SOVIET PHYSICS JETP VOLUME 35(8), NUMBER 6 JUNE, 1959

ON COORDINATE CONDITIONS IN EINSTEIN'S GRAVITATION THEORY

I. G. FIKHTENGOL'TS

Submitted to JETP editor June 17, 1958

J. Exptl. Theoret. Phys. (U.S.S.R.) **35**, 1457-1465 (December, 1958)

Certain relations connected with the covariance of field equations under transformation of variables are derived. The connection between coordinate conditions and invariance of the field Lagrangian is established. The geometrical and physical properties of the coordinate systems corresponding to the coordinate conditions thus derived are considered.

As is well known, only six of the ten equations in Einstein's theory of gravitation are independent, as a consequence of the conservative character of the Einstein tensor (and mass tensor). Therefore, when obtaining the fundamental field tensor, four additional relations among its components must be stated beside the Einstein gravitation equations and boundary conditions.* Since the choice of these four additional relations fixes the coordinate system in which the gravitational field will be studied, they are usually referred to as the coordinate conditions.

In contradistinction to the covariant character of the Einstein gravitation equations, the aggregate of additional conditions should not be covariant, since a covariant system of equations admits of arbitrary coordinate transformations and therefore determines the quantities sought only within four arbitrary functions. At the same time the covariant character of the Einstein gravitation equations permits one to choose arbitrary coordinate conditions (provided they do not contradict each other or the gravitation equations) and so the question arises: which of the permissible coordinate conditions are preferable when the peculiarities of the given physical problem are taken into account? Closely connected with this is the study of the physical and geometrical properties of the coordinate system chosen; it is scarcely possible to give a consistent physical interpretation of the solution of the field equations without the knowledge of these properties.

The above problems, which have been repeatedly investigated (see, e.g., reference 1), form the subject of the present work (Secs. 3 to 5). At the same time certain general problems connected with the covariance of field equations under transformation of variables are investigated (Secs. 1 and 2).

*It is also necessary to give the equation of state of the masses under study (see, e.g. references 1 and 2).

1. CONSEQUENCES OF COVARIANCE OF FIELD EQUATIONS

We start with the investigation of arbitrary fields whose equations may be written in the Lagrangian formalism

$$\frac{\partial \mathscr{L}}{\partial q_l} - \sum_{k=1}^{n} \frac{\partial}{\partial x_k} \frac{\partial \mathscr{L}}{\partial (\partial q_l / \partial x_k)} = 0 \quad (l = 1, 2, \ldots, m), \tag{1}$$

where $\mathscr{L} = \mathscr{L}([x_i], [q_j], [\partial q_j/\partial x_i])$ is the field Lagrangian, $[x_i]$ stands for the aggregate of the independent variables $(i = 1, 2 \ldots n)$, and $[q_j]$ stands for the aggregate of the unknown functions $(j = 1, 2 \ldots m)$.

To establish just what consequences follow from covariance of the field equations let us pass in Eq. (1) from the variables $x_k (k = 1, 2 \ldots n)$ and $q_l (l = 1, 2 \ldots m)$ to the new variables x'_k, q'_l defined by

$$x'_k = x'_k([x_i], \varepsilon), \quad q'_l = q'_l([x_i], [q_j], \varepsilon), \tag{2}$$

where ϵ is some parameter, and

$$(x'_k)_{\varepsilon=0} = x_k, \quad (q'_l)_{\varepsilon=0} = q_l. \tag{3}$$

It is easy to show, following the methods of references 3 to 5, that if Eq. (1) is covariant under the one-parameter family of transformations defined by Eqs. (2) and (3) then the following is true:

$$\sum_{k=1}^{n} \frac{\partial}{\partial x_k} \left\{ \mathscr{L} \left(\frac{\partial x'_k}{\partial \varepsilon} \right)_{\varepsilon=0} - \sum_{i=1}^{n} \sum_{j=1}^{m} \frac{\partial \mathscr{L}}{\partial (\partial q_j / \partial x_k)} \frac{\partial q_j}{\partial x_i} \left(\frac{\partial x'_i}{\partial \varepsilon} \right)_{\varepsilon=0} + \sum_{j=1}^{m} \frac{\partial \mathscr{L}}{\partial (\partial q_j / \partial x_k)} \left(\frac{\partial q'_j}{\partial \varepsilon} \right)_{\varepsilon=0} - \left(\frac{\partial f_k}{\partial \varepsilon} \right)_{\varepsilon=0} \right\} = 0, \tag{4}$$

