

ON THE THEORY OF SUPERCONDUCTING ALLOYS

I. THE ELECTRODYNAMICS OF ALLOYS AT ABSOLUTE ZERO

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In this paper we give the theory of superconductors containing impurities at the absolute zero. The dependence of the penetration depth on the impurity concentration is considered for small atomic concentrations. We have obtained the electrodynamic equations in an alternating field for superconductors with a mean free path which is smaller than the correlation length.

It is well known that Bardeen, Cooper, and Schrieffer¹ succeeded in formulating the electrodynamics of superconductors with the aid of their theory of superconductivity. They obtained equations connecting the current with the vector potential which replaced the old equations of the phenomenological theory of H. and F. London. Of most interest was the non-local character of the connection between the current and the field for the majority of pure superconductors.

The non-local character is put into the very basis of the theory which is based on the physical idea of Cooper's² about the formation of bound singlet pairs of electrons near the Fermi surface under the action of the phonon interaction. The dimensions of such a pair causes the existence of a correlation between electrons at distances of the order $\xi_0 \sim 10^{-4}$, which becomes apparent in the form of a non-local connection between the field and the current, if the penetration depth of the field is less than ξ_0 . The majority of pure superconductors belong to that class.

The creation of a theory of superconductivity made it possible to consider a whole series of phenomena. With its aid they could construct the thermodynamics and electrodynamics of superconductors¹ and, in particular, determine the behavior of superconductors in a high-frequency field.³ A very interesting problem is that of "alloys," i.e., of superconductors with impurity atoms of other elements or other lattice defects. In the normal state these defects cause the so-called residual resistance.

At very small concentrations the role of impurities is small. Increasing the concentration, however, leads to a decrease of the spatial correlation of the electrons in the superconductor. For a sufficiently concentrated alloy the role of the

correlation parameter goes over from ξ_0 to the electron mean free path. Thanks to this, it is natural to expect the appearance of a local connection between the current and the vector potential for such concentrations where the mean free path becomes less than the penetration depth. The difference with the Londons' theory will lie in a change in the constant of proportionality of \mathbf{j} to \mathbf{A} . Below we shall consider this problem, assuming, however, that the atomic impurity concentration is small.

The problem of the penetration of a weak electromagnetic field into a medium is solved if we know the Green function of the field in the medium

$$D_{ik}(x-x') = \frac{4\pi i}{c} \langle T(A_i(x) A_k(x')) \rangle. \quad (1)$$

The function D_{ik} is directly connected with experimentally observed quantities. Let $D_{ik}(\mathbf{k}, \omega_0)$ be the Fourier transform of $D_{ik}(x-x')$:

$$D_{ik}(x-x') = \frac{4\pi}{(2\pi)^4} \int \exp\{ik \cdot (x-x') - i\omega_0(t-t')\} D_{ik}(\mathbf{k}, \omega_0) d^3\mathbf{k} d\omega_0.$$

In view of the symmetry of the tensor D_{ik} we have

$$D_{ik}(\mathbf{k}, \omega_0) = D_1(\mathbf{k}^2, \omega_0) (\delta_{ik} - k_i k_k / k^2) + D_2(\mathbf{k}^2, \omega_0) k_i k_k / k^2.$$

The function D_2 which determines the longitudinal part of D_{ik} is connected with the gauge of the vector potential and is arbitrary. The physical quantities depend only on the transverse part, i.e., on D_1 .

With the aid of the general theory of the anomalous skin effect we can arrive at the following formula for the surface impedance in the case of dif-

fuse scattering at the surface*

$$Z = \frac{4\pi^2 i \omega_0}{c^2} \int_0^\infty \ln [k^2 D_1(k^2, \omega_0)] d|k|. \quad (2)$$

We recall that the penetration depth of the field is

$$\delta = \text{Re} \left[-\frac{c^2}{4\pi\omega_0} iZ \right]. \quad (3)$$

Usually, instead of finding the function D_{ik} one finds the connection between the current and the vector potential

$$j_i = -Q_{ik}(k, \omega_0) A_k(k, \omega_0). \quad (4)$$

To find the connection between the tensors D_{ik} and Q_{ik} . It is sufficient to substitute (4) into the Maxwell equation and to write down with its aid an equation for D_{ik} . This gives

$$(D^{-1})_{ik} = (k^2 \delta_{ik} - k_i k_k) + \frac{4\pi}{c} Q_{ik}. \quad (5)$$

The first term corresponds to the "zero" value of the tensor D_{ik} , obtained by neglecting the interaction of the field with the electrons. In Eqs. (2) to (5) we have implied $\omega_0 > 0$.

From Eq. (5) it is clear that the tensor Q_{ik} must be symmetric. Also, since the current is conserved ($k_j j_j = 0$) it follows from Eq. (4) that it must be transverse.† Equation (5) determines thus only the transverse part of D_{ik} and its longitudinal part remains arbitrary, as it should be.

