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CALCULATION OF THE PHONON PART OF THE MUTUAL FRICTION FORCE IN SUPERFLUID HELIUM

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The part of the mutual friction force in superfluid helium due to the scattering of phonons by vortex filaments is calculated. It is shown that the phonon part of the mutual friction force becomes comparable with the roton part at temperatures on the order of 0.5°K.

It has been shown by Hall and Vinen[1] that in rotating helium II there exists a force of mutual friction between the normal and the superfluid parts of the liquid. To calculate this force it is necessary to study the scattering of the elementary excitations that make up the normal part of the liquid, i.e., of the phonons and rotons, by the vortex filaments. At temperatures above about 0.6°K the phonon part of ρ_n is considerably smaller than the roton part. Because of this, at such temperatures the entire mutual friction force is due to the rotons. Calculations of this part of the friction force have been carried out by Hall and Vinen[2] and by Lifshitz and Pitaevskii.[3] At the very lowest temperatures, however, the number of rotons decreases sharply, and the contribution of the phonons to the mutual friction force can become important.

To calculate this contribution we must use the ordinary hydrodynamical equations to calculate the scattering of sound by a vortex line. We point out to start with that the theory given below is essentially based on the smallness of the dimensionless quantity

$$\nu = \hbar k / mc = \hbar\omega / mc^2, \tag{1}$$

where k is the wave vector of the phonon, ω is the frequency, c is the speed of sound in the helium, and m is the mass of a helium atom. The main part in all phenomena is played by the phonons for which $\hbar\omega \sim \kappa T$ (κ is Boltzmann's constant), so that we have for ν:

$$\nu \sim \kappa T / mc^2 \approx 4 \cdot 10^{-2} T, \tag{2}$$

i.e., $\nu \sim 0.02$ for $T \sim 0.5°$. Physically the quantity ν is the ratio of the distance from the axis of the vortex at which the speed would equal the speed of sound to the wavelength of the phonon.

We write out the hydrodynamical equations:

$$\partial \mathbf{v} / \partial t + (\mathbf{v} \cdot \nabla)\mathbf{v} + \nabla p / \rho = 0, \tag{3}$$

$$\partial \rho / \partial t + \operatorname{div} \rho \mathbf{v} = 0, \tag{4}$$

and take the velocity $\mathbf{v}$ in the form $\mathbf{v} = \mathbf{v}_0 + \mathbf{v}'$, where $\mathbf{v}_0$ is the velocity of the liquid around the vortex

$$\mathbf{v}_0 = (\hbar / m) [\boldsymbol{\beta} \times \mathbf{r}] / r^2 \tag{5}$$

($\boldsymbol{\beta}$ is the unit vector along the axis of the vortex), and $\mathbf{v}'$ is the velocity in the sound wave. We use similar expressions for the density ρ and the pressure p: $\rho = \rho_0 + \rho'$, $p = p_0 + p'$. Linearization of Eqs. (3) and (4) in the primed quantities leads to the equations of propagation of sound in the presence of the vortex:

$$\frac{\partial \mathbf{v}'}{\partial t} = -(\mathbf{v}_0 \cdot \nabla)\mathbf{v}' - (\mathbf{v}' \cdot \nabla)\mathbf{v}_0 - \frac{c^2}{\rho_0}\nabla\rho' + \frac{\nabla\rho_0}{\rho_0^2}\rho'; \tag{6}$$

$$\frac{\partial \rho'}{\partial t} + \rho_0 \operatorname{div} \mathbf{v}' + \mathbf{v}_0 \cdot \nabla\rho' + \nabla\rho_0 \cdot \mathbf{v}' = 0. \tag{7}$$

We now put $\mathbf{v}' = \mathbf{v}_1 + \mathbf{v}_2$, where $\mathbf{v}_1 = (\mathbf{k}/k) v_1 \times e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)}$ is the velocity in the incident wave,* $\mathbf{v}_2$ that in the scattered wave; and similarly we put $\rho' = \rho_1 + \rho_2$ and $p' = p_1 + p_2$.

We now make the assumption, the correctness of which will be verified later, that at all relevant distances the scattered wave is weaker than the incident wave. This enables us to treat the scattering by perturbation theory, i.e., in the Born approximation.