where we understand by $\mathscr{L}$ the untransformed $\mathscr{L}([x_i], [q_j], [\partial q_j/\partial x_i])$ and the functions $f_k = f_k([x_i], [q_j], \epsilon)$ are determined, corresponding to the structure of $\mathscr{L}$ and of the transformation (2), by the equality

$$\mathscr{L}^*([x_i],\ [q_j],\ [\partial q_j/\partial x_i],\ \varepsilon) - \mathscr{L}([x_i],\ [q_j],\ [\partial q_j/\partial x_i]) = \sum_{k=1}^{n} \partial f_k([x_i],\ [q_j],\ \varepsilon)/\partial x_k, \qquad (5)$$

where

$$\mathscr{L}^*\left([x_i],\ [q_j],\ \left[\frac{\partial q_j}{\partial x_i}\right],\ \varepsilon\right) = \mathscr{L}\left([x_i'],\ [q_j'],\ \left[\frac{\partial q_j'}{\partial x_i'}\right]\right)\frac{D(x_1',\ x_2',\ x_3',\ \ldots,\ x_n')}{D(x_1,\ x_2,\ x_3,\ \ldots,\ x_n)} \qquad (6)$$

with x_i', q_j' given by the transformation (2).

The existence of a system of functions f_k which satisfy condition (5) is equivalent to the requirement of covariance of Eq. (1) under the transformation (2). In the special case when the function $\mathscr{L}$ and the transformation (2) are such that the equality $\mathscr{L}^* = \mathscr{L}$ holds, it is possible to put all f_k equal to zero. The function $\mathscr{L}$ is then a relative invariant under the transformation (2) — relative, because the expression for $\mathscr{L}^*$ contains as a factor the Jacobian of the transformation.

We now specify that the field under discussion is a tensor field. Although the results to be established are applied only to the Einstein gravitation equations (in the region outside of masses) to begin with we shall not use the explicit form of the Lagrangian $\mathscr{L}$ but will take $\mathscr{L} = \mathscr{L}([x_\alpha],\ [g_{\mu\nu}],\ [\partial g_{\mu\nu}/\partial x_\alpha])$ where $g_{\mu\nu}$ is the fundamental (metric) tensor.* Equations (1) become

$$\frac{\partial \mathscr{L}}{\partial g_{\mu\nu}} - \frac{\partial}{\partial x_\alpha}\frac{\partial \mathscr{L}}{\partial(\partial g_{\mu\nu}/\partial x_\alpha)} = 0. \qquad (1')$$

In these equations and in the following greek indices take on the values 0, 1, 2, 3 and repeated indices are to be summed over from 0 to 3.

If Eq. (1′) is covariant under a certain family of coordinate transformations defined by

$$x_\beta' = x_\beta'([x_\alpha],\ \varepsilon), \qquad (2')$$

$$(x_\beta')_{\varepsilon=0} = x_\beta, \qquad (3')$$

then Eq. (4) becomes

$$\frac{\partial}{\partial x_\alpha}\left\{\mathscr{L}\left(\frac{\partial x_\alpha'}{\partial \varepsilon}\right)_{\varepsilon=0} - \frac{\partial \mathscr{L}}{\partial(\partial g_{\mu\nu}/\partial x_\alpha)}\frac{\partial g_{\mu\nu}}{\partial x_\beta}\left(\frac{\partial x_\beta'}{\partial \varepsilon}\right)_{\varepsilon=0} + \frac{\partial \mathscr{L}}{\partial(\partial g_{\mu\nu}/\partial x_\alpha)}\left(\frac{\partial g_{\mu\nu}'}{\partial \varepsilon}\right)_{\varepsilon=0} - \left(\frac{\partial f_\alpha}{\partial \varepsilon}\right)_{\varepsilon=0}\right\} = 0. \qquad (4')$$

The equality (4′) can be substantially simplified due to the fact that the transformation of the dependent variables is, as a consequence of their tensor character, determined entirely by the transformation of the independent variables:

$$g_{\mu\nu}' = \frac{\partial x_\alpha}{\partial x_\mu'}\frac{\partial x_\beta}{\partial x_\nu'}g_{\alpha\beta}. \qquad (7)$$

Consequently the derivatives of $g_{\mu\nu}'$ with respect to ϵ can be eliminated from Eq. (4′).