We evaluate below the function $D_{ik}(k, \omega_0)$ for a superconductor with impurities, using the techniques of the quantum field theory‡ (see reference 4, which in contradistinction to other methods^{1,5} makes it possible to state the problem very clearly). We assume that we have already taken into account the interaction of the electrons with the impurities and consider their interaction with the electromagnetic field. The Hamiltonian of this interaction is

*The correspondence of this equation with the usual expression is easily established using Eqs. (4) and (5).

†The method of finding the tensor Q_{ik} usually employed in various problems of the theory of superconductivity (see references 1 and 3) leads to an expression which is not purely transverse. One can, however, show by means of a more rigorous calculation based on the equations of reference 4 that to get the correct result for the given problem it is sufficient to retain only the transverse part of the tensor Q_{ik} obtained by the usual method, or to use a vector potential satisfying the condition $\text{div } \mathbf{A} = 0$.

‡We must emphasize that the method used here is suitable only at $T = 0$. In the case where $T \neq 0$ there will occur, instead of simple time-ordered products, more complicated combinations which make this method practically inapplicable to the problem under consideration.

$$H'_1 = \frac{e^2}{mc^2} \psi^\dagger(x) \psi(x) \mathbf{A}^2(x) + \frac{ie\mathbf{A}(x)}{2mc} (\nabla_x - \nabla_{x'}) \psi^\dagger(x') \psi(x). \quad (6)$$

Dyson's equation for the function D will be of the form

$$D^{-1} = D_{(0)}^{-1} + \text{diagram 1} + \text{diagram 2}, \quad (7)$$

where a solid line corresponds to an electron and a wavy one to a photon. More complicated diagrams with internal photon lines can be neglected in the non-relativistic approximation.

Comparing Eqs. (7) and (5) we see that, apart from a factor $4\pi/c$ the term added to $D_{(0)}$ corresponds to the tensor Q_{ik} . The first term of this addition gives simply $(Ne^2/mc) \delta_{ik}$ in Q_{ik} . The second diagram is more complicated. For the following it is convenient to separate, as is usually done, one of the photon vertices, for instance p_i , from the remaining part which, apart from a coefficient, can be considered as the result of including the photon vertex p_k in the electron line. To find the coefficient is not difficult (the factor 2 comes from taking the trace over the spin). In this way we get

$$Q_{ik}(k, \omega_0) = \frac{Ne^2}{mc} \delta_{ik} - \frac{2ie^2}{m^2 c} \int p_i \Pi_k(p_+, p_-) \frac{d^4 p}{(2\pi)^4}, \quad (8)$$

where $p_\pm = (\mathbf{p} \pm \mathbf{k}/2, \omega \pm \omega_0/2)$.

As far as the interaction of the electrons with the impurities is concerned, the corresponding term in the Hamiltonian is of the form

$$H'_2 = \sum_a H_a, \quad H_a = u(\mathbf{x} - \mathbf{x}_a) \psi^\dagger(x) \psi(x), \quad (9)$$

where $u(\mathbf{x} - \mathbf{x}_a)$ is the interaction potential with an impurity atom at the point \mathbf{x}_a . We shall assume that the impurity atoms are randomly distributed in the metal and that their distance apart is large compared to the lattice period. We assume, of course, that the impurity concentration is so small that the interaction between the electrons is not changed. We shall denote the impurity vertex in the Feynman diagrams by a cross. It corresponds to a factor

$$u(\mathbf{q}) e^{i\mathbf{q} \cdot \mathbf{x}_a} \delta(\omega_1 - \omega_2), \quad (10)$$

where $u(\mathbf{q})$ is the Fourier component of the potential $u(\mathbf{x})$ and \mathbf{q} is the momentum transferred.

For the sake of clarity we first give the calculations for the application to a normal metal with a residual resistance (Sec. 1). We then consider superconducting alloys (Sec. 2). All calculations refer only to the absolute zero.

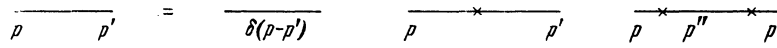


FIG. 1

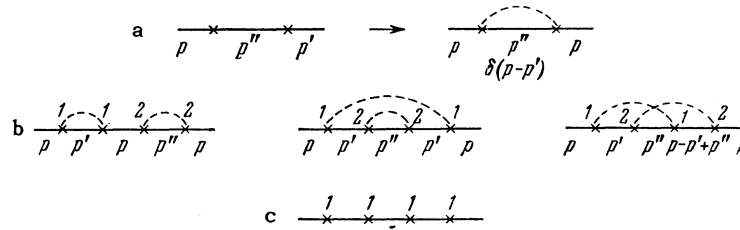


FIG. 2

1. RESIDUAL RESISTANCE OF A NORMAL METAL

We introduce the Green function of a system of fermions

$$G_{\alpha\beta}(x, x') = -i \langle T(\psi_\alpha(x) \psi_\beta^\dagger(x')) \rangle. \tag{11}$$

