PASSAGE OF HIGH-ENERGY MUONS THROUGH MATTER

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Submitted to JETP editor June 14, 1958

J. Exptl. Theoret. Phys. (U.S.S.R.) **35**, 1440-1446 (December, 1958)

The passage of high-energy muons through thick layers of matter is studied with account of ionization losses, bremsstrahlung, pair production, and "star" production. The energy loss distributions are taken into account for the last three processes. It is shown that taking account of the distribution functions leads to a smaller muon flux at sea level than the calculation on the basis of mean energy losses.

1. The passage of muons through thick layers of matter is related to the characteristics of nuclear interactions of high-energy particles. We are thus able to state that in a nuclear interaction most of the primary energy is carried away by a nucleon.[1,2] The importance of this conclusion increases the interest of the present problem, since the energy spectrum of high-energy muons ($\gtrsim 10^{11}$ ev) can be obtained only by measuring the dependence of muon intensity on depth underground, followed by a calculation based on a specific form of the range-energy relation for muons, which has been calculated in a number of papers. In his fundamental paper George[3] did not use the most accurate cross section obtainable for some of the processes that accompany the passage of muons through matter; moreover, he used mean energy losses instead of basing the calculation on distribution functions. The methods used by Belen'kii[4] and by Mando and Sona,[5] which take only bremsstrahlung into account in addition to ionization losses, are applicable only to not very high energies $< 10^{11}$ ev. In the present work we consider the passage of both low- and high-energy muons through matter, taking account of all known forms of muon interactions and energy-loss fluctuations.

2. It is convenient to divide interactions of relativistic muons into processes that result in continuous energy losses and others that result in infrequent large energy losses. The first type is represented by ionization losses (including Cerenkov radiation), while the second type includes bremsstrahlung, the direct production of electron-positron pairs, and collisions that result in the production of nuclear-interacting particles (which we shall call nuclear processes). In accordance with this division, we shall solve our problem in an approximation that takes into account the distribution of losses in only the second type of processes. Ionization losses will always be assumed equal to their mean value.

Following these preliminary remarks we turn to the equations for energy losses incurred in different processes by fast particles bearing the charge e. Ionization losses, with thickness measured in g/cm^2, are described by (see reference 6)

$$-\frac{dE_i}{dx} = \frac{2\pi NZ}{A}\frac{e^4}{mc^2}\ln\frac{E_1 m^2c^2Z}{\hbar^2e^2NA}, \tag{1}$$

where N is Avogadro's number, m is the electron mass, while Z and A are the charge and atomic weight of the matter, which will be taken as 10 and 20 in our numerical calculations. Strictly speaking, Eq. (1) describes ionization losses subject to the condition that the δ-electron energy is $E_1 \ll E$, where E is the muon energy. However, since E_1 is in the logarithm, which is on the order of 30 or 40 in the energy region of greatest interest for us, we can replace E_1 by E with an error of about 5%. As we shall see, all other cross sections except nuclear losses are taken with the same degree of error. Nuclear losses introduce the principal uncertainty. Equation (1) can conveniently be written as*

$$-dE_i/dx = a + b\ln(E_{\mathrm{Mev}}/1000),$$
$$a = 2.1\ \mathrm{Mev\cdot cm^2/g},\quad b = 0.077\ \mathrm{Mev\cdot cm^2/g}. \tag{2}$$

Ionization losses depend only slightly on energy and will hereafter be regarded as constant. Since we are interested in the passage of particles with energies above some $E_{\min}$, we average (2) over the energy spectrum, which we assume to be represented by a power law with the exponent $(\gamma+1)$:

$$\overline{\frac{dE_i}{dx}} = \int_{E_{\min}}^{\infty}\frac{dE_i}{dx}E^{-(\gamma+1)}dE \Big/ \int_{E_{\min}}^{\infty}E^{-(\gamma+1)}dE. \tag{3}$$

*Here and hereafter energies will be expressed in Mev.

