we shall obtain in the system of coordinates I:

$$\mathbf{V}^{gr} = \pm\left(\varkappa V + \frac{dV}{d\cos\theta}\frac{[\mathbf{k}\times[\mathbf{H}\times\mathbf{k}]]}{Hk^2}\right) + \mathbf{v}. \qquad (5.1)$$

For the classification of waves we need the component V_x^{gr} normal to the discontinuity. By going over to the angle α we can easily obtain

$$V_x^{gr} \equiv \frac{\partial\omega}{\partial k_x} = v_x \pm \left(V(\alpha)\cos\alpha - \frac{\partial V}{\partial\alpha}\sin\alpha\right). \qquad (5.2)$$

We turn once again to the ψ circles to obtain a geometrical interpretation. We now regard equation (4.4) as the equation for ψ in terms of α and we take the derivative

$$\frac{\partial\psi}{\partial\alpha} = \frac{\sin^2\psi}{v_x\sin^2\alpha}\left[v_x \pm \left(V(\alpha)\cos\alpha - \frac{\partial V}{\partial\alpha}\sin\alpha\right)\right]. \qquad (5.3)$$

By comparing formulas (5.2) and (5.3) we arrive at the relation

$$\frac{\partial\psi}{\partial\alpha} = \frac{\sin^2\psi}{v_x\sin^2\alpha}V_x^{gr}, \qquad (5.4)$$

which is of importance to us, where the group velocity is determined in the coordinate system in which the surface of the shock wave is at rest. The divergent waves in II and $\overline{\text{II}}$ must satisfy the inequalities:

$$\begin{aligned} V_x^{gr} &\equiv v_x \pm \left[V(\alpha)\cos\alpha - \frac{\partial V}{\partial\alpha}\sin\alpha\right] > 0, \\ \bar{V}_x^{gr} &\equiv \bar{v}_x \pm \left[\bar{V}(\bar{\alpha})\cos\bar{\alpha} - \frac{\partial\bar{V}}{\partial\bar{\alpha}}\sin\bar{\alpha}\right] < 0. \end{aligned} \qquad (5.5)$$

Similarly the incident waves will be determined by the inverse inequalities. In accordance with (5.4), we need merely determine the sign of $\partial\psi/\partial\alpha$ to determine the sign of the group velocity.

As can be seen from the expression for the group velocity (5.3), its projection on the direction of the propagation vector coincides with the phase velocity in the coordinate system in which the liquid is at rest.*

Consequently, in the case of normal incidence of the disturbance, the components of the phase and of group velocities along the normal to the discontinuity coincide, and in the case of normal incidence the division into incident and divergent waves can be carried out by making use of the components of the phase velocity (taking drift into account).[1] In the diagram (Fig. 4) these components are given by the points of intersection of the curves $V(\alpha)$ with the x axis, into which the ψ circle degenerates. The end of the segment v_x separates the waves into incident and divergent ones in accordance with the inequalities:

$$v_x \pm V_x \gtrless 0, \quad \bar{v}_x \lesseqgtr \bar{V}_x\ 0, \qquad (5.6)$$

where the upper sign in the inequality corresponds to divergent, and the lower sign corresponds to incident waves.

In the case of oblique incidence, the ψ circle intersects both parts of the curve $V_A(\alpha)$, since they are also circles [cf. (2.7)] and have a point common with the ψ circle at the origin. A similar assertion can be made with respect to the curve $V_-(\alpha)$, both parts of which pass through the origin and have a common tangent at that point. Two of the aforementioned four points of intersection of the ψ circle with $V_A(\alpha)$ and $V_-(\alpha)$ may lie on the smaller arc of the ψ circle and correspond in that case to $\omega_0 < 0$ (for $\psi < \pi/2$). We now turn to the intersection of the ψ circle with the curve $V_+(\alpha)$. We consider the case $v_x < V_+(\pi)$. In the case of normal incidence ($\psi = 0$) we have two waves, one ($\alpha = 0$) divergent and the other ($\alpha = \pi$) incident. As ψ increases the points of intersection with the ψ circle move along the curve $V_+(\alpha)$ towards each other. At the same time the angle α for the incident wave decreases, while for the divergent (reflected) wave the angle α increases as ψ increases, i.e., in accordance with (5.4) and (5.5), the condition $\partial\psi/\partial\alpha < 0$ is fulfilled for the former and the condition $\partial\psi/\partial\alpha > 0$ is fulfilled for the latter wave. When $\partial\psi/\partial\alpha = 0$ the ψ circle is tangent to the curve $V_+(\alpha)$ and both roots α of Eq. (4.4) coincide. We denote the corresponding values of ψ and α by ψ_m and α_m. We see from (5.7) that at this point for both