We further note that at all distances much larger than $\hbar/mc \approx 0.5 \times 10^{-8}$ cm the condition $v_0 \ll c$ is satisfied, so that the change of density in the vor-

*Since the component of a wave propagated along the vortex is not scattered, we can confine ourselves to a discussion of waves in the plane perpendicular to the vortex line. Accordingly in all formulas **r**, **k**, and so on are two-dimensional vectors.

tex is relatively small, $\Delta\rho_0 \ll \rho_0$, and can be evaluated from Bernoulli's equation:

$$\Delta\rho_0 \approx c^{-2}\Delta p_0 = -\rho_0 v^2/2c^2 = \rho(\hbar/mc^2)^2. \quad (8)$$

It can be seen from Eq. (8) that near the axis of the vortex the change of the density of the liquid is quadratic in the "perturbation" v_0, and consequently in the Born approximation we can regard ρ_0, p and c as constants in Eqs. (7) and (8). A detailed calculation shows that the terms coming from the change of density would make a contribution to the scattered wave containing an additional factor of the small parameter $\hbar k/mc$. Thus we have (from now on we shall always write ρ instead of ρ_0):

$$\frac{\partial \mathbf{v}_2}{\partial t} + \frac{c^2}{\rho}\nabla\rho_2 = -(\mathbf{v}_0\cdot\nabla)\mathbf{v}_1 - (\mathbf{v}_1\cdot\nabla)\mathbf{v}_0; \quad (9)$$

$$\frac{\partial\rho_2}{\partial t} + \rho\,\mathrm{div}\,\mathbf{v}_2 = -\mathbf{v}_0\cdot\nabla\rho_1. \quad (10)$$

Eliminating the velocity $\mathbf{v}_2$ from Eqs. (9) and (10), we get an equation for ρ_2:

$$\Delta\rho_2 + k^2\rho_2 = c^{-2}\rho\,\{k^2(\mathbf{v}_0\cdot\mathbf{v}_1) - ik\cdot\nabla(\mathbf{v}_0\cdot\mathbf{v}_1) - \mathrm{div}\,[(\mathbf{v}_1\cdot\nabla)\mathbf{v}_0]\}. \quad (11)$$

As is well known, the solution of the equation

$$\Delta\rho + k^2\rho = Q(\mathbf{r}) \quad (12)$$

(Δ is the two-dimensional Laplacian operator) that contains only an outgoing wave at infinity has the form[4]

$$\rho(\mathbf{r}) = -\frac{i}{4}\int H_0^{(1)}(k|\mathbf{r}-\mathbf{r}'|)\,Q(\mathbf{r}')\,d^2\mathbf{r}' \quad (13)$$

($H_0^{(1)}$ is the Hankel function). Solving Eq. (11) by this formula and using integration by parts in the terms containing the divergence, we get:

$$\rho_2(\mathbf{r}) = \frac{i\rho}{4}\Big\{\frac{k^2}{c^2}\int H_0^{(1)}(k|\mathbf{r}-\mathbf{r}'|)(\mathbf{v}_0\cdot\mathbf{v}_1)\,d^2\mathbf{r}'$$
$$-\,\mathrm{div}\Big[\int H_0^{(1)}(k|\mathbf{r}-\mathbf{r}'|)\Big(\frac{i}{c}\mathbf{k}(\mathbf{v}_0\cdot\mathbf{v}_1) + c^{-2}(\mathbf{v}_1\cdot\nabla)\mathbf{v}_0\Big)d^2\mathbf{r}'\Big]\Big\}. \quad (14)$$

First let us convince ourselves that the condition for the applicability of the Born approximation, which we have used for the calculation, is actually fulfilled. For the validity of the Born approximation it is necessary that the condition

$$\rho_2 \ll \rho_1 \quad (15)$$

hold at such distances as are of importance in all the integrations. It is clear that the region $r' \sim 1/k$ is the important one for the integrals of Eq. (14). But for $r \sim 1/k$

$$\rho_2 \sim (\hbar k/mc)\,\rho_1 \ll \rho_1, \quad (16)$$

which indeed justifies the approximations that have been made.*

Let us now find the asymptotic expression for ρ_2 for

$$r \gg \lambda \sim 1/k$$

(λ is the wavelength of the phonon). To do this we use the asymptotic formula

$$H_0^{(1)}(kR) = \sqrt{2/\pi ikR}\,e^{ikR}, \quad R \gg 1/k. \quad (17)$$

for $r \gg 1/k$ values $r' \ll r$ are the important ones in the integrals. We have approximately

$$|\mathbf{r}-\mathbf{r}'| \approx r - \mathbf{r}'\cdot\mathbf{n};$$
$$H_0^{(1)}(k|\mathbf{r}-\mathbf{r}'|) \approx \sqrt{2/\pi ikr}\,\exp(ikr - i\mathbf{k}'\cdot\mathbf{r}') \quad (18)$$

