ASYMPTOTIC THEORY OF A ONE-DIMENSIONAL FOUR-FERMION INTERACTION

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We study a fermion field interacting with itself, in a world with one space and one time dimension. Asymptotic expressions are obtained for the vertex part and Green's function by summing an infinite series of graphs. The theory is free of divergences, and there is no charge-renormalization. It turns out that the single-limit technique gives the correct form of the leading term in the asymptotic expansion of the exact solution, but the double-limit technique gives an incorrect result. The theory is compared with the exact solution of Thirring.

1. INTRODUCTION

LANDAU, Abrikosov, and Khalatnikov[1] solved the equations of quantum electrodynamics in the asymptotic region $p^2 \gg m^2$, where p is the momentum 4-vector of a particle and m is its mass. They obtained the following relation between the bare charge e_0 and the renormalized charge e_c:

$$e_c^2 = e_0^2 / [1 + (e_0^2/3\pi)L], \tag{1}$$

where $L = \ln(\Lambda^2/m^2)$ and Λ is the cut-off momentum. If we pass to the limit $L \to \infty$ in Eq. (1), then, no matter how e_0 varies with L, e_c tends to zero and consequently the interaction disappears. This happens provided only that $e_0^2 > 0$, that is to say, if e_0 is real. Also, the renormalized photon Green's function $d_c(\xi)$, $\xi = \ln(p^2/m^2)$, is given by

$$d_c(\xi) = [1 - (e_c^2/3\pi)\xi]^{-1} \tag{2}$$

and has a pole at $p^2 = m^2 \exp(3\pi \cdot 137)$. This contradicts the well-known theorem of Lehmann.[2] The appearance of the plus-sign in the denominator of Eq. (1) produces these unpleasant effects, and is a reflection of the fact, proved by Lehmann, that the renormalization constant Z_3^{-1} (in this case equal to $1 + (e_0^2/3\pi)L$) must exceed unity.*

The relation (1) between the bare and renormalized charge was obtained on the assumption of weak coupling, $e_0^2 \ll 1$. The double-limit technique proves the correctness of Eq. (1) for any value of e_0^2, if we pass to the limit of a point interaction in a suitable way.[7] The question then arises, whether the final result depends on the method by which one passes to the limit. In the present paper we consider a simple model of a one-dimensional four-fermion interaction. This example shows that the double-limit technique can give a completely different result from the single-limit technique. The application of the double-limit technique to this problem leads to a violation of the Pauli principle in the cut-off theory before passing to the limit.

Thirring[8] has recently obtained an exact solution to the problem of a one-dimensional four-fermion interaction. It is interesting to compare his result with the results of methods which have been developed in the study of other types of field theory. We shall find that the single-limit technique leads to the correct expression for the leading term in the expansion of the vertex part in a series of the form*

$$\alpha_0(\xi) = f_0[g_0(L-\xi)] + g_0^2 f_2[g_0(L-\xi)] + g_0^4 f_4[g_0(L-\xi)] + \ldots, \tag{3}$$

but the two-limit technique gives an incorrect result. We leave open the question, whether this fact is a consequence of the violation of the Pauli principle in the cut-off theory, or whether it indicates a general deficiency of the two-limit technique.

The study of this example is interesting for another reason. We have here a completely different relation between renormalized and bare charge

*Abrikosov, Galanin, and Khalatnikov[3] obtained a result similar to Eq. (1) for pseudoscalar meson theory. If we give up the requirement that e_0^2 be positive, so that the Hamiltonian becomes non-Hermitian, we are forced to introduce an indefinite metric. This problem has been studied in detail for the example of the Lee model[4] by Källèn and Pauli[5] and by Heisenberg.[6]

*Terms of the form $g_0 f_1[g_0(L-\xi)]$, $g_0^3 f_3[g_0(L-\xi)$, etc. are absent. This can be seen from a detailed study of the graphs, or from the exact solution of Thirring.[8]