For a pure metal we have:*

$$G_{0\alpha\beta}(x - x') = \delta_{\alpha\beta} (2\pi)^{-4} \int G_0(p) e^{ip(x-x')} d^4p, \tag{12}$$

$$G_0(p) = (\omega - \xi + i\delta \text{sign } \omega)^{-1},$$

where $\xi = v(|\mathbf{p}| - p_0)$, $v = p_0/m$, $\delta \rightarrow +0$. When impurities are present the Green function (11) is not the same as expression (12). We shall write it in the form

$$G_{\alpha\beta}(x, x') = \delta_{\alpha\beta} (2\pi)^{-7} \int G(\mathbf{p}, \mathbf{p}', \omega) \times \exp \{i\mathbf{p} \cdot \mathbf{x} - i\mathbf{p}' \cdot \mathbf{x}' - i\omega(t - t')\} d^3p d^3p' d\omega. \tag{13}$$

The function $G(\mathbf{p}, \mathbf{p}', \omega)$ is expressed as a sum of the diagrams given in Fig. 1 according to the well-known rules of field theory. Each line corresponds to $G_0(\mathbf{p})$.

Summing over the diagrams gives an integral equation for $G(\mathbf{p}, \mathbf{p}', \omega)$:

$$G(\mathbf{p}, \mathbf{p}', \omega) = \delta(\mathbf{p} - \mathbf{p}') G_0(p) + (2\pi)^{-3} \int G_0(p) \sum_a u(\mathbf{p} - \mathbf{p}'') e^{i(\mathbf{p} - \mathbf{p}'') \cdot \mathbf{x}_a} G(\mathbf{p}'', \mathbf{p}', \omega) d^3p'' \tag{14}$$

We are not interested in the exact solution of (14), since the impurity atoms are randomly distributed over the metal. We must average all expressions over the positions of all impurity atoms. It is then essential that by virtue of our assumption about the smallness of the atomic concentration the average distance apart of the impurity atoms is much larger than the atomic distances in the metallic lattice, so that the averaging can take place over volumes with dimensions large compared to the interatomic distances. After such an averaging, the Green function $G(\mathbf{p}, \mathbf{p}', \omega)$ is clearly of the form

*Here and henceforth we use units with $\hbar = 1$.

$$\overline{G(\mathbf{p}, \mathbf{p}', \omega)} = G(\mathbf{p}, \omega) \delta(\mathbf{p} - \mathbf{p}'). \tag{15}$$

The absolute magnitude of the momenta \mathbf{p}, \mathbf{p}' which are of interest to us are of the order of magnitude of the momentum p_0 at the Fermi surface which in turn is of the order of the inverse of the interatomic distances. This fact makes it at once easy to carry out the averaging.

We shall perform the calculation in the Born approximation, i.e., we shall assume that $p_0^3 \int u(\mathbf{x}) d^3\mathbf{x} \ll E_{ph}$. Once can show that the final results expressed in terms of the collision time will also be correct in the general case.

The simplest diagram for G contains only one cross. The value obtained by averaging over the position of the impurity atom is a constant

$u(\mathbf{q}) e^{i(\mathbf{q} \cdot \mathbf{x}_\alpha)} = u(0)$, which can be included in the ground state energy and which in the following we assume to be equal to zero. The next diagram, as far as complexity is concerned, contains two crosses referring to one atom (Fig. 2a). The average value of these diagrams (without external G_0) is equal to

$$\frac{1}{V} \int |u(\mathbf{p} - \mathbf{p}')|^2 G_0(p') \frac{d^3p'}{(2\pi)^3}, \tag{16}$$

where V is the volume of the system. In the following we shall be interested in values of $|\mathbf{p}|$ which in absolute magnitude are close to p_0 . The integral in Eq. (16) can be split into two parts: an integral over values of $|\mathbf{p}'|$ far from the Fermi surface and one over values near the Fermi surface (the limits of the second integral over $|\mathbf{p}'|$ can be taken symmetric with respect to $|\mathbf{p}'| = p_0$). The integral over the far region gives a real constant which together with $u(0)$ is a renormalization of the chemical potential and need not be considered. One may assume that in the second integral $u(\mathbf{p} - \mathbf{p}')$ is a slowly-varying function. Substituting Eq. (12) for $G_0(p)$ and summing over the impurity atoms (which simply means multiplying by the number of atoms) we get the essen-

tial contribution to the G -function:

$$i \operatorname{sign} \omega / 2\tau,$$

where

$$\frac{1}{\tau} = n \frac{p_0 m}{(2\pi)^2} \int |u(\theta)|^2 d\Omega \quad (17)$$

is the time between collisions, n the number of impurity atoms per unit volume. From this it is clear that the region near the Fermi surface plays the leading role in the integrals.