($\mathbf{k}' = k\mathbf{n}' = k\mathbf{r}/r$ is the wave vector of the scattered wave). Substituting Eq. (18) into Eq. (14), we find

$$\rho_2 = \frac{i}{4}\rho\frac{v_1}{c}\sqrt{2/\pi ikr}\,\frac{e^{ik\cdot r}}{c}\{k^2\mathbf{k} + (\mathbf{k}\cdot\mathbf{k}')\mathbf{k} + (\mathbf{k}\cdot\mathbf{q})\mathbf{k}'\}\cdot\frac{\mathbf{v}_\mathbf{q}}{k}. \quad (19)$$

Here $\mathbf{q} = \mathbf{k} - \mathbf{k}'$, and

$$\mathbf{v}_\mathbf{q} = \int e^{i\mathbf{q}\cdot\mathbf{r}'}\mathbf{v}_0\,d^2\mathbf{r}'. \quad (20)$$

We can calculate the integral in Eq. (20) most simply by using the fact that

$$\mathrm{div}\,\mathbf{v}_0 = 0, \qquad \mathrm{curl}\,\mathbf{v}_0 = 2\pi\frac{\hbar}{m}\beta\delta(\mathbf{r}). \quad (21)$$

By the relations (21) one can easily find $(\mathbf{v}_\mathbf{q}\cdot\mathbf{q})$ and $\mathbf{v}_\mathbf{q}\times\mathbf{q}$, and consequently also $\mathbf{v}_\mathbf{q}$. In fact,

$$(\mathbf{v}_\mathbf{q}\cdot\mathbf{q}) = \int \mathbf{q}e^{i\mathbf{q}\cdot\mathbf{r}}\cdot\mathbf{v}_0\,d^2\mathbf{r}$$
$$= -i\int \mathbf{v}_0\cdot\nabla e^{i\mathbf{q}\cdot\mathbf{r}}d^2\mathbf{r} = i\int e^{i\mathbf{q}\cdot\mathbf{r}}\,\mathrm{div}\,\mathbf{v}_0\,d^2\mathbf{r} = 0; \quad (22)$$

$$[\mathbf{v}_\mathbf{q}\times\mathbf{q}] = \int[\mathbf{v}_0\times\mathbf{q}]\,e^{i\mathbf{q}\cdot\mathbf{r}}d^2\mathbf{r}$$
$$= -i\int[\mathbf{v}_0\times\nabla e^{i\mathbf{q}\cdot\mathbf{r}}]\,d^2\mathbf{r} = -i\int e^{i\mathbf{q}\cdot\mathbf{r}}\,\mathrm{curl}\,\mathbf{v}_0 d^2\mathbf{r} = -2\pi i\frac{\hbar}{m}\beta. \quad (23)$$

We take the vector products of the two sides of Eq. (23) by $\mathbf{q}$. Thus we get

$$2\pi i\frac{\hbar}{m}[\mathbf{q}\times\beta] = [\mathbf{q}\times[\mathbf{q}\times\mathbf{v}_\mathbf{q}]] - q^2\mathbf{v}_\mathbf{q}$$

or, using Eq. (22),

$$\mathbf{v}_\mathbf{q} = 2\pi\frac{\hbar}{m}i[\beta\times\mathbf{q}]\,q^{-2}. \quad (24)$$

*We note that $\rho_2 \ll 1$ also at all distances down to those between atoms. In fact, as can be seen from Eq. (14), for small r we have $\rho_2 \sim \hbar/mcr$, i.e., $\rho_2 \ll \rho_1$ for $r \sim a$.

Substituting this expression into Eq. (19), we find:

$$\rho_2 = \rho \frac{v_1}{c^2} \frac{\pi e^{ikr}}{\sqrt{2\pi ikr}} \frac{\hbar}{m} \left(1 - \frac{2k^2}{q^2}\right) \frac{(\mathbf{k}\cdot[\beta\times\mathbf{k}'])}{k} \tag{25}$$

Differentiating, we obtain the velocity in the scattered wave

$$\mathbf{v}_2 = \mathbf{k}' \frac{v_1}{c} \frac{\pi e^{ikr}}{\sqrt{2\pi ikr}} \frac{\hbar}{m} \frac{\sin\varphi\cos\varphi}{\sin^2(\varphi/2)}, \tag{26}$$

where φ is the angle between $\mathbf{k}$ and $\mathbf{k}'$. We note that for values of φ close to zero v_2 increases as $1/\varphi$. This means that the Born approximation is not valid for small scattering angles. Such angles, however, are unimportant for the calculation of the friction force.