Since

$$\left(\frac{\partial g_{\mu\nu}'}{\partial \varepsilon}\right)_{\varepsilon=0} = -g_{\mu\beta}\frac{\partial}{\partial x_\nu}\left(\frac{\partial x_\beta'}{\partial \varepsilon}\right)_{\varepsilon=0} - g_{\nu\beta}\frac{\partial}{\partial x_\mu}\left(\frac{\partial x_\beta'}{\partial \varepsilon}\right)_{\varepsilon=0},$$

we conclude that if Eq. (1′) is covariant under a certain one-parameter family of transformations defined by Eqs. (2′) and (3′) [and, of course, (7)] then the following relation holds

$$\frac{\partial}{\partial x_\alpha}\left\{\mathscr{L}\left(\frac{\partial x_\alpha'}{\partial \varepsilon}\right)_{\varepsilon=0} - \frac{\partial \mathscr{L}}{\partial(\partial g_{\mu\nu}/\partial x_\alpha)}\frac{\partial g_{\mu\nu}}{\partial x_\beta}\left(\frac{\partial x_\beta'}{\partial \varepsilon}\right)_{\varepsilon=0} - 2\frac{\partial \mathscr{L}}{\partial(\partial g_{\mu\nu}/\partial x_\alpha)}g_{\mu\beta}\frac{\partial}{\partial x_\nu}\left(\frac{\partial x_\beta'}{\partial \varepsilon}\right)_{\varepsilon=0} - \left(\frac{\partial f_\alpha}{\partial \varepsilon}\right)_{\varepsilon=0}\right\} = 0, \qquad (8)$$

where f_α is determined from an equation analogous to Eq. (5), corresponding to the structure of $\mathscr{L}$ and of the transformation (2′).

Taking into account the identity

$$\frac{\partial \mathscr{L}}{\partial(\partial g_{\mu\nu}/\partial x_\alpha)} = -g^{\mu\sigma}g^{\nu\tau}\frac{\partial \mathscr{L}}{\partial(\partial g^{\sigma\tau}/\partial x_\alpha)},$$

it is possible to express the relation (8) in a different form, which is more convenient for applications, if the unknown in the field equations is taken to be the contravariant fundamental tensor $g^{\mu\nu}$ rather than the covariant tensor $g_{\mu\nu}$.

2. TENSOR FIELD WITH A LINEARLY-INVARIANT LAGRANGIAN

Using Eq. (8) we now establish just what consequences follow from the assumption that the Lagrangian $\mathscr{L}$ is a relative invariant* under arbitrary linear coordinate transformations. We shall refer to such an $\mathscr{L}$, for brevity, as linearly invariant (omitting the word "relative").

In an n-dimensional space the most general linear coordinate transformation contains $n(n+1)$ parameters. We are interested in the case of space-time, which is four-dimensional and, consequently, the number of parameters is 20.

It is easy to show that the most general linear transformation of the coordinates x_0, x_1, x_2, x_3 can be expressed as a linear combination with constant coefficients of the following twenty simple (one-parameter) linear transformations:

*The discussion is analogous for the case $\mathscr{L} = \mathscr{L}([x_\alpha],\ [A_{(\mu)}^{(\nu)}],\ [\partial A_{(\mu)}^{(\nu)}/\partial x_\alpha])$ where $A_{(\mu)}^{(\nu)} = A_{\mu_1 \ldots \mu_l}^{\nu_1 \ldots \nu_m}$ is a tensor of rank $(l + m)$.