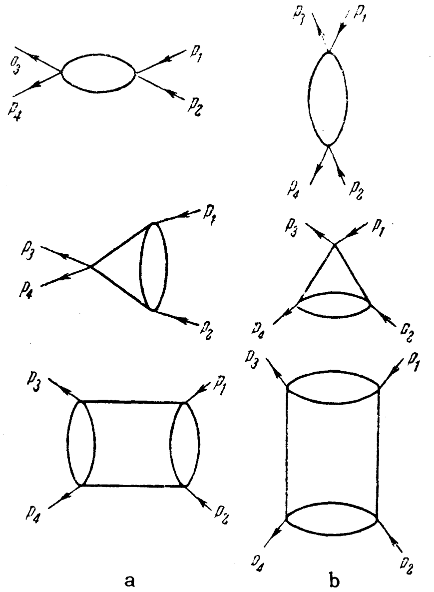

FIG. 1

from that which arises in the interaction of a fermion with a boson field. Examination of the structure of the graphs shows that the insertion of an additional vertex produces one extra factor g_0 and two fermion Green's functions. This means that the quantity which is invariant under renormalization is now $g_0\alpha_0(\xi)\,\beta_0^2(\xi)$, where α_0 is the vertex part and β_0 is the Green's function. The series for the vertex part has the form of Eq. (3), where the leading term contains the product of L with the first power of g_0. The renormalized charge in this theory is presumably determined by the behavior of the vertex part, since the leading terms in the expansion of β_0 have the form $(g_0^2 L)^n$. Therefore the relation between renormalized and bare charge must have the form

$$g_c = g_0/(1 \pm a g_0 L) \qquad (4)$$

where g_0 is positive, and a is a positive number of the order of unity.

If the sign is plus in the denominator of Eq. (4), we meet with the same difficulties as before, but if the sign is minus the renormalized charge can be different from zero. Equation (4) is obtained from the double-limit technique, but only after throwing away a certain number of divergent graphs, which in the single-limit technique are cancelled by diagrams which are retained in Eq. (4). In fact, thanks to this cancellation, the theory does not contain any real divergences. Thus Eq. (4) is incorrect, and the problem of a zero charge does not arise. The correct form of the series (3) is

$$\alpha_0(\xi') = f_0(g_0\xi') + g_0^2 f_2(g_0\xi') + g_0^4 f_4(g_0\xi') + \ldots, \qquad (3')$$

Here $\xi' = \ln(p_{\rm in}^2/p_{\rm out}^2)$, and $p_{\rm in}$ and $p_{\rm out}$ are

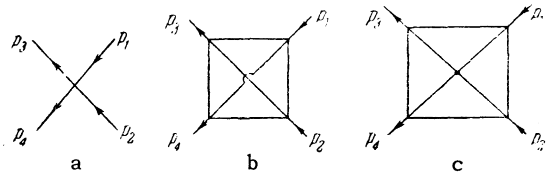

FIG. 2

the momenta of the incoming and outgoing particles in the vertex part.

All these results are confirmed by the calculation of Thirring.[8]

2. EXPRESSION FOR THE INTERACTION ENERGY

We consider a fermion field $\psi_\alpha(x)$, depending on one space and one time coordinate $x = (x_1, x_0)$. The operator $\psi_\alpha(x)$ satisfies a Dirac equation, in which there are two anticommuting matrices γ_μ ($\mu = 1, 4$), representable with two rows and columns.* For definiteness we shall write

$$\gamma_1 = \sigma_x,\ \gamma_4 = \sigma_z,\ \gamma_5 = i\gamma_1\gamma_4 = \sigma_y \qquad (5)$$

where σ are the Pauli matrices.

Instead of the usual five types of four-fermion interaction we now have three, scalar, pseudoscalar and vector. The most general form of the Hamiltonian is

$$H = c_S(\bar\psi\psi) + c_P(\bar\psi\sigma_y\psi)(\bar\psi\sigma_y\psi) + c_V(\bar\psi\sigma_\mu\psi)(\bar\psi\sigma_\mu\psi)$$
$$= (g_0/4)\,Q_{\alpha\beta\gamma\delta}\bar\psi_\alpha\bar\psi_\beta\psi_\gamma\psi_\delta, \qquad (6)$$
$$\bar\psi = \psi^*\gamma_4 = \psi^*\sigma_z, \quad \mu = (x, z).$$

Since the operators ψ_γ, ψ_δ, $\bar\psi_\alpha$, $\bar\psi_\beta$ in Eq. (6) anticommute, we must have

$$Q_{\alpha\beta\gamma\delta} = -Q_{\beta\alpha\gamma\delta} = -Q_{\alpha\beta\delta\gamma}. \qquad (7)$$

Therefore

$$-c_S = c_P = c_V \equiv g_0/4, \qquad (8)$$

which shows that in the one-dimensional case there exists only the single antisymmetric interaction: (−S + P + V).