The effective scattering cross section for the sound, per unit length of the vortex filament, is given by:

$$d\sigma = \frac{|\mathbf{v}_2|^2}{|\mathbf{v}_1|^2} r d\varphi = \frac{\pi}{2}\left(\frac{\hbar}{mc}\right)^2 k \frac{\sin^2\varphi\cos^2\varphi}{\sin^4(\varphi/2)} d\varphi. \tag{27}$$

Let us now introduce the "transport cross section"

$$\sigma^* = \int (1 - \cos\varphi)\, d\sigma.$$

The physical meaning of this quantity lies in the fact that the momentum transferred to unit length of a vortex in unit time by a sound wave of energy density E_0 is given by

$$\sigma^* E_0 \sin^2\vartheta \tag{28}$$

(where ϑ is the angle of incidence of the sound wave on the plane perpendicular to the axis of the vortex). Integrating, we find

$$\sigma^* = (\pi^2\hbar^2/m^2c^2)\, k. \tag{29}$$

Let us now go on to the calculation of the phonon part of the mutual friction force. The energy density of the phonons of given frequency and direction of motion is $E_0 = \hbar\omega\, dN$, where dN is the number of phonons in unit volume. When the relative velocity is $\mathbf{u} = \mathbf{v}_R - \mathbf{v}_L^2$ ($\mathbf{v}_R$ is the velocity of the normal part near the vortex filament, and $\mathbf{v}_L$ is the velocity of the filament), dN has the form

$$dN = \frac{1}{\exp\{\hbar(\omega - \mathbf{k}\cdot\mathbf{u})/\varkappa T\} - 1} \frac{d^3\mathbf{k}}{(2\pi)^3}$$
$$\approx dN_0 - \frac{\hbar}{\varkappa T} \frac{(\mathbf{k}\cdot\mathbf{u})\, e^{\hbar\omega/\varkappa T}}{(e^{\hbar\omega/\varkappa T} - 1)^2} \frac{d^3\mathbf{k}}{(2\pi)^3} \tag{30}$$

(dN_0 is the phonon distribution for $u = 0$).

Projecting the momentum transfer (28) onto the direction $\mathbf{u}$ and integrating over the angles, we get for the friction force:

$$F = \frac{3\pi^2}{8} \frac{\hbar^2 cu}{\varkappa T} \int \sigma^* \frac{e^{\hbar kc/\varkappa T}}{(e^{\hbar kc/\varkappa T} - 1)^2} \frac{k^4 dk}{(2\pi)^3}.$$

Let us insert here the expression (29) for σ^* and make the change of variable $\hbar kc/\kappa T = x$. Then

$$F = \frac{3\pi^4}{8(2\pi)^3} \frac{(\varkappa T)^5 u}{\hbar^2 c^7 m^2} \int_0^\infty \frac{x^5 e^x}{(e^x - 1)^2} dx. \tag{31}$$

The remaining integral reduces to a Riemann zeta function. We get finally:

$$F = \frac{3\pi^4\, 5!\, \zeta(5)}{8(2\pi)^3} \frac{(\varkappa T)^5 u}{\hbar^2 c^7 m^2} = \frac{5!\,\zeta(5)\cdot 135}{128\pi} u\rho_{n\mathrm{ph}} \approx 42 \frac{\hbar\varkappa T}{c^2 m^2} \rho_{n\mathrm{ph}} u. \tag{32}$$

As for the lift force, it is zero in this approximation, because of the symmetry of the scattering in the Born approximation.

The effect calculated here can be important only at low temperatures, at which the free paths of the elementary excitations (and therefore also the viscosity of the normal part) are large and $\rho_n \ll \rho$. In this case we can neglect all effects associated with the viscosity and the Magnus effect (cf. reference 2), and simply assume that $\mathbf{u} = \mathbf{v}_n - \mathbf{v}_s$. Then formula (32) gives directly the coefficient B in the Hall-Vinen expression for the mutual friction force:

$$B_\Phi \approx 42\,\varkappa T/\pi mc^2 \approx 0.42\,T. \tag{33}$$

Noting that the phonon and roton coefficients B_{ph} and B_r give the total coefficient B by the formula

$$B = (B_{\mathrm{ph}}\rho_{n\mathrm{ph}} + B_r\rho_{n\,r})/(\rho_{n\mathrm{ph}} + \rho_{n\,r})$$

(for $\rho_n \ll \rho$), and recalling that $B_r \sim 1$, we see that B_{ph} can be of importance only at temperatures $\lesssim 0.5°$. Such experiments are of course very difficult, but they are of interest for testing the theoretical ideas about the vortex filaments.

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