*I.e. $\mathscr{L}^* = \mathscr{L}$ where $\mathscr{L}^*$ is determined by an equation of the same type as Eq. (6).

$$
\begin{array}{llll}
1)\ x_0' = x_0 + \varepsilon & x_1' = x_1 & x_2' = x_2 & x_3' = x_3, \\
2)\ x_0' = x_0 & x_1' = x_1 + \varepsilon & x_2' = x_2 & x_3' = x_3, \\
3)\ x_0' = x_0 & x_1' = x_1 & x_2' = x_2 + \varepsilon & x_3' = x_3, \\
4)\ x_0' = x_0 & x_1' = x_1 & x_2' = x_2 & x_3' = x_3 + \varepsilon, \\
5)\ x_0' = x_0 & x_1' = x_1 - \varepsilon x_2 & x_2' = \varepsilon x_1 + x_2 & x_3' = x_3, \\
6)\ x_0' = x_0 & x_1' = x_1 + \varepsilon x_3 & x_2' = x_2 & x_3' = -\varepsilon x_1 + x_3, \\
7)\ x_0' = x_0 & x_1' = x_1 & x_2' = x_2 - \varepsilon x_3 & x_3' = \varepsilon x_2 + x_3, \\
8)\ x_0' = x_0 - \varepsilon x_1 & x_1' = -\varepsilon x_0 + x_1 & x_2' = x_2 & x_3' = x_3, \\
9)\ x_0' = x_0 - \varepsilon x_2 & x_1' = x_1 & x_2' = -\varepsilon x_0 + x_2 & x_3' = x_3, \\
10)\ x_0' = x_0 - \varepsilon x_3 & x_1' = x_1 & x_2' = x_2 & x_3' = -\varepsilon x_0 + x_3, \\
11)\ x_0' = (1+\varepsilon) x_0 & x_1' = x_1 & x_2' = x_2 & x_3' = x_3, \\
12)\ x_0' = x_0 & x_1' = (1+\varepsilon) x_1 & x_2' = x_2 & x_3' = x_3, \\
13)\ x_0' = x_0 & x_1' = x_1 & x_2' = (1+\varepsilon) x_2 & x_3' = x_3, \\
14)\ x_0' = x_0 & x_1' = x_1 & x_2' = x_2 & x_3' = (1+\varepsilon) x_3, \\
15)\ x_0' = x_0 & x_1' = x_1 + \varepsilon x_2 & x_2' = \varepsilon x_1 + x_2 & x_3' = x_3. \\
16)\ x_0' = x_0 & x_1' = x_1 + \varepsilon x_2 & x_2' = x_2 & x_3' = \varepsilon x_1 + x_3, \\
17)\ x_0' = x_0 & x_1' = x_1 & x_2' = x_2 + \varepsilon x_3 & x_3' = \varepsilon x_2 + x_3, \\
18)\ x_0' = x_0 - \varepsilon x_1 & x_1' = \varepsilon x_0 + x_1 & x_2' = x_2 & x_3' = x_3, \\
19)\ x_0' = x_0 - \varepsilon x_2 & x_1' = x_1 & x_2' = \varepsilon x_0 + x_2 & x_3' = x_3, \\
20)\ x_0' = x_0 - \varepsilon x_3 & x_1' = x_1 & x_2' = x_2 & x_3' = \varepsilon x_0 + x_3.
\end{array}
\qquad (9)
$$

We note that the transformations $(9)_1$ to $(9)_{10}$ are special cases of infinitesimal Lorentz transformations, whereas the remaining transformations $(9)_{11}$ to $(9)_{20}$ do not belong to the Lorentz transformations group.

Let us now assume that the Lagrangian $\mathscr{L}$ is linearly invariant. It is then, in particular, a relative invariant under each of the twenty simple transformations (9). Using Eq. (8), we deduce from this fact the following consequences:

From the relative invariance of $\mathscr{L}$ under the translation of the origin of time coordinate x_0 [transformation $(9)_1$] and of the space coordinates x_1, x_2, x_3 [transformations $(9)_2$ to $(9)_4$] it follows that $\mathscr{L}$ is not an explicit function of x_0, x_1, x_2, x_3, i.e., $\mathscr{L} = \mathscr{L}([g_{\mu\nu}], [\partial g_{\mu\nu}/\partial x_\alpha])$.