From this point of view not all diagrams are equivalent. Let us compare, for instance, the three diagrams of Fig. 2b. (Dotted lines connect crosses referring to one atom.) We see easily that in the first two expressions the integration over \mathbf{p}' and \mathbf{p}'' can be performed over arbitrary angles between the momenta near the Fermi surface. On the other hand, in the third integral the requirement that all arguments of the G -functions are near the Fermi surface leads to a limitation of the angles. As a result the contribution from such a graph turns out to be less by a factor $(v p_0 l)^{-1}$ as compared to the others. Since in the following we need values of ω and $\xi \sim 1/\tau$ where τ is the time between collisions, the smallness of the "intersecting" graphs can be estimated to be $(p_0 l)^{-1}$ where $l = v\tau$ is the mean free path.

One can easily show that a small contribution is also given by diagrams containing more than two crosses from one impurity atom. Let us, for instance, compare the total contribution (from all impurity atoms) of diagrams of the kind of the first one of Fig. 2b with diagrams of the type 2c. The first ones give a quantity of the order of magnitude

$$\frac{1}{\tau^2} G_0 \sim \frac{1}{\tau}.$$

The second ones give, as far as order of magnitude is concerned

$$\frac{1}{\tau} \frac{u^2(q) p_0^4}{v^2} \sim \frac{1}{\tau} \left[\frac{\int u dV p_0^3}{E_{ph}} \right]^2 \ll \frac{1}{\tau}.$$

It is thus clear that we need consider only diagrams containing up to two crosses on one impurity atom.

Summing all essential diagrams (i.e., only "pair" diagrams and those which do not contain "intersections" like the third diagram of Fig. 2b) we get the following expression for the G -function:

$$G(p) = G_0(p) + n G_0(p) \int |u(\mathbf{p} - \mathbf{p}')|^2 G(p') \frac{d^3 p'}{(2\pi)^3} G(p). \quad (18)$$

If we do not wish to use the Born approximation, it is necessary to take into account diagrams with many crosses at one impurity atom. One can show

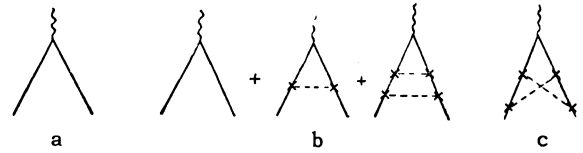


FIG. 3

that all changes which this entails are taken into account by replacing the Born amplitude $u(\theta)$ by the total scattering amplitude. This remains true also for all calculations in the following. We can therefore imply in all formulae by $u(\theta)$ the total scattering amplitude.

The solution of Eq. (18) is

$$G(p) = \frac{1}{\omega - \xi - \bar{G}(\omega)},$$

where $\bar{G}(\omega)$ satisfies the equation

$$\bar{G}(\omega) = n \int |u(\mathbf{p} - \mathbf{p}')|^2 \frac{1}{\omega - \xi - \bar{G}(\omega)} \frac{d^3 p'}{(2\pi)^3}$$

Assuming $\bar{G}(\omega)$ to be purely imaginary ($\bar{G}(\omega) = -i\beta$) and applying to the integral on the right hand side the same approach as was applied to (16) we find

$$\beta = \operatorname{sign} \beta / 2\tau,$$

where τ is determined by Eq. (17). Comparing G which is obtained in this manner with the case of small amounts of impurities ($G \rightarrow G_0$) we find that $\beta = \operatorname{sign} \omega / 2\tau$ or

$$G = (\omega - \xi + i \operatorname{sign} \omega / 2\tau)^{-1}. \quad (19)$$

Going over to the \mathbf{x} -representation we see easily that the whole change in G as compared to G_0 is obtained by multiplying the latter by an exponential screening factor

$$G = G_0 e^{-|\mathbf{x} - \mathbf{x}'| / 2l}, \quad (20)$$

where $l = v\tau$.

We now proceed to evaluate Q_{ijk} . Putting the photon vertex into the electron Green function $G(\mathbf{p}, \mathbf{p}', \omega)$ we get:

$$\begin{aligned} & \Pi(p_+, p_-) \\ &= (2\pi)^{-3} \int G(\mathbf{p}_+, \mathbf{p}', \omega + \frac{\omega_0}{2}) \mathbf{p}' G(\mathbf{p}', \mathbf{p}_-, \omega - \frac{\omega_0}{2}) d^3 p'. \end{aligned} \quad (21)$$

The function $G(\mathbf{p}, \mathbf{p}', \omega)$ entering here corresponds to the sum of the diagrams in Fig. 1 and is described by Eq. (14). To average over the positions of the impurity atoms we must take into account that the average of two Green functions is not equal to the product of the two average values.