3. DETERMINATION OF THE VERTEX PART IN THE CASE WHEN ALL EXTERNAL MOMENTA ARE OF THE SAME ORDER

The vertex part $\Gamma(p_4\xi_4, p_3\xi_3, p_2\xi_2, p_1\xi_1)$ is the sum of all diagrams with two incoming and two outgoing lines (Figs. 1 and 2). We divide the graphs into two classes. The first class contains graphs in

*Although the field is quantized with Fermi statistics, it does not posses spin. The spin operator is connected with rotations in three dimensions and therefore cannot exist in this example. The two values of the index α in the operator ψ_α correspond to the two signs of the particle energy.

which some pair of external lines can be separated from the remaining pair by a cut crossing two lines (Fig. 1). The second class contains graphs in which such a cut is impossible (Fig. 2). Graphs of the second class can be considered as units out of which graphs of the first class are constructed by joining together the outgoing lines from one graph with the ingoing graph of another. Figure 1 shows graphs of the first class in which one unit consists of the simplest graph of the second class, a single point (Fig. 2a). Each point can in general be replaced by any graph of the second class. To find out which terms in the expansion (3) of the vertex part contain contributions from a given graph, it is sufficient to study the units from which the graph is composed. For example all graphs whose units are points give contributions* $[g_0(L-\xi)]^n$. If one unit is an "unsealed envelope" (Fig. 2b), the contribution is of order $g_0^2[g_0(L-\xi)]^n$, and so on. To determine the leading term of the series (3) in the approximation $g_0 \ll 1$, we need to sum the graphs of the first class whose units are points, which are those illustrated in Fig. 1.

We denote by $f(p_4\xi_4, p_3\xi_3, p_2\xi_2, p_1\xi_1)$ the sum of those diagrams in which the incoming momenta (p_1, p_2) can be separated from the outgoing momenta (p_3, p_4) as described above (for example, the "lying bricks" in Fig. 1a). We denote by $\varphi(p_4\xi_4, p_3\xi_3, p_2\xi_2, p_1\xi_1)$ the sum of all graphs in which (p_1, p_3) can be separated from (p_2, p_4) (for example the "standing bricks" in Fig. 1b). In this approximation the vertex part becomes

$$\Gamma(p_4\xi_4, p_3\xi_3, p_2\xi_2, p_1\xi_1) = Q_{\xi_4\xi_3\xi_2\xi_1} + f(p_4\xi_1, p_3\xi_3, p_2\xi_2, p_1\xi_1) + \varphi(p_4\xi_4, p_3\xi_3, p_2\xi_2, p_1\xi_1) - \varphi(p_4\xi_4, p_3\xi_3, p_1\xi_1, p_2\xi_2). \quad (9)$$

If all the momenta p_4, p_3, p_2, p_1 are of the same order $(\sim p)$, then because of the logarithmic structure of the theory Γ must have the form:

$$\Gamma(p_4\xi_4, p_3\xi_3, p_2\xi_2, p_1\xi_1) = Q_{\xi_4\xi_3\xi_2\xi_1}\alpha(\xi),$$
$$f(p_4\xi_4, p_3\xi_3, p_2\xi_2, p_1\xi_1) = Q_{\xi_4\xi_3\xi_2\xi_1}f(\xi),$$
$$\varphi(p_4\xi_4, p_3\xi_3, p_2\xi_2, p_1\xi_1) - \varphi(p_4\xi_4, p_3\xi_3, p_1\xi_1, p_2\xi_2) = Q_{\xi_4\xi_3\xi_2\xi_1}\varphi(\xi), \quad (10)$$
$$\alpha(\xi) = 1 + f(\xi) + \varphi(\xi), \quad \xi = \ln(p^2/m^2).$$