*This result depends in an essential way on the fact that the dispersion is a purely spatial one. If V depends not only on $\varkappa$, but also on ω, the component of the group velocity does not coincide with the phase velocity.

waves we have $V_x^{gr} = 0$.*

As ψ increases further, the ψ circle no longer intersects the curve V_+. The real solutions are now replaced by complex conjugate solutions, which correspond to traveling waves. One of these must be discarded since it does not satisfy the boundary conditions at ∞, while the other one represents a surface wave that is attenuated as it penetrates into the fluid. We note that while in an ordinary fluid surface waves can arise only in the case of propagation of sound,[3,4] in the present case surface V_+ waves can arise also in the case of reflection of V_- or V_A waves.

6. STABILITY OF SHOCK WAVES

In accordance with the results of Akhiezer, Liubarskii, and Polovin[1] a shock wave is stable with respect to splitting up only in the case when the problem of the incidence of an arbitrary small disturbance on the discontinuity has a unique solution.† The system of equations for the amplitudes of the divergent waves in terms of the amplitudes of the incident waves (3.3) contains (after the amplitude of the shock wave itself has been eliminated) six equations, from which it follows that six waves must diverge from a stable discontinuity. The entropy wave always gives rise to a diverging wave in Π and to an incident one in $\overline{\Pi}$. Therefore in the vector diagram a stable wave must correspond in Π and in $\overline{\Pi}$ to five divergent waves of types $V_\pm$ and V_A for aribtrary ψ. When $\psi = 0$ (normal incidence) this requirement yields the conditions of Akhiezer, Liubarskii, and Polovin,[1] leading to waves of three kinds:

$$\begin{array}{lll} \text{a)}\ v_x < V_-, & \overline{V}_- < \overline{v}_x < \overline{V}_A & \\ \text{b)}\ V_- < v_x < V_A, & \overline{V}_A < \overline{v}_x < \overline{V}_+ & (6.1) \\ \text{c)}\ V_A < v_x < V_+, & \overline{V}_+ < \overline{v}_x. & \end{array}$$

We shall show that the number of diverging waves for any ψ (any α) is equal to the number of divergent waves for $\psi = 0$.

The component of the group velocity along any given direction in the case of Alfven waves is equal to the phase velocity in this direction, as can be seen directly from the dispersion law $\omega_0 = \pm \mathbf{k} \cdot \mathbf{u}$. Therefore the component along the normal, $V_A^{gr} \cdot \mathbf{n} = V_x^{ph}$, and the inequality that determines the divergent wave are generally independent of the angle.

In accordance with the results of the preceding section the number of divergent waves does not change when $\psi < \psi_m$ in the case of magnetoacoustic waves. As ψ increases one divergent (and one incident) wave disappears at the point ψ_m, but a surface wave appears which, from the point of view of counting up the number of unknowns in the boundary conditions, must be included among the divergent waves. Thus, the number of unknowns is, as before, equal to the number of equations, only we must now consider all quantities to be complex. [Similarly, in the case when the ψ circle for a certain value $\psi_{m'}$ is also tangent to the curve V_- in addition to intersecting two curves, then in passing through $\psi_{m'}$ the surface wave disappears, and in addition to the waves present previously two traveling V_- waves appear, one of which corresponds to $\partial\psi/\partial\alpha > 0$ and is incident, while the other, for which $\partial\psi/\partial\alpha < 0$, is diverging (the converse holds in $\overline{\Pi}$). Leaving aside the question as to when such a case can be realized, we see that it does not change the number of divergent waves.] Thus, the number of diverging waves does not depend on the angle of incidence, and in order to have stability with respect to splitting up for an arbitrary angle of incidence of the disturbance on the discontinuity (provided only that the equations are not separable) it is sufficient that the conditions (6.1) should be satisfied.

If the disturbances were classified by means of their phase velocities (taking into account the drift of the disturbance with the stream) we would have arrived at a "paradox" of the instability of shock waves in magnetohydrodynamics. Indeed, in virtue of the spatial dispersion, there will be such a range of angles $\psi_0 < \psi < \psi_m$ in which the components of the phase and of the group velocities of fast sound along the normal to the discontinuity have opposite signs. In connection with this, if a classification with respect to phase velocity is carried out, the number of diverging waves for this angular interval differs from the number of diverging waves determined according to their group velocity, i.e., from the number of diverging waves for $\psi = 0$.*

*We note that the ψ circle is tangent to the curve V_+ at only one point, since in the opposite case there could be four points of intersection of the ψ circle with the curve V_+, which is impossible. Indeed, the equation for the points of intersection of the ψ circle with V_+ and V_- is of the fourth degree, but two points of intersection always lie on the curve V_-.