For a Lagrangian $\mathscr{L}$, which is not an explicit function of x_α, we obtain, after some calculations, the following relations $(10)_k$ corresponding to relative invariance under the transformations $(9)_k$ ($k = 5, 6, \ldots, 20$) respectively

$$
\begin{array}{ll}
5)\ N_1^2 = N_2^1, & 6)\ N_1^3 = N_3^1, \\
7)\ N_2^3 = N_3^2, & 8)\ N_0^1 = -N_1^0, \\
9)\ N_0^2 = -N_2^0, & 10)\ N_0^3 = -N_3^0, \\
11)\ N_0^0 = \mathscr{L}, & 12)\ N_1^1 = \mathscr{L}, \\
13)\ N_2^2 = \mathscr{L}, & 14)\ N_3^3 = \mathscr{L}, \\
15)\ N_1^2 = -N_2^1, & 16)\ N_1^3 = -N_3^1, \\
17)\ N_2^3 = -N_3^2, & 18)\ N_0^1 = N_1^0, \\
19)\ N_0^2 = N_2^0, & 20)\ N_0^3 = N_3^0,
\end{array}
\qquad (10)
$$

where

$$N_\alpha^\beta = \frac{\partial g_{\mu\nu}}{\partial x_\alpha} \frac{\partial \mathscr{L}}{\partial(\partial g_{\mu\nu}/\partial x_\beta)} + 2 g_{\mu\alpha} \frac{\partial \mathscr{L}}{\partial g_{\mu\beta}} + 2 \frac{\partial g_{\mu\alpha}}{\partial x_\nu} \frac{\partial \mathscr{L}}{\partial(\partial g_{\mu\beta}/\partial x_\nu)}.$$

Collecting the relations (10) we conclude

$$N_\alpha^\beta = \mathscr{L} \delta_{\alpha\beta}.$$

Consequently, the Lagrangian $\mathscr{L}$ is linearly invariant if it is not an explicit function of the variables x_α and if it satisfies the following system of equations:

$$\frac{\partial g_{\mu\nu}}{\partial x_\alpha} \frac{\partial \mathscr{L}}{\partial(\partial g_{\mu\nu}/\partial x_\beta)} + 2 g_{\mu\alpha} \frac{\partial \mathscr{L}}{\partial g_{\mu\beta}} + 2 \frac{\partial g_{\mu\alpha}}{\partial x_\nu} \frac{\partial \mathscr{L}}{\partial(\partial g_{\mu\beta}/\partial x_\nu)} = \mathscr{L} \delta_{\alpha\beta}.$$

We next assume that Eq. (1′) with a linearly invariant Lagrangian is, in addition, covariant under a certain family of nonlinear transformations defined by Eqs. (2′) and (3′). Again starting from the relation (8) we now obtain

$$\frac{\partial \mathscr{L}}{\partial(\partial g_{\mu\nu}/\partial x_\alpha)} g_{\beta\nu} \frac{\partial^2}{\partial x_\alpha \partial x_\mu} \left(\frac{\partial x_\beta'}{\partial \varepsilon}\right)_{\varepsilon=0} = -\frac{1}{2} \frac{\partial}{\partial x_\alpha} \left(\frac{\partial f_\alpha}{\partial \varepsilon}\right)_{\varepsilon=0}, \qquad (11)$$

where the function f_α is determined, as before, by an equation analogous to Eq. (5), corresponding to the structure of $\mathscr{L}$ and of the transformation (2′).

It follows from Eq. (11) that the linearly invariant Lagrangian under discussion will also be a relative invariant under this family of nonlinear transformations if

$$\frac{\partial \mathscr{L}}{\partial(\partial g_{\mu\nu}/\partial x_\alpha)} g_{\beta\nu} \frac{\partial^2}{\partial x_\alpha \partial x_\mu}\left(\frac{\partial x'_\beta}{\partial \varepsilon}\right)_{\varepsilon=0} = 0. \tag{12}$$

In the case of infinitesimal transformations

$$x'_\beta = x_\beta + \varepsilon a^\beta(x_0, x_1, x_2, x_3) \quad (\beta = 0, 1, 2, 3), \tag{13}$$

Equation (12) becomes

$$a^\beta_{\alpha\mu} g_{\beta\nu} \frac{\partial \mathscr{L}}{\partial(\partial g_{\mu\nu}/\partial x_\alpha)} = 0, \tag{14}$$

where we have introduced the abbreviation

$$a^\beta_{\alpha\mu} = \partial^2 a^\beta(x_0, x_1, x_2, x_3)/\partial x_\alpha \partial x_\mu. \tag{15}$$

We note that relation (14) is not only a necessary but also a sufficient condition for the linearly invariant Lagrangian $\mathscr{L}$ to be a relative invariant under the infinitesimal transformation (13).