Using the method by which Diatlov, Sudakov, and Ter-Martirosian[9] determined the meson-meson scattering amplitude, it is easy to write down integral equations for the functions f, φ, and α. No new difficulties of principle arise. We therefore omit the algebra and state the result

$$f(\xi) = -\frac{g_0}{2\pi}\int_\xi^L \alpha^2(z)\beta^2(z)\,dz, \quad \varphi(\xi) = \frac{g_0}{2\pi}\int_\xi^L \alpha^2(z)\beta^2(z)\,dz, \quad (11)$$

where $(\beta/i\hat{p})$ is the fermion Green's function. The meaning of these equations is that both the functions f and φ are composed of two vertex parts joined by two lines. The numerical coefficients in front of the integrals are most easily determined by first-order perturbation theory. Equation (11) implies

$$\alpha(\xi) = 1, \quad (12)$$

and if also $\beta = 1$, then

$$f(\xi) = -\frac{g_0}{2\pi}(L-\xi), \quad \varphi(\xi) = \frac{g_0}{2\pi}(L-\xi). \quad (13)$$

The result of this analysis is that each "lying brick" cancels in the asymptotic region against a similar "standing brick." Consequently, the vertex part is unchanged by the interaction, and the "bricks" themselves are equal to their lowest-order perturbation-theory approximations.

4. DETERMINATION OF THE GREEN'S FUNCTION

The main result of the preceding section, that $\alpha(\xi) = 1$, did not depend on the form of the fermion Green's function. This fact arose from the special cancellation of graphs which we discussed; otherwise we might have expected that $\alpha(\xi)$ would contain terms proportional to $[g_0(L-\xi)]^n$. The Green's function might have been assumed to be free, since the corrections to it are at least of order $g_0^2(L-\xi)$. In that case the connection between renormalized and bare charge would have had the form of Eq. (4). However, it turned out that the vertex part is not renormalized by the interaction, and we must now determine the Green's function. We shall show that in the asymptotic region the expression for the Green's function also coincides with its lowest-order approximation, independently of the value of the vertex part. Thus the whole theory becomes free of divergences, and there is no renormalization of charge. There remains the question of the higher approximations to Γ, for example the contributions from graphs of the type shown in Fig. (2b) and (2c). We shall see later, from the exact solution of Thirring,[8] that these diagrams also contain no divergent parts, and consequently do not give rise to charge renormalization.

We shall derive the Dyson-Schwinger equation for this problem:

$$G^{-1}(p) = G_0^{-1}(p) - \Sigma^*(p). \quad (14)$$

*It is convenient to divide the vertex part by a factor $(2\pi)^2 g_0/i$. Then a single point gives simply the contribution $Q_{\xi_4\xi_3\xi_2\xi_1}$.

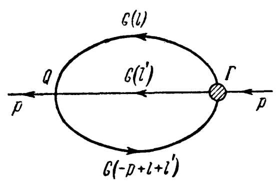

FIG. 3

It is easy to write down an expression for the self-energy operator Σ^*, by considering the graph illustrated in Fig. 3. We find

$$G^{-1}_{\xi\xi'}(p) = (i\hat{p} + m)_{\xi\xi'} - (g_0^2/2)\int Q_{\xi\eta\alpha\gamma}\, G_{\alpha\beta}(l)\, G_{\gamma\delta}(l') \quad (15)$$
$$\times \Gamma(l\beta, l'\delta, p\xi', -p + l + l'\varepsilon)\, G_{\varepsilon\eta}(-p + l + l')\, \frac{d^2 l}{(2\pi)^2}\frac{d^2 l'}{(2\pi)^2}.$$

In the region $p^2 \gg m^2$, we can write

$$G(p) = \beta(\xi)/i\hat{p}, \quad \xi = \ln(p^2/m^2),$$
$$\Gamma(p_4\xi_4, p_3\xi_3, p_2\xi_2, p_1\xi_1) = Q_{\xi_4\xi_3\xi_2\xi_1}\,\alpha(\xi, \eta, \zeta). \quad (16)$$