†When the number of diverging waves can only exceed the number of equations, for example, in the case of a perpendicular shock wave, the conclusion with respect to instability also follows from the well-known argument given in the book by Landau and Lifshitz.[13]

*We note that while opposite signs for the phase and the group velocities occur very infrequently in the usual problems involving a boundary (cf. reference 12), in the case of a moving boundary three always exists such a range of velocities of this motion, that in a system of reference in which the boundary is at rest the phase and the group velocities have opposite signs (if $V^{ph} > V^{gr}$, it is sufficient, for example, that $V^{ph} > v > V^{gr}$). In spatial problems, such a difference in sign may arise for arbitrary velocities v within a definite range of angles of incidence if spatial dispersion is present, as in our case.

We now consider a parallel shock wave in which the magnetic field is directed along the normal to the surface of discontinuity. In the case of exact normal incidence the perturbed equations can be separated into those for purely acoustic and for Alfven waves. Akhiezer, Liubarskii, and Polovin,[1] in considering unidimensional disturbances, have naturally drawn the conclusion that parallel shock waves that do not satisfy the inequalities $\bar{v} > \bar{s}$, $v < s$ [type b in (6.1)] are unstable. However, since the separation of a purely acoustic wave corresponds to an isolated point in the space of propagation vectors, it is necessary to investigate obliquely-incident disturbances in order to obtain a final answer to this problem. In such an investigation the boundary equations for the Alfven wave can be separated from the equations for the magnetoacoustic waves, and in order for stability to exist it is necessary to have two diverging Alfven waves. The waves of type b are unstable in agreement with the results of Akhiezer, Liubarskii and Polovin,[1] since among the divergent waves they have only one Alfven wave.

7. FREQUENCY SHIFT

When monochromatic disturbances interact with a shock wave, the Doppler effect and the discontinuity in the propagation vector at the surface of discontinuity cause the generator frequency Ω_0 and the received frequency Ω' to be different. In the case of magnetohydrodynamics there are more possibilities in this connection than in an ordinary fluid,[4] since we can compare the frequencies of different types of disturbances, for example, the frequency shift of the reflected V_+, V_-, V_A waves with respect to the frequency of the incident V_+ wave, etc. We shall restrict ourselves to the case of normal incidence, for which there is no need to use the graphic method. We assume that the receiver and the transmitter are at rest with respect to $\overline{\Pi}$. In accordance with (2.6), we obtain for the absolute value of the propagation vector in Π (limiting ourselves to the solution involving the plus sign)

$$k = \omega / (V(\alpha) + \varkappa \cdot \mathbf{v}). \tag{7.1}$$

Here, as before, ω is the frequency in the system I. In Π we should place a bar over all the quantities with the exception of ω. The relation between ω and Ω_0 and Ω' is given by:

$$\omega - k^{(i)} \bar{v}_x \cos \alpha^{(i)} = \Omega_0, \quad \omega - k^{(r)} \bar{v}_x \cos \alpha^{(r)} = \Omega'. \tag{7.2}$$

Here $\alpha^{(i)}$ and $\alpha^{(r)}$ are equal to 0 or π, depending on the direction of the incident and the reflected waves, $\sin \alpha \equiv 0$. In $\overline{\Pi}$ we should place a bar over the quantities k and α. From (7.1) and (7.2) we obtain formulas for the relative shifts $\Delta = (\Omega_0 - \Omega')/\Omega_0$. For the sake of definiteness we shall limit ourselves to those incident waves which have a propagation vector directed towards the boundary, and to divergent waves which have a propagation vector directed away from the boundary. In Π the incident wave corresponds to $\alpha = \pi$, the diverging wave corresponds to $-\alpha = 0$; the converse holds in $\overline{\Pi}$. The relative frequency shift is given by the following expressions — in the case of reflection in Π:

$$\Delta = \frac{\bar{v}_x [V(0) + V(\pi)]}{[V(\pi) - \{v_x\}][V(0) + \bar{v}_x]}, \quad \{v_x\} \equiv v_x - \bar{v}_x; \tag{7.3}$$

in the case of reflection in $\overline{\Pi}$:

$$\Delta = -\frac{\bar{v}_x}{\bar{V}(0)} \frac{\bar{V}(\pi) + \bar{V}(0)}{\bar{V}(\pi) - v_x}; \tag{7.4}$$

in the case of transmission from Π into $\overline{\Pi}$:

$$\Delta = \frac{\bar{v}_x [\bar{V}(\pi) - V(\pi) + \{v_x\}]}{V(\pi) - \{v_x\}][\bar{V}(\pi) - \bar{v}_x]}; \tag{7.5}$$

in the case of transmission from $\overline{\Pi}$ into Π:

$$\Delta = \frac{\bar{v}_x}{\bar{V}(0)} \frac{\bar{V}(0) - V(0) - \{v_x\}}{V(0) + \bar{v}_x}. \tag{7.6}$$