Relation (14) will be taken as the starting point in the discussion of the problem of coordinate conditions in the Einstein theory of gravitation.

3. RELATION BETWEEN COORDINATE CONDITIONS AND INVARIANCE OF THE FIELD LAGRANGIAN

As is well known, the Einstein gravitation equations, in a region outside of masses, may be expressed in the Lagrangian formalism (1′) with the following Lagrangian:

$$\mathscr{L} = \sqrt{-g}L, \quad \text{where} \quad L = g^{\mu\nu}(\Gamma^\beta_{\mu\alpha}\Gamma^\alpha_{\nu\beta} - \Gamma^\alpha_{\mu\nu}\Gamma^\beta_{\alpha\beta}). \tag{16}$$

In these formulas g stands for the determinant formed from the components of the fundamental tensor $g_{\mu\nu}$ and $\Gamma^\alpha_{\mu\nu}$ stands for the Christoffel symbol of the second kind.

Let us show that the above Lagrangian $\mathscr{L}$ is linearly invariant.

From the general transformation properties of the determinant g (see, e.g., reference 1)

$$\sqrt{-g'}\,\frac{D(x'_0, x'_1, x'_2, x'_3)}{D(x_0, x_1, x_2, x_3)} = \sqrt{-g},$$

we conclude that the requirement $\mathscr{L}^* = \mathscr{L}$, when applied to the present $\mathscr{L}$ [see Eq. (16)], is equivalent to the condition $L' = L$. Consequently the requirement of relative invariance for $\mathscr{L}$ is equivalent in this case to the requirement of (absolute) invariance of L in the usual sense. However, it follows directly from the known transformation properties of the Christoffel symbols that the equality $L' = L$ holds for arbitrary linear transformations of the coordinates. That is, the Lagrangian $\mathscr{L}$ under discussion is indeed linearly invariant.

We now introduce Eq. (16) into Eq. (14). Since in the present case

$$\frac{\partial \mathscr{L}}{\partial(\partial g_{\mu\nu}/\partial x_\alpha)} = \frac{\sqrt{-g}}{2}\{(2g^{\mu\sigma}g^{\nu\tau} - g^{\mu\nu}g^{\sigma\tau})\Gamma^\alpha_{\sigma\tau} + (g^{\mu\nu}g^{\alpha\sigma} - g^{\mu\alpha}g^{\nu\sigma} - g^{\mu\sigma}g^{\nu\alpha})\Gamma^\tau_{\sigma\tau}\},$$

we can write Eq. (14) as

$$a^\beta_{\alpha\beta} g^{\mu\nu}\Gamma^\alpha_{\mu\nu} - a^\alpha_{\mu\nu}(2g^{\mu\beta}\Gamma^\nu_{\alpha\beta} - g^{\mu\nu}\Gamma^\beta_{\alpha\beta}) = 0,$$

which with the help of a simple transformation becomes finally:

$$a^\alpha_{\mu\nu}\partial \mathfrak{G}^{\mu\nu}/\partial x_\alpha - a^\mu_{\alpha\mu}\,\partial \mathfrak{G}^{\alpha\nu}/\partial x_\nu = 0, \tag{17}$$

where, as usual, $\mathfrak{G}^{\mu\nu} = \sqrt{-g}\; g^{\mu\nu}$.

Equation (17), which is a necessary and sufficient condition for invariance of the function* L under the infinitesimal transformation (13), can also be obtained directly from the transformation properties of the function L.

If linear coordinate transformations are considered, Eq. (17) becomes an identity. Thus it does not impose any additional conditions on the fundamental tensor $g_{\mu\nu}$ as is to be expected from the linear invariance of the Lagrangian $\mathscr{L}$. If, however, nonlinear coordinate transformations are also considered besides linear ones and relative invariance of $\mathscr{L}$ under these transformations is demanded, then Eq. (17) gives additional conditions which the tensor $g_{\mu\nu}$ must satisfy. Furthermore, the additional (coordinate) conditions resulting from Eq. (17) represent the necessary and sufficient conditions for the field Lagrangian to be a relative invariant under the given family of nonlinear coordinate transformations. For brevity we shall refer to these families of transformations and to the resultant coordinate conditions as corresponding to each other.