Here ξ, η, and ζ are the logarithms of the three independent momenta on which Γ depends. The integral (15) is formally linearly divergent. After integrating over the angles, only a logarithmic divergence remains. We may therefore make the replacement

$$G(-p + l + l') \to -i\beta \frac{1}{\hat{l} + \hat{l}'}\, \hat{p}\, \frac{1}{\hat{l} + \hat{l}' - \hat{p}}. \quad (17)$$

After this we have on the right of Eq. (15) an integral of the form

$$\int \frac{l_\mu l'_\lambda (l + l')_\sigma (l + l' - p)_\rho\, d^2l d^2l'}{l^2 l'^2 (l + l')^2 (l + l' - p)^2} = A\delta_{\mu\lambda}\delta_{\sigma\rho} + B(\delta_{\mu\sigma}\delta_{\lambda\rho} + \delta_{\mu\rho}\delta_{\lambda\sigma}), \quad (18)$$

where A and B are some invariant quantities. This integral is multiplied by the following combination of matrices:

$$Q_{\xi\eta\alpha\gamma}(\gamma_\mu)_{\alpha\beta}(\gamma_\lambda)_{\gamma\delta}\, Q_{\beta\delta\xi'\varepsilon}(\gamma_\sigma\hat{p}\gamma_\rho)_{\varepsilon\eta}. \quad (19)$$

A simple calculation shows that the product of (18) and (19) is equal to zero. Therefore in the asymptotic region

$$\beta = 1, \quad G(p) = G_0(p) = 1/i\hat{p}. \quad (20)$$

If the spinor combination in front of the integral had not been zero, then with a free vertex part we would have obtained the following expression for β:

$$\beta = [1 + a g_0^2 (L - \xi)]^{-1/4}. \quad (20)$$

Here a is a number of the order of unity (which happens to be zero). The relation between bare and renormalized charge would have had the form of Eq. (1).

5. CALCULATION OF THE VERTEX PART WHEN THE EXTERNAL MOMENTA HAVE DIFFERENT ORDERS OF MAGNITUDE

To compare our theory with the result of Thirring, it is useful to calculate the vertex part for the case in which the incident momenta (p_1, p_2) and the outgoing momenta (p_3, p_4) are of different orders of magnitude. Following the method of Diatlov et al.,[9] we assume that in the asymptotic region the "lying brick" depends on the logarithms of three momenta:

$$f(p_4\xi_4, p_3\xi_3, p_2\xi_2, p_1\xi_1) = f(\xi, \eta, \zeta; \xi_4, \xi_3, \xi_2, \xi_1),$$
$$\xi = \ln\frac{p^2_{\text{out}}}{m^2}, \quad \eta = \ln\frac{(p_1 + p_2)^2}{m^2} = \ln\frac{(p_3 + p_4)^2}{m^2}, \quad \zeta = \ln\frac{p^3_{\text{in}}}{m^2} \quad (21)$$
$$p^2_{\text{out}} = \max\{p_3^2, p_4^2\}, \quad p^2_{\text{in}} = \max\{p_1^2, p_2^2\}.$$

In the most general case

$$\xi > \zeta > \eta, \quad p^2_{\text{out}} \gg p^2_{\text{in}} \gg (p_1 + p_2)^2. \quad (22)$$

In the same way the "standing brick" gives

$$\varphi(p_4\xi_4, p_3\xi_3, p_2\xi_2, p_1\xi_1) = \varphi(\xi', \eta', \zeta'; \xi_4, \xi_3, \xi_2, \xi_1), \quad (23)$$
$$\xi' = \ln\frac{p^2_{\text{I}}}{m^2}, \quad \eta' = \ln\frac{(p_4 - p_2)^2}{m^2} = \ln\frac{(p_3 - p_1)^2}{m^2}, \quad \zeta' = \ln\frac{p^2_{\text{II}}}{m^2},$$
$$p^2_{\text{I}} = \max\{p_4^2, p_2^2\}, \quad p^2_{\text{II}} = \max\{p_3^2, p_1^2\}.$$