When $E > 10^4$ only very slight dependence on γ is exhibited by $\overline{dE_i}/dx$, which can be represented by

$$-\frac{\overline{dE_i}}{dx} = a + b \ln \frac{E_{\min}}{1000}. \tag{4}$$

The effective bremsstrahlung cross section is known to depend on the parameter

$$\delta = 100 \frac{\mu c^2}{E} \frac{E'}{E-E'} Z^{-1/3}, \tag{5}$$

where E' is the energy of the emitted photon and μ is the muon mass (see references 4 and 7, for example). In the limiting cases $\delta \gg 1$ (no screening) and $\delta \ll 1$ (complete screening) the differential cross section $\varphi_r(E, v)$, where $v = E'/E$, is given analytically by

$$\varphi_r^{(1)}(E, v)\, dv = 4\alpha \frac{N}{A} Z^2 r_e^2 \left(\frac{m}{\mu}\right)^2 \times \left\{\left[1 + (1-v)^2 - \frac{2}{3}(1-v)\right] \times \ln\left(\frac{\mu}{m} 183 Z^{-1/3}\right) + \frac{1}{9}(1-v)\right\}\frac{dv}{v} \tag{6}$$

(complete screening) and

$$\varphi_r^{(2)}(E, v)\, dv = \frac{16}{3} \frac{N}{A} \alpha Z^2 r_e^2 \left(\frac{m}{\mu}\right)^2 \left(\frac{3}{4} v + \frac{1-v}{v}\right) \times \left[\ln\left(\frac{12}{5} \frac{1-v}{v} \frac{E}{\mu c^2 Z^{1/3}}\right) - \frac{1}{2}\right] dv \tag{7}$$

(no screening); here

$$\alpha = 1/137, \quad r_e = e^2/mc^2.$$

For our further calculations it will be convenient to introduce the meson radiation unit of length defined by

$$\frac{1}{t_\mu} = 4\alpha \frac{N}{A} Z^2 r_e^2 \left(\frac{m}{\mu}\right)^2 \ln\left(\frac{\mu}{m} 183 Z^{-1/3}\right). \tag{8}$$

In the present case this radiation length is equal to 4.5×10^5 g/cm². These units can be used to write (6) and (7) with good accuracy as follows:

$$\varphi_r^{(1)}(E, v) = 1/v, \tag{9}$$

$$\varphi_r^{(2)}(E, v) = 0.3\left(\frac{3}{4} v + \frac{1-v}{v}\right)\left[\ln \frac{E(1-v)}{v} - 5\right]. \tag{10}$$

We shall now determine the range of applicability of these equations. For this purpose we estimate the average value of the factor $E'/(E-E')$, using the distribution (9), which diverges when the average is taken over the interval $0 < E' < E$, so that (6) and (7) cannot be used for very large energy losses. The average must therefore be taken over the interval $0 < E' < \kappa E$, where κ is close to but not equal to unity. $E'/(E-E')$ exhibits very weak (logarithmic) dependence on κ. Thus when $\kappa = 0.9$ we have $E'/(E-E') \sim 1$ and when $\kappa = 0.99$ we have $E'/(E-E') \sim 3$. We can therefore use (9) when $E \gg \mu c^2 Z^{-1/3} \sim 10^4 Z^{-1/3}$ and (10) when $E \ll 10^4 Z^{-1/3}$. For a rough estimate in the intermediate region $E \sim 10^4 Z^{-1/3}$ we can interpolate the cross section assuming that for $Z = 10$ it is represented by (9) in the entire region $E > 10^4$ and by (10) when $E < 10^4$. From (9) we easily obtain the average energy loss in the region $E > 10^4$:

$$-dE_r/dx = 2.2 \cdot 10^{-6} E \ \text{cm}^2/\text{g}. \tag{11}$$

We shall now consider direct pair production. According to Bhabha[8] the differential cross section $\varphi_p(E, v)$ differs in four energy regions as follows:

$$\varphi_p(E, v)\, dv = \frac{56}{9\pi} (\alpha Z r_e)^2 \frac{N}{A} f(E, v) \frac{dv}{v}, \tag{12}$$

where

$$\begin{aligned}
f_1 &= \ln(k_1 E/mc^2) \ln(k_1' m/v\mu), && 2mc^2/E < v < 2mc^2/E\alpha Z^{1/3}, \\
f_2 &= \ln(k_2 \alpha Z^{1/3}) \ln(k_2' m/\mu v), && 2mc^2/E\alpha Z^{1/3} < v < m/\mu, \\
f_3 &= {}^9/_7 (m/\mu v)^2 \ln(k_3 \mu v/\alpha Z^{1/3} m), && m/\mu < v < 2mE/\alpha Z^{1/3} \mu^2 c^2, \\
f_4 &= {}^9/_7 (m/\mu v)^2 \ln(k_4 E/\mu c^2), && 2mE/\alpha Z^{1/3} \mu^2 c^2 < v < 1,
\end{aligned} \tag{13}$$

where all k are of the order of unity but their exact values are unknown. Continuity of the differential cross sections at the boundaries between the intervals is satisfied when $k_1 = 1/2$, $k_1' = 3$, $k_2 = k_3' = 1$, $k_2' = 3$, $k_4 = 2$. In radiation lengths we obtain