For a pure metal expression (21) corresponds to the diagram of Fig. 3a. After averaging over the positions of the impurity atoms, it turns out

that apart from the simple diagrams corresponding to the transition from the zeroth Green functions $G_0(\mathbf{p})$ to the function $G(\mathbf{p})$ of (19) the essential diagrams for the quantity (21) are those of Fig. 3b. The large contribution from these corrections is connected with the fact that the photon momentum occurring in the vertex $|\mathbf{k}| \ll p_0$. Diagrams of another kind, for instance the graph of Fig. 3c, are far smaller since one of the integrations in them is over a region of momenta far from the Fermi surface. The averaging of the quantity (21) is thus performed by the summation of the "ladder" diagrams of Fig. 3b.

The integral equation for $\Pi(\mathbf{p}_+, \mathbf{p}_-)$ is

$$\begin{aligned} & \Pi(\mathbf{p}_+, \mathbf{p}_-) \\ &= G(\mathbf{p}_+) G(\mathbf{p}_-) \left[\mathbf{p} + \frac{n}{(2\pi)^3} \int |u(\mathbf{p} - \mathbf{p}')|^2 \Pi(\mathbf{p}'_+, \mathbf{p}'_-) d^3\mathbf{p}' \right]. \end{aligned} \quad (22)$$

There are two limiting cases possible:

(a) The skin effect is anomalous, $(kv) \gg \tau$; it is easy to verify that the integral on the right hand side of (22) will be negligibly small in that case ($\sim 1/kv\tau$).

(b) The skin effect is normal, $(kv) \ll 1/\tau$; this case is just of the greatest interest to us. In Eq. (22) we can then assume $\mathbf{p}_+ = \mathbf{p}_-$.

The vector obtained from the integral on the right hand side of (22) will clearly be parallel to \mathbf{p} . We introduce the notation

$$\mathbf{p}\Lambda(\omega, \omega_0) = \frac{n}{(2\pi)^3} \int |u(\mathbf{p} - \mathbf{p}')|^2 \Pi(\mathbf{p}'_+, \mathbf{p}'_-) d^3\mathbf{p}'. \quad (23)$$

Since $|\mathbf{p}| \approx p_0$, we can assume $\Lambda(\omega, \omega_0)$ to be independent of $|\mathbf{p}|$. We multiply (22) by $n(2\pi)^{-3} \times |u(\mathbf{1} - \mathbf{p})|^2$ and integrate over $d^3\mathbf{p}$:

$$\begin{aligned} & \Lambda(\omega, \omega_0) \\ &= \frac{n}{(2\pi)^3} \int |u(\mathbf{1} - \mathbf{p})|^2 \mathbf{p} G(\mathbf{p}_+) G(\mathbf{p}_-) [1 + \Lambda(\omega, \omega_0)] d^3\mathbf{p}. \end{aligned} \quad (24)$$

Replacing $G(\mathbf{p})$ by Eq. (19) we find easily that $\Lambda(\omega, \omega_0)$ is only different from zero if $|\omega| < \omega_0/2$, since in the opposite case both poles in (24) for the integration over ξ according to (20) lie in the same half-plane. In that integral Λ does not depend on ω . Integrating over ξ and using the relation

$$\cos \theta = \cos \theta' \cos \theta'' + \sin \theta' \sin \theta'' \cos(\varphi' - \varphi''), \quad (25)$$

we get

$$\Lambda(\omega, \omega_0) = \begin{cases} \frac{i}{\tau_1} \frac{1}{\omega_0 + i/\tau_{tr}} & \omega^2 < \omega_0^2/4, \\ 0, & \omega^2 > \omega_0^2/4, \end{cases} \quad (26)$$

where

$$\frac{1}{\tau_1} = \frac{1}{\tau} - \frac{1}{\tau_{tr}}; \quad \frac{1}{\tau_{tr}} = \frac{nmv_0}{(2\pi)^2} \int |u(\theta)|^2 (1 - \cos \theta) d\Omega. \quad (27)$$

After substituting (26) into (22) and (7) and integrating, assuming that $\omega_0\tau \ll 1$, we find

$$Q_{ik} = -\frac{4\pi i \omega_0 \sigma}{c^2} \delta_{ik}. \quad (28)$$

The "transport" collision time enters into the conductivity $\sigma = Ne^2\tau_{tr}/m$, as should be the case.