The condition (22) is the most general case for f, but is a special case for φ. When Eq. (22) holds,

$$\xi' = \eta' = \zeta' = \xi. \quad (24)$$

We calculate the vertex part assuming these conditions to be satisfied. We find

$$\Gamma(p_4\xi_4, p_3\xi_3, p_2\xi_2, p_1\xi_1) = Q_{\xi_4\xi_3\xi_2\xi_1}\,\alpha(\xi, \eta, \zeta),$$
$$f(p_4\xi_4, p_3\xi_3, p_2\xi_2, p_1\xi_1) = Q_{\xi_4\xi_3\xi_2\xi_1}\, f(\xi, \eta, \zeta),$$
$$\varphi(p_4\xi_4, p_3\xi_3, p_2\xi_2, p_1\xi_1) - \varphi(p_4\xi_4, p_3\xi_3, p_1\xi_1, p_2\xi_2) \quad (25)$$
$$= Q_{\xi_4\xi_3\xi_2\xi_1}\,\varphi(\xi),$$
$$\alpha(\xi, \eta, \zeta) = 1 + f(\xi, \eta, \zeta) + \varphi(\xi).$$

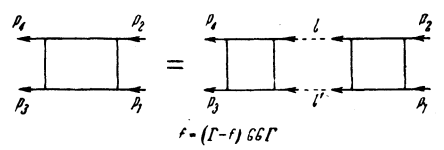

FIG. 4

An examination of Fig. 4 leads to the following integral equation:

$$f(p_4\xi_4, p_3\xi_3, p_2\xi_2, p_1\xi_1)$$
$$= (ig_0/2)\int[Q_{\xi_4\xi_2\alpha\gamma} + \varphi(p_4\xi_4, p_3\xi_3, l\alpha, l'\gamma) - \varphi(p_4\xi_4, p_3\xi_3, l'\gamma, l\alpha)] \quad (26)$$
$$\times G_{\alpha\beta}(l)G_{\gamma\delta}(l')\Gamma(l\beta, l'\delta, p_2\xi_2, p_1\xi_1)d^2l/(2\pi)^2,$$
$$l' = p_1 + p_2 - l.$$

This equation is derived by dividing the "lying brick" in such a way that on the left side there is only a point and "standing bricks". A similar equation can be written down for the "standing brick," but it is not essential for what follows. We eliminate the spinor indices from Eq. (26), transform the integration to euclidean space, and change to logarithmic variables, with the result

$$f(\xi, \eta, \zeta) = -(g_0/2\pi)\int_\eta^\zeta[1+\varphi(\xi)][1+f(z, \eta, \zeta)+\varphi(\zeta)]\,dz$$
$$-(g_0/2\pi)\int_\zeta^\xi[1+\varphi(\xi)][1+f(z, \eta, \zeta)+\varphi(z)]\,dz \quad (27)$$
$$-(g_0/2\pi)\int_\xi^L[1+\varphi(z)][1+f(z, \eta, \zeta)+\varphi(z)]\,dz.$$

We have set $\beta = 1$ in accordance with the results of section 3. When $\eta = \zeta$, we obtain an equation for the function $f(\xi, \eta) = f(\xi, \eta, \eta)$. We use Eq. (13) for φ. It is convenient to change to the new variables

$$x = 1+(g_0/2\pi)(L-\xi),\ y = 1+(g_0/2\pi)(L-\eta),\ t = 1+(g_0/2\pi)(L-z), \quad (28)$$

in terms of which the integral equation becomes

$$f(x, y) = \frac{1}{6}x^3 - \frac{1}{2}xy^2 + \frac{1}{3} + x\int_y^x f(t, y)\,dt + \int_x^1 tf(t, y)\,dt. \quad (29)$$