$$\varphi_p^{(1)}(E, v)\, dv = -16\left[\ln v^2 + (4.3 + \ln E) \ln v + 4.3 \ln E\right] dv/v, \tag{14}$$

$$\varphi_p^{(2)}(E, v)\, dv = -(280 + 64 \ln v)\, dv/v, \tag{15}$$

$$\varphi_p^{(3)}(E, v)\, dv = (0.0045 + 0.0005 \ln v)\, dv/v^3, \tag{16}$$

$$\varphi_p^{(4)}(E, v)\, dv = 0.0005 \ln(E/50)\, dv/v^3. \tag{17}$$

Equations (14) to (17) apply to the same energy intervals as (13). Multiplying by v and integrating over these intervals, we easily obtain the energy losses corresponding to the functions $\varphi_p^{(1-4)}$. Table I shows these energy losses expressed in terms of $10^{-6}E$ cm²/g for the different intervals representing four different $dE_p^{(i)}/dx$.

TABLE I

	E: $6\cdot10^4$	10^6	$6\cdot10^6$	10^8
$-dE_p^{(1)}/dx$	0.3	0.2	0.1	—
$-dE_p^{(2)}/dx$	1.7	1.4	1.4	1.4
$-dE_p^{(3)}/dx$	0.8	1.0	1.0	1.0
$-dE_p^{(4)}/dx$	0.1	0.02	—	—
$-\sum_{i=1}^{4} dE_p^{(i)}/dx = -dE_p/dx$	2.9	2.6	2.5	2.4

The direct production of nuclear-interacting particles by muons can be treated consistently as their creation by virtual photons.[3,9] A calculation by the method of Weizsäcker and Williams yields the following cross section for the production of a "star" or electron-nuclear shower with total energy in the interval E′, E′ + dE′:

$$\varphi_n(E,\ E')\,dE' = \frac{2\alpha}{\pi} N\sigma_\gamma(E') \frac{dE'}{E'} \ln\frac{E}{E'}, \tag{18}$$

where $\sigma_\gamma(E')$ is the total cross section for the production of "stars" and showers by a gamma quantum with energy E′. A somewhat more exact calculation[9] leads to the analogous expression

$$\varphi_n(E,\ E')\,dE' \approx \frac{2\alpha}{\pi} N \ln\frac{E}{\mu c^2}\, \sigma_\gamma(E')\, \frac{dE'}{E'} \tag{19}$$

The different forms of the logarithm in (18) and (19) do not represent an essential difference between the two expressions for the following reasons. The first and less important reason is that in reference 9 the entire calculation was performed with accuracy to within a logarithmic factor of the order of unity. The indeterminacy of the cross section $\sigma_\gamma(E')$ is the second and decisive consideration in estimating the accuracy. At sufficiently high energies $(E' > 1000)$ this cross section is usually taken as constant, although this hypothesis is certainly not confirmed by data on $\sigma_\gamma(E')$ at lower energies (see reference 10, for example). Since we do not possess experimental data on the energy-dependence of this cross section we can only compare (18) and (19) with data on the production of showers and "stars" by muons. However the experimental data are not accurate enough to permit a distinction between the two formulas with $\sigma_\gamma(E')$ assumed constant. Following George[3] and Kessler and Maz[11] we assume $\sigma_\gamma(E')$ to have the constant value $d \times 10^{-28}$ cm^2 per nucleon, with the constant d being varied in the numerical computations. Thus by means of (18) and (19) we write the cross section in radiation lengths as follows:

$$\varphi_n(E,\ v)\,dv = 0.13d \frac{\ln(1/v)}{v}\,dv \tag{20}$$

or

$$\varphi_n(E,\ v)\,dv = 0.13d \frac{\ln(E/\mu c^2)}{v}\,dv. \tag{21}$$

It follows that the mean energy loss is*

$$-dE_n/dx = 0.13d\cdot E 10^{-6}\,\mathrm{cm}^2/\mathrm{g} \tag{22}$$

or

$$-dE_n/dx = 0.13dE \ln(E/\mu c^2)\, 10^{-6}\,\mathrm{cm}^2/\mathrm{g}. \tag{23}$$

3. To describe the passage of muons through matter, we shall now set up the kinetic equation. It is convenient to introduce the critical energy ϵ_μ, which represents the ionization losses per radiation length. From (2) and (4) we obtain $\epsilon_\mu = 1.0 \times 10^6$, 1.0×10^6 and 1.2×10^6 for $E_{min} = 10^4$, 10^5, and 10^6. Using the notation $\epsilon_\mu t_\mu = y$, we easily arrive at an equation to describe the muon spectrum $\pi(E, y)$ by analogy with the kinetic equations of ordinary cascade theory:

$$\frac{\partial\pi(E,\ y)}{dy} - \frac{\partial\pi(E,\ y)}{dE} = -\frac{1}{\varepsilon_\mu}\int_0^1 \Big[\pi(E,\ y)\,\varphi(E,\ v) - \frac{1}{1-v}\,\pi\Big(\frac{E}{1-v},\ y\Big)\,\varphi\Big(\frac{E}{1-v},\ v\Big)\Big]\,dv, \tag{24}$$

where $\varphi = \varphi_r + \varphi_p + \varphi_n$.