Without stopping to consider the various consequences of the invariance of the field Lagrangian under nonlinear coordinate transformations we pass now to the study of some of the problems related to coordinate conditions and corresponding families of transformations.

4. COORDINATE CONDITIONS PREFERRED FOR A STATIONARY GRAVITATIONAL FIELD

A gravitational field is called stationary in some (not small) region, if a coordinate system exists (the same one for the whole region) for which the components of the fundamental tensor $g_{\mu\nu}$ are independent of the time coordinate x_0. We shall call

*When applied to the function $\mathscr{L}$, invariance in the usual sense is understood.

those coordinate conditions "preferred for a stationary gravitational field" (more accurately, preferred with respect to the stationary property of the gravitational field) whose use yields time independent solutions of the Einstein gravitational equations.

In order to obtain such coordinate conditions starting from relation (17) we assume that the values of the components of the fundamental tensor $g_{\mu\nu}$ are independent of the time coordinate x_0 and that the field Lagrangian is a relative invariant under arbitrary transformations of the variable x_0. In that case we can put in Eq. (13)

$$a^{\beta}(x_0, x_1, x_2, x_3) = \varphi(x_0, x_1, x_2, x_3)\,\delta_{0\beta} + \psi^{\beta}(x_0, x_1, x_2, x_3), \tag{18}$$

where $\varphi(x_0, x_1, x_2, x_3)$ is an arbitrary (twice differentiable) function and $\psi^{\beta}(x_0, x_1, x_2, x_3)$ is an arbitrary linear function of the coordinates. Taking into account that in the present case [see Eq. (15)]

$$a^{\beta}_{\mu\nu} = \frac{\partial^2\varphi(x_0, x_1, x_2, x_3)}{\partial x_\mu \partial x_\nu}\,\delta_{0\beta},$$

we deduce from Eq. (17) the following as the necessary and sufficient condition for the field Lagrangian to be a relative invariant under the indicated family of transformations (with the condition $\partial g_{\mu\nu}/\partial x_0 = 0$):

$$\frac{\partial \mathfrak{G}^{\alpha k}}{\partial x_k}\,\frac{\partial^2\varphi(x_0, x_1, x_2, x_3)}{\partial x_\alpha \partial x_0} \doteq 0,$$

where summation from 1 to 3 over the latin index k is understood. Owing to the arbitrary nature of the function $\varphi(x_0, x_1, x_2, x_3)$, this leads to the following four relations

$$\partial \mathfrak{G}^{\alpha k}/\partial x_k = 0 \quad (\alpha = 0, 1, 2, 3), \tag{19}$$

Relations (19) represent the coordinate conditions preferred for a stationary gravitational field. It also follows from the above that the coordinate conditions (19) and the family of infinitesimal coordinate transformations, defined by Eqs. (13) and (18), correspond to each other when $\partial g_{\mu\nu}/\partial x_0 = 0$.

The following conditions, which are a special case of the coordinate conditions (19),

$$g^{0i} = 0, \quad \partial \mathfrak{G}^{ik}/\partial x_k = 0 \quad (i = 1, 2, 3),$$

were used by Fock[1] to obtain spherically symmetric solutions of the Einstein gravitation equations in the static case.

However one should keep in mind that conditions (19) do not exhaust the class of all possible coordinate conditions preferred for a stationary gravitational field. For instance, the coordinate conditions used by Schwarzchild[6] in obtaining a spherically symmetric solution of Einstein's equations are preferred for a stationary gravitational field and do not coincide with conditions (19).

We will say no more about the difference between the Schwarzchild and Fock solutions; this matter is discussed in the book by Fock.[1]

The well known harmonic coordinate conditions

$$\partial \mathfrak{G}^{\alpha\beta}/\partial x_\beta = 0 \quad (\alpha = 0, 1, 2, 3), \tag{20}$$

widely used by Fock in his work on the astronomical problem of an isolated mass system also belong to the class of coordinate conditions preferred for a stationary gravitational field.