Differentiating Eq. (29) twice with respect to x, we find

$$d^2f(x, y)/dx^2 = x + f(x, y), \quad (30)$$

which has the solution

$$f(x, y) = C_1(y)e^x + C_2(y)e^{-x} - x. \quad (31)$$

The initial condition

$$(df(x, y)/dx)_{x=y} = 0, \quad (32)$$

gives

$$C_2(y) = C_1(y)e^{2y} - e^y. \quad (33)$$

Substituting Eq. (33) back into Eq. (29), we find $C_1(y) = e^{-y}$. Thus our final result is

$$f(x, y) = e^{x-y} - x,\ f(\xi, \eta) = \exp\{(g_0/2\pi)(\eta-\xi)\} - 1 - (g_0/2\pi)(L-\xi). \quad (34)$$

When $\xi = \eta$, this expression reduces to Eq. (13). The vertex part then becomes

$$\alpha(\xi, \eta) = \exp\{(g_0/2\pi)(\eta-\xi)\} = (p_{in}/p_{out})^{g_0/\pi}. \quad (35)$$

Equation (12) shows that this expression is valid not only when $p_{in}^2 \gg p_{out}^2$, but also when $p_{in}^2 \approx p_{out}^2$. It is easy to prove that it is valid in general for any values of p_{in}^2 and p_{out}^2. For this purpose one uses not Eq. (26), but the analogous equation obtained by cutting the "lying brick" in such a way that only a point and a "standing brick" is left on the right side.

6. THE DOUBLE-LIMIT TECHNIQUE

In this section we shall derive an expression for the vertex part, using a special version of the double-limit technique, which was recently used[10] for proving the vanishing of the charge in a real three-dimensional theory with a four-fermion interaction.

Consider any vertex at which the operator

$$H = (g_0/4)(\bar\psi_\alpha(O_j)_{\alpha\gamma}\psi_\gamma)(\bar\psi_\beta(O_j)_{\beta\delta}\psi_\delta)$$

annihilates two particles with momenta p_1p_2 and creates a pair of particles with momenta p_3p_4. Obviously, the particle p_1 may be annihilated either by ψ_γ or by ψ_δ, and the particle p_3 may be created either by $\bar\psi_\alpha$ or by $\bar\psi_\beta$, and so on. In a local theory, the result of adding together these four possibilities is only to make the contribution of a simple vertex diagram equal to $g_0O_j \times O_j$ instead of $(g_0/4)O_j \times O_j$. This simple result is connected with the indentity of the particles, since the operator ψ_δ can be paired either with $\bar\psi_\alpha$ or with $\bar\psi_\beta$, etc. ($Q_{\alpha\beta\gamma\delta} = -Q_{\alpha\beta\delta\gamma} = \ldots\ldots$).

In a cut-off theory, we have to destroy the identity of particles, and distinguish cases in which p_1 and p_3 are annihilated and created in the same pair or in different pairs. Likewise also for p_2 and p_4. To represent these two possibilities, we draw the vertex either in the way shown in Fig. 5a or as in Fig. 5b. We cut off the divergent integrals over momentum variables which follow along the fermion lines at the momentum Λ, while the integrals over momenta which cross from line to line are cut off at the momentum λ.

We suppose that the limiting process is carried out with $\lambda/\Lambda \ll 1$. It has been shown[10] that in this case, of all the graphs which contribute to the vertex part, only those illustrated in Fig. 6 remain.

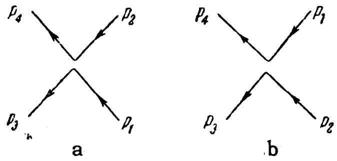

FIG. 5

Also the Green's function can be considered to be free ($\beta = 1$). The graphs of Fig. 6 are easily summed, using Eq. (11), and we thus obtain

$$\alpha(\xi) = \frac{1}{1-(g_0/4\pi)(L-\xi)}, \quad g_c = \frac{g_0}{1-(g_0/4\pi)L}, \quad L = \ln\frac{\Lambda^2}{m^2}. \tag{36}$$

If $g_0 < 0$, the renormalized charge g_c tends to zero, no matter how g_0 varies with Λ and λ. Thus the double-limit technique leads to a zero charge in a theory completely free from divergences. The whole picture can be seen with unusual clarity from this example.

By using a special form of limiting process with $\Lambda \to \infty$, $\lambda \to \infty$, $\lambda/\Lambda \to 0$, we have thrown away a set of graphs (a part of the "standing brick") which in the local theory cancel against the remaining graphs. In this way we artificially introduced divergences into a non-divergent theory.