Assuming that for y = 0 the spectrum is represented by the power function

$$\pi(E,\ 0) = CE^{-(\gamma+1)}, \tag{25}$$

we first solve (24) for small depths, when $y \ll E$. Substituting the solution without the right member, $\pi_0(E, y) = C/(E + y)^{\gamma+1}$, into the right member of (24), we obtain

$$\partial\pi/\partial y - \partial\pi/\partial E = \varphi(E,\ y); \tag{26}$$

$$\varphi(E,\ y) = -\frac{1}{\varepsilon_\mu}\int_0^1 \Big[\pi_0(E,\ y)\,\varphi(E,\ v) - \frac{1}{1-v}\,\pi\Big(\frac{E}{1-v}, y\Big)\,\varphi\Big(\frac{E}{1-v},\ v\Big)\Big]\,dv. \tag{27}$$

We shall seek a solution in the form

$$\pi(E,\ y) = \pi_0(E,\ y) + \pi_1(E,\ y)$$

subject to the initial conditions (25) and $\pi_1(E, 0) = 0$. π_1 satisfies (26) when π is replaced by π_1. The modified equation is easily solved by means of the substitutions $E + y = u$, $E - y = w$. After some simple transformations we obtain

$$\pi_1 = \frac{1}{2}\int_{E-y}^{E+y} \varphi\Big(\frac{E+y+w}{2},\ \frac{E+y-w}{2}\Big)\,dw$$

$$= -\frac{1}{2\varepsilon_\mu}\int_0^1 \Big\{\frac{2\varphi(E,\ v)\,y}{(E+y)^{\gamma+1}} - \frac{(1-v)^\gamma 2^{\gamma+1}}{\gamma v}\,[(E+y-yv)^{-\gamma} - (E+y)^{-\gamma}]\,\varphi\Big(\frac{E}{1-v},\ v\Big)\Big\}\,dv. \tag{28}$$

An expansion of $(E + y - yv)^{-\gamma}$ in powers of $yv/(E + y)$ gives

$$\pi_1 = -\frac{y}{\varepsilon_\mu(E+y)^{\gamma+1}}\int_0^1 \Big\{\varphi(E,\ v) - (1-v)^\gamma\,\varphi\Big(\frac{E}{1-v},\ v\Big) \times \Big[1 + \frac{yv(\gamma+1)}{E+y} + \ldots\Big]\Big\}\,dv. \tag{29}$$

*We shall not consider the production of penetrating pairs separately (as was done by George), because the most accurate experiments have shown that this process is only a special case of shower production by muons.[12]

The method used here is valid when the condition $\pi_1 \ll \pi_0$ is satisfied, for which it is sufficient that

$$y \ll \varepsilon_\mu \qquad (30)$$

(or $t_\mu \ll 1$). These conditions are both sufficient and necessary, since when $y \gtrsim \epsilon_\mu$ it can be shown that $\pi_1 \gtrsim \pi_0$, which makes the method entirely unsuitable. It follows from (30) that in calculating π_1 we can limit ourselves to the first term in the expansion (29).

4. In this section we obtain a solution for $E > \epsilon_\mu$, for which purpose the expressions for the cross sections φ must be simplified so that they depend only on the ratio v. When $E > \epsilon_\mu$, the cross section for pair production is then approximated with good accuracy ($\sim 1\%$). In this approximation we can assume

$$\varphi_p^1 = \varphi_p^{(4)} = 0; \qquad (31)$$
$$\varphi_p = \varphi_p^{(2)} \text{ for } 0 < v < m/\mu, \quad \varphi_p = \varphi_p^{(3)} \text{ for } m/\mu < v < 1.$$

When (9) and (18) are used as cross sections of the other processes, (24) becomes

$$\frac{\partial \pi(E, y)}{\partial y} - \frac{\partial \pi(E, y)}{\partial E} = -\frac{1}{\varepsilon_\mu}\int_0^1 \left[\pi(E, y) - \frac{1}{1-v}\pi\left(\frac{E}{1-v}, y\right)\right]\varphi(v)\,dv. \qquad (32)$$