2. SUPERCONDUCTING ALLOYS

The method given above will now be applied to a study of a superconductor with impurities. The main difference as compared to what was given above lies in the fact that a superconductor is described by three Green functions:

$$\begin{aligned} \delta_{\alpha\beta} G(x, x') &= -i \langle T(\psi_\alpha(x) \psi_\beta^\dagger(x')) \rangle; \\ \hat{I}_{\alpha\beta} F^+(x, x') &= \langle T(\psi_\alpha^\dagger(x) \psi_\beta^\dagger(x')) \rangle; \\ -\hat{I}_{\alpha\beta} F(x, x') &= \langle T(\psi_\alpha(x) \psi_\beta(x')) \rangle, \end{aligned} \quad (29)$$

where $\hat{I}_{\alpha\beta} = -\hat{I}_{\beta\alpha}$, $\hat{I}^2 = -E$. These functions were introduced in reference 4 for a pure superconductor and are equal to

$$\begin{aligned} G(x - x') &= (2\pi)^{-4} \int G(p) e^{ip(x-x')} d^4p, \\ F^+(x - x') &= (2\pi)^{-4} \int F^+(p) e^{ip(x-x')} d^4p, \\ F(x - x') &= F^+(x - x'), \end{aligned}$$

where

$$G_0(p) = \frac{\omega + \xi}{\omega^2 - \xi^2 - \Delta^2 + i\delta}; \quad F_0^+(p) = \frac{i\Delta}{\omega^2 - \xi^2 - \Delta^2 + i\delta} \quad (\delta \rightarrow +0). \quad (30)$$

The occurrence of three Green functions makes it necessary to change the diagram technique.* The functions G , F^+ , and F can be represented by the lines with arrows given in Fig. 4a. If vertices are included in an electron line it is necessary to put an arrow next to the vertex. If the interaction Hamiltonian includes products $\psi_\alpha^\dagger(y')\psi_\alpha(y)$ (this always occurs in our problem) the arrows around the vertex must be aimed in the same direction. There are then two possibilities: both arrows can be directed towards the left or towards the right. From Fig. 4c it is clear what will be the result of including vertices $\psi^\dagger\psi$ in the functions G , F^+ , and F . The result can be written in the form

$$\begin{aligned} G(x, x') &\rightarrow G(x, y')G(y, x') + F(x, y)F^+(y', x'); \\ F^+(x, x') &\rightarrow F^+(x, y')G(y, x') + G(y, x)F^+(y', x'); \\ F(x, x') &\rightarrow G(x, y')F(y, x') + F(x, y)G(x', y'). \end{aligned} \quad (31)$$

*The technique with three Green functions was worked out by S. T. Beliaev⁶ in an application to the Bose gas.

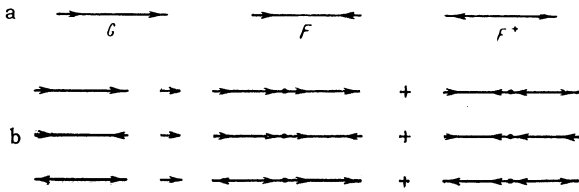


FIG. 4

The inclusion of vertices leads in this way to the appearance of F and F^+ lines in the diagrams for G and similarly to G lines in the diagrams for F and F^+ . The equations for G , F^+ , and F are therefore more complicated than Eq. (14), obtained by summing the diagrams of Fig. 1 with one G -function.

The derivation of these equations and their solution are given in the Appendix. It turns out that the averaging over the positions of the impurity atoms of the functions $G(p)$ and $F^+(p) = F(p)$ can be obtained from the expressions (30) for a pure superconductor by the substitution

$$\omega; \Delta \rightarrow \omega\tau; \Delta\tau_i, \tag{32}$$

where

$$\tau = \begin{cases} 1 + i/2\tau\sqrt{\omega^2 - \Delta^2}; & \omega^2 > \Delta^2, \\ 1 + 1/2\tau\sqrt{\Delta^2 - \omega^2}; & \omega^2 < \Delta^2; \end{cases}$$

τ is the collision time for the metal in its normal state. One can easily verify that, as in Eq. (18) of Sec. 1, these equations signify in coordinate space a multiplication of the function $G_0(x)$ and $F_0^+(x)$ by $e^{-|\mathbf{x}|/2l}$

We now turn to the determination of the function Q_{ik} . This time we have two diagrams corresponding to the simple loop for a normal metal. They are given in Fig. 5. This means that instead of a product of two G 's we have now the combination $GG - FF^+$ (the sign is due to the fact that the operator $\nabla_{\mathbf{y}} - \nabla_{\mathbf{y}'}$ occurs in the electromagnetic interaction, which changes sign if the direction of the arrows at the vertex is changed). For the following it will again be convenient for us, taking the two loops of Fig. 5 together, and taking one of the photon vertices p_i separately, to consider the remaining factor as the result of including the photon vertex p_k in the function G . After this Q_{ik} will be of the form (8) and the problem lies in evaluating the vector Π .

To average Π over the impurities we note at once that since superconductivity changes the Green functions only near the Fermi surface we need again only sum over ladder graphs of the type given in Fig. 3b. Owing to the modified diagram technique, we must consider, as can easily be seen from Fig. 6, not one, but four different vectors $\Pi^{(i)}$ ($i = 1, \dots, 4$) which occur as a result of including

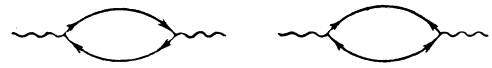


FIG. 5

photon vertices in G , F^+ , $G^{(-)}$, and F (where $G^{(-)}$ denotes G with the arguments interchanged). The first of these quantities enters directly into the integral (8) for Q_{ik} .