From these results we cannot decide the question, whether the double-limit technique is generally useless, or whether the trouble could be avoided by introducing the cut-off in such a way as not to violate general principles such as the Pauli principle, gauge invariance, etc. At least one can say that the double-limit technique might lead to incorrect results also in the three-dimensional case,[10] where one is dealing with a field interacting with itself or with several fields interacting in an antisymmetric combination ($A - V$, $S + P - T$, $2(S - P) - (A + V)$).

7. CONCLUSIONS

Thirring[8] considered the problem of a field with the interaction Hamiltonian given by Eqs. (6) and (8), but instead of studying the asymptotic region $p^2 \gg m^2$ he omitted the mass term from the equations from the beginning. He obtained an exact solution of the problem in the Schrödinger representation, without using the condition $g_0 \ll 1$. He obtained an expression for the matrix element of a Schrödinger operator between the physical vacuum and a physical three-particle state:

$$\langle 3|\psi_1(x)|0\rangle = \frac{\exp\{-i(p_b-p_c+k_m)x\}}{L^{3/2}(p_c-p_b)}\,\frac{1}{\sqrt{1+\lambda^2}}\left(\frac{p_b}{|p_c|}\right)^{(\arctan\lambda)/\pi} \tag{37}$$

Here p_b, p_c and k_m are the momenta of the par-

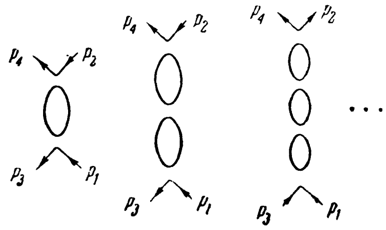

FIG. 6

ticles in the state $\langle 3|$, L is the normalization volume, and λ is the coupling constant (g_0 in our notations). The matrix element (37) has a similar meaning to the vertex part which we studied earlier. The fact that $\psi_1(x)$ is a Schrödinger operator is unimportant, since it can easily be transformed into a Heisenberg operator $\psi_1(x) = e^{-iHt}\psi_1(x)e^{iHt}$, the only effect being to multiply the matrix element by a factor e^{-iE_3t} where E_3 is the energy of the three-particle state. The quantity $(1+\lambda^2)^{-1/2} \times (p_b/|p_c|)^{(\arctan\lambda)/\pi}$, which tends to unity as $\lambda \to 0$, can be expanded in a series of the form (3′):

$$\frac{1}{\sqrt{1+\lambda^2}}\left(\frac{p_b}{|p_c|}\right)^{(\arctan\lambda)/\pi} = e^{\lambda\xi'/2\pi} - \lambda^2 e^{\lambda\xi'/2\pi}\left[\frac{1}{2} + \frac{1}{6\pi}(\lambda\xi')\right] + \lambda^4 e^{\lambda\xi'/2\pi}\left[\frac{3}{8} + \frac{11}{60\pi}(\lambda\xi') + \frac{1}{72\pi^2}(\lambda\xi')^2\right] + \cdots, \quad \xi' = \ln(p_l^2/p_c^2). \tag{38}$$

The first term of this series coincides with Eq. (35). The general structure of the expression confirms the correctness of our assumption that the higher approximations to the vertex part do not contain divergences and consequently do not lead to any charge renormalization.

It is a very curious fact, that the double-limit technique leads to an incorrect result for the renormalized vertex part when the charge has one sign, and leads to a zero charge when the charge has the other sign. One may suppose that this is all the result of using a cut-off which destroys the symmetry between identical particles. An example of a similar kind* is provided by electrodynamics with all vacuum-polarization effects omitted. In this case all the divergences cancel by virtue of Ward's identity $Z_1 = Z_2$. In this case also, if one uses a cut-off which is not gauge invariant, one can obtain a renormalized charge equal to zero.

After this work was finished, we received a preprint of a paper by M. E. Maier and D. V. Shirkov, in which the method of the renormalization group led to a result similar to Eq. (35). This is not surprising, because the single-limit method is essentially identical with ordinary renormaization.

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