We shall use Snyder's method[13] to obtain a solution, which will be sought in the form

$$\pi(E, y) = C\exp\{\lambda(\gamma)t_\mu\}F(E, \gamma). \qquad (33)$$

A Mellin transformation in the variable E gives

$$F(E, \gamma) = \varepsilon_\mu^{-(\gamma+1)}\frac{1}{2\pi i} \times \int_{-\delta-i\infty}^{-\delta+i\infty} \frac{\Gamma(-r)\Gamma(\gamma+r+1)}{\Gamma(\gamma+1)} K(\gamma, r)\left(\frac{E}{\varepsilon_\mu}\right)^{-(r+\gamma+1)} dr, \qquad (34)$$

where $\delta \gtrsim 0$, Γ is the gamma function, and $K(\gamma, r)$ satisfies the recursion relation

$$K(\gamma, r) = \frac{rK(\gamma, r-1)}{A(\gamma+r) - A(\gamma)},$$
$$A(x) = \int_0^1 [1-(1-v)^x]\varphi(v)\,dv, \quad K(\gamma, 0) = 1. \qquad (35)$$

When $E/\epsilon_\mu > 1$ we can close the contour with an arc of infinitely large radius and calculate (34) by the method of residues:

$$F(E, \gamma) = E^{-(\gamma+1)}\sum_{n=0}^{\infty} C_{\gamma+n}^n K(\gamma, n)\left(\frac{\varepsilon_\mu}{E}\right)^n(-1)^n. \qquad (36)$$

Comparing (36) and (33) with (28) and noting that when $t_\mu \to 0$ and $\epsilon_\mu/E \to 0$ (33) and (28) must coincide, we obtain $\lambda(\gamma) = -A(\gamma)$. It can be shown that our functions φ make (36) converge in the entire region $E > \epsilon_\mu$.

5. It is useful to compare the results obtained when meson-intensity fluctuations at different depths are taken into account by means of distribution functions, with the results based on mean energy losses. We find the ratio between the number $N(E, t_\mu)$ of mesons with energies above E at depth t_μ, obtained by means of (33) and (36), and the analogous number $N_s(E, t_\mu)$, computed neglecting distribution functions. We use the fact that

$$N_s(E, t_\mu) = \int_{E_t}^{\infty} F(E, \gamma)\,dE, \qquad (37)$$

where E_t is the minimum energy which a muon can expend in passing through the depth t_μ when mean energy losses are used and the final muon energy is E. E_t is obtained from

$$t_\mu = \int_E^{E_t}\left[\varepsilon_\mu + \frac{dE_r}{dt_\mu} + \frac{dE_p}{dt_\mu} + \frac{dE_n}{dt_\mu}\right]^{-1} dE. \qquad (38)$$

We finally obtain

$$\frac{N(E, t_\mu)}{N_s(E, t_\mu)} = \frac{\left(\frac{E_t}{E}\right)^\gamma e^{-A(\gamma)t_\mu}\sum_{n=0}^{\infty}(-1)^n C_{\gamma+n}^n K(\gamma, n)(\varepsilon_\mu/E)^n}{\sum_{n=0}^{\infty}(-1)^n \frac{C_{\gamma+n}^n}{\gamma+n} K(\gamma, n)(\varepsilon_\mu/E_t)^n} \qquad (39)$$

Numerical values of this ratio for $E = 2\epsilon_\mu$ and $3\epsilon_\mu$ are given in Table II.

TABLE II

t_μ	$d=1$			$d=5$		
	0.3	0.6	1.0	0.3	0.6	1.0
		$\gamma=2$				
$\varepsilon_\mu/E = 1/2$	1.1	1.2	1.4	1.1	1.25	1.6
$\varepsilon_\mu/E = 1/3$	1.1	1.25	1.5	1.15	1.35	1.7
		$\gamma=3$				
$\varepsilon_\mu/E = 1/2$	1.2	1.7	2.7	1.4	2.1	4.0
$\varepsilon_\mu/E = 1/3$	1.3	1.9	2.9	1.5	2.2	4.8

Table II shows that N/N_s depends only slightly on ϵ_μ/E and d and that it is greater than unity (see also the general formula (39)). This indicates that a measured intensity at a given depth represents a smaller muon flux than is obtained by means of a calculation neglecting fluctuations.

In conclusion the authors wish to thank G. T. Zatsepin for a discussion of the questions which were touched on in this paper and Z. S. Maksimova for the numerical computations.

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Translated by I. Emin
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