We obtain easily for the $\Pi^{(i)}$ a set of four integral equations. The first one of them is obtained with the aid of Fig. 6 and has the form:

$$\begin{aligned} \Pi^{(1)}(p_+, p_-) = & \mathbf{p} [G(p_+)G(p_-) - F(p_+)F(p_-)] \tag{33} \\ & + G(p_+)G(p_-)\Lambda^{(1)} + F(p_+)G(p_-)\Lambda^{(2)} \\ & + F(p_+)F(p_-)\Lambda^{(3)} + G(p_+)F(p_-)\Lambda^{(4)}. \end{aligned}$$

The quantities $\Lambda^{(i)}(\omega, \omega_0, \mathbf{k}, \mathbf{p})$ are the integrals (23) of the $\Pi^{(i)}(p_+, p_-)$. The remaining equations are obtained in a similar way. We shall not write them down since, as was pointed out above, we only need $\Pi^{(i)}(p_+, p_-)$.

As was already noted, for high impurity concentrations the role of the correlation length is assumed here not by $\xi_0 = v/\Delta$ but by the mean free path l . As the impurity concentration increases this length decreases. We shall consider here the case where l is much less than the penetration depth. It is then natural to expect a local connection between the current and the vector potential. The case considered is characterized by the condition $kl \ll 1$, which leads to a considerably simplified equation.

Indeed, neglecting \mathbf{k} , we obtain $\mathbf{p}_+ = \mathbf{p}_-$. It is easy to check that in that case $\Lambda^{(i)}(\omega, \omega_0, \mathbf{k}, \mathbf{p}) \approx \mathbf{p}\Lambda^{(i)}(\omega, \omega_0)$ and

$$\Lambda^{(3)}(\omega, \omega_0) = -\Lambda^{(1)}(\omega, \omega_0); \quad \Lambda^{(2)}(\omega, \omega_0) = \Lambda^{(4)}(\omega, \omega_0). \tag{34}$$

The set of equations for $\Lambda^{(1,2)}$ takes the form

$$\begin{aligned} \mathbf{p}\Lambda^{(1)}(\omega, \omega_0) = & \frac{n}{(2\pi)^3} \int |u(\mathbf{p} - \mathbf{p}')|^2 \mathbf{p}' \{ [G(p'_+)G(p'_-) \\ & - F(p'_+)F(p'_-)] \times [1 + \Lambda^{(1)}(\omega, \omega_0)] \\ & + [F(p'_+)G(p'_-) + G(p'_+)F(p'_-)] \Lambda^{(2)}(\omega, \omega_0) \} d^3\mathbf{p}; \tag{35} \\ \mathbf{p}\Lambda^{(2)}(\omega, \omega_0) = & \frac{n}{(2\pi)^3} \int |u(\mathbf{p} - \mathbf{p}')|^2 \mathbf{p}' \{ [F(p'_+)G(p'_-) \\ & - G(-p'_+)F(p'_-)] [1 + \Lambda^{(1)}(\omega, \omega_0)] \\ & + [G(-p'_+)G(p'_-) + F(p'_+)F(p'_-)] \Lambda^{(2)}(\omega, \omega_0) \} d^3\mathbf{p}. \end{aligned}$$

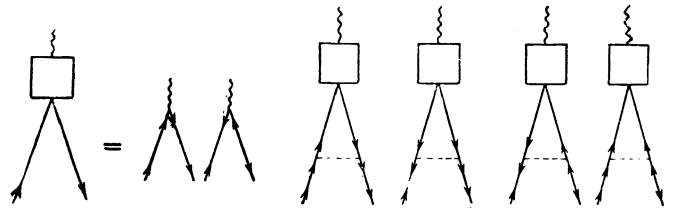


FIG. 6

Substituting into (35) the expressions (30) and (32) for the functions G and F, integrating over ξ ,

and using the relation (25) we arrive at the following expressions for the quantities of interest to us:

$$\Lambda^{(1)}(\omega, \omega_0) = \frac{i}{2\tau_1} \frac{1 - (\omega_+\omega_- + \Delta^2) / \sqrt{\omega_+^2 - \Delta^2} \sqrt{\omega_-^2 - \Delta^2}}{\sqrt{\omega_+^2 - \Delta^2} + \sqrt{\omega_-^2 - \Delta^2} + i/\tau_{tr}};$$

$$\Lambda^{(2)}(\omega, \omega_0) = \frac{\Delta}{2\tau_1} \frac{2\omega}{\sqrt{\omega_+^2 - \Delta^2} \sqrt{\omega_-^2 - \Delta^2} [\sqrt{\omega_+^2 - \Delta^2} + \sqrt{\omega_-^2 - \Delta^2} + i/\tau_{tr}]}.$$