In all the above expressions we must everywhere in place of $V(\pi)$ and $\bar{V}(0)$ substitute the velocity of the incident disturbance, in place of $V(0)$, $\bar{V}(\pi)$ we must take the velocity of the disturbance of the type with which we are concerned. By measuring the different Δ we obtain from (7.3) to (7.6) a system of equations from which $\bar{v}/v$, $\bar{V}_{(i)}/\bar{v}$, $V_{(i)}/v$, ($i = \pm$, A) may be found. If we know V_i/v, we can easily obtain s/v, u/v, and H_x/H (similarly in Π and in $\overline{\Pi}$):

$$s^2 = V_+^2 V_-^2 / V_A^2, \quad u^2 = V_+^2 - V_-^2 - s^2, \quad (H_x/H)^2 = V_A^2/u^2. \tag{7.7}$$

If the values of $\bar{s}$ and $\bar{\rho}$ (or $\bar{H}$) are known, we can obtain not only the ratios, but the quantities that characterize the discontinuity themselves.

The author takes this opportunity to thank A. I. Akhiezer, G. Ia. Liubarskii, and R. V. Polovin for communicating the results of their work prior to publication, and for useful discussions.

Note added in proof (September 30, 1958). As shown by Syrovatskii [J. Exptl. Theoret. Phys. **35**, (1958), Soviet Phys. JETP **8** (in press)] because of plane motion in shock waves (in contradistinction to the essentially spatial rotational discontinuities) there always exists a special plane of incidence (which coincides with the plane of motion in the shock wave), in which the boundary conditions for the Alfven and the magnetoacoustic waves become separable. In particular, such a separation always occurs in the case of normal incidence. As a result of this shock waves of type b are unstable (cf. the case of a parallel shock wave), while rotational discontinuities which should also be included in type b, are stable with respect to splitting up.

[1] Akhiezer, Liubarskii, and Polovin, J. Exptl. Theoret. Phys. (U.S.S.R.) **35**, 731 (1958), Soviet Phys. JETP **8** 507 (1959).

[2] D. I. Blokhintsev, Dokl. Akad. Nauk S.S.S.R. **47**, 22 (1945); J. M. Burgers, Proc. Ned. Akad. Wet. **49**, 273 (1946).

[3] J. Brillouin, Acustica **5**, 149 (1955).

[4] V. M. Kontorovich, J. Exptl. Theoret. Phys. (U.S.S.R.) **33**, 1527 (1957), Soviet Phys. JETP **6**, 1180 (1958); Акустический журнал Acoust. Journal) (in press).

[5] S. P. D'iakov, J. Exptl. Theoret. Phys. (U.S.S.R.) **33**, 948, 962 (1957), Soviet Phys. JETP **6**, 729 (1958).

[6] L. D. Landau and E. M. Lifshitz, Электродинамика сплошных сред (The Electrodynamics of Continuous Media), Ch. VIII, Tekhteoretizdat, 1957.

[7] S. I. Syrovatskii, Usp. Fiz. Nauk **62**, 247 (1957).

[8] S. I. Syrovatskii, Труды ИФАН (Trans. Physics Institute, Academy of Sciences, U.S.S.R.) **8**, 15 (1956).

[9] S. P. D'iakov, J. Exptl. Theoret. Phys. (U.S.S.R.) **27**, 288 (1954).

[10] V. M. Kontorovich, J. Exptl. Theoret. Phys. (U.S.S.R.) **34**, 186 (1958), Soviet Phys. JETP **7**, 127 (1958).

[11] L. D. Landau and E. M. Lifshitz, Механика сплошных сред (The Mechanics of Continuous Media), Gostekhizdat, 1953, p. 311.

[12] L. I. Mandel'shtam, J. Exptl. Theoret. Phys. (U.S.S.R.) **15**, 475 (1945); Собрание трудов (Collected Works) Acad. of Sci. Press, **5**, 463-465 (1950), **2**, 334 (1948).

[13] L. D. Landau and E. M. Lifshitz, Механика сплошных сред (The Mechanics of Continuous Media), Gostekhizdat, 1953, p. 405.

Translated by G. Volkoff
254