The harmonic coordinate conditions (20) coincide with the conditions (19) under the assumed independence of the components of the fundamental tensor $g_{\mu\nu}$ on the time coordinate x_0; if, however, $\partial g_{\mu\nu}/\partial x_0 \neq 0$ then the conditions (19) and (20) differ substantially. In this connection, it is of interest to find the family of coordinate transformations corresponding to the conditions (20).

5. HARMONIC COORDINATE CONDITIONS

To find the family of coordinate transformations corresponding to harmonic coordinate conditions we put in Eq. (13)

$$a^{\beta}(x_0, x_1, x_2, x_3) = w(x_0, x_1, x_2, x_3)\,x_\beta + \psi^{\beta}(x_0, x_1, x_2, x_3) \tag{21}$$

where $w(x_0, x_1, x_2, x_3)$ is a certain (twice differentiable) function and $\psi^{\beta}(x_0, x_1, x_2, x_3)$ is an arbitrary linear function of the coordinates. In this case [see Eq. (15)]

$$a^{\beta}_{\mu\nu} = w_{\mu\nu}x_\beta + w_\mu\delta_{\nu\beta} + w_\nu\delta_{\mu\beta},$$

where we have used the abbreviations

$$w_\mu = \partial w/\partial x_\mu \quad w_{\mu\nu} = \partial^2 w/\partial x_\mu \partial x_\nu.$$

Consequently, the following is a necessary and sufficient condition [see Eq. (17)] for the field Lagrangian to be a relative invariant under the family of infinitesimal coordinate transformations defined by Eqs. (13) and (21):

$$3w_\mu \partial \mathfrak{G}^{\mu\beta}/\partial x_\beta + x_\mu w_{\mu\nu}\partial \mathfrak{G}^{\nu\beta}/\partial x_\beta - x_\beta w_{\mu\nu}\partial \mathfrak{G}^{\mu\nu}/\partial x_\beta = 0. \tag{22}$$

Now suppose that $w(x_0, x_1, x_2, x_3)$ in Eq. (21) is an arbitrary linear function, then w^{μ} are some arbitrary constants and $w_{\mu\nu} = 0$. Under these circumstances Eq. (22) provides us directly with the harmonic coordinate conditions of interest.

The harmonic coordinate conditions thus represent the necessary and sufficient conditions for the field Lagrangian to be a relative invariant under the family of all transformations

$$x'_\beta = (1 + \varepsilon w)\, x_\beta + \varepsilon \psi^\beta \quad (\beta = 0,\ 1,\ 2,\ 3), \tag{23}$$

where w and ψ^β are some arbitrary linear functions of the coordinates.

If we take into account that the transformations (23) are the infinitesimal analogue of fractional linear coordinate transformations with the same denominator,[1,7] we conclude that in coordinate systems defined by the harmonic coordinate conditions rectilinear and uniform motion in any one of them always corresponds to the same kind of motion in any other. This fundamental property of harmonic coordinate conditions, together with the fact they are preferred for a stationary gravitational field, explains the privileged character of the corresponding coordinate systems in the astronomical problem of an isolated mass system.[1]

We note in conclusion that the harmonic coordinate conditions can be written as follows with the help of the field Lagrangian:

$$g_{\mu\nu}\left(\frac{\partial \mathscr{L}}{\partial\,(\partial g_{\mu\nu}/\partial x_\alpha)} + \frac{\partial \mathscr{L}}{\partial\,(\partial g_{\mu\alpha}/\partial x_\nu)}\right) = 0. \tag{24}$$

[1] V. A. Fock, Теория пространства, времени и тяготения (Theory of Space, Time and Gravitation), GTTI, 1955.

[2] L. D. Landau and E. M. Lifshitz, Теория поля (Field Theory), GTTI, 1948 [Engl. Transl. Addison-Wesley, 1951].

[3] E. Noether, Nachr. kgl. Ges. Wiss. Göttinger, 235 (1918).

[4] E. L. Hill, Revs. Modern Phys. **23**, 253 (1951).

[5] R. Courant and D. Hilbert, Methods of Mathematical Physics (Russ. Transl.) **1**, GTTI, 1933).

[6] K. Schwarzchild, Sitzber, preuss. Akad. Wiss. **7**, 189 (1916).

[7] H. Weyl, Mathematische Analyse des Raumproblems, Berlin, Springer, 1923.

Translated by A. M. Bincer
307