Substituting (34) and (36) into (33) we find $\Pi^{(1)}(p_+, p_-)$ and substituting this into Eq. (8) we get an expression for Q:

$$Q_{ik}(\omega_0) = \delta_{ik} \left\{ \frac{Ne^2}{mc} - \frac{i e^2 N}{2\pi mc} \int d\xi \int_C d\omega [(\omega_+\omega_- + \Delta^2) \eta_+ \eta_- + \xi^2] \right. \\ \left. \times (1 + \Lambda^{(1)}(\omega, \omega_0)) + 2i\Delta\omega\Lambda^{(2)}(\omega, \omega_0) \right\} \quad (37)$$

$$\times \frac{1}{[\xi^2 - (\omega_+^2 - \Delta^2) \eta_+^2][\xi^2 - (\omega_-^2 - \Delta^2) \eta_-^2]}.$$

The contour around the singular points in the integral over ω is given in Fig. 7a for the case $\omega_0 < 2\Delta$ and in Fig. 7c for the case $\omega_0 > 2\Delta$. Before performing the integration (37) we note that the integral of the first term in (33) diverges. One must therefore observe the well-known caution in the problem of the order of integration over ω and ξ . A closer inspection shows that one must first integrate by parts.

We first find the penetration depth of a static magnetic field into a superconductor. The latter is connected with Q_{ik} by the relation

$$\delta = \sqrt{\frac{c}{4\pi Q(0)}}; \quad \text{if } Q_{ik} = \delta_{ik}Q. \quad (38)$$

The contour C in Fig. 7a can in the case $\omega_0 = 0$ conveniently be deformed so that the integration over ω is performed along the imaginary axis (this is possible because of the decrease of the expression under the integral sign as $\omega \rightarrow \infty$):

$$Q_{ik}(0) = \delta_{ik}Q(0) = \frac{Ne^2}{mc} \delta_{ik} \left\{ 1 + \frac{\Delta^2}{2\tau_1(\omega^2 + \Delta^2)(\sqrt{\omega^2 + \Delta^2} + 1/2\tau_{tr})} \right. \\ \left. + \frac{1}{2\pi} \int_{-\infty}^{\infty} d\xi d\omega \frac{1 + \frac{\Delta^2}{2\tau_1(\omega^2 + \Delta^2)(\sqrt{\omega^2 + \Delta^2} + 1/2\tau_{tr})}}{\xi^2 + (\sqrt{\omega^2 + \Delta^2} + 1/2\tau_{tr})^2} \right. \\ \left. - \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{d\xi d\omega \omega^2 \left(1 + \frac{1}{2\tau_1 \sqrt{\omega^2 + \Delta^2}}\right)^2}{[\xi^2 + (\sqrt{\omega^2 + \Delta^2} + 1/2\tau_{tr})^2]^2} \right\}.$$

Integrating by parts over ω in the first term of the first integral makes it possible formally to cancel the diverging terms in that expression. In

the remaining double integral one can change the order of integration. The integral over ξ is at once performed:

$$Q(0) = \frac{Ne^2}{mc} \left\{ 1 - \frac{\Delta^2}{4\tau_{tr}} \int_{-\infty}^{\infty} \frac{d\omega}{(\omega^2 + \Delta^2)^{3/2} (\sqrt{\omega^2 + \Delta^2} + 1/2\tau_{tr})} \right\}.$$

After that we integrate over ω and find

$$Q(0) = \frac{Ne^2}{mc} \tau_{tr} \Delta \begin{cases} \pi - \frac{4\tau_{tr}\Delta}{\sqrt{1 - (2\tau_{tr}\Delta)^2}} \cosh^{-1} \frac{1}{2\tau_{tr}\Delta}; & 2\tau_{tr}\Delta < 1, \\ \pi - \frac{4\tau_{tr}\Delta}{\sqrt{(2\tau_{tr}\Delta)^2 - 1}} \arccos \frac{1}{2\tau_{tr}\Delta}; & 2\tau_{tr}\Delta > 1. \end{cases}$$

Substituting this into Eq. (38) we can from this obtain the penetration depth. If $\tau_{tr}\Delta \ll 1$ it is (in the usual units) of the form

$$\delta = \frac{c}{2\pi} \sqrt{\hbar/\sigma\Delta}, \quad (40)$$

where $\sigma = Ne^2\tau_{tr}/m$ is the conductivity of the metal in the normal state. In the case $\tau_{tr}\Delta \gg 1$ the expression for the penetration depth goes over into the usual London formula:

$$\delta = \sqrt{mc^2/4\pi Ne^2}.$$

It is well known¹ that pure superconductors fall into two classes: in one we have $\delta > v/\Delta$ for all temperatures; they can be called London superconductors. In that case there is no limit whatever for the quantity $\tau_{tr}\Delta$. For the other superconductors, which can be called Pippard superconductors, we have $\delta < v/\Delta$ at $T = 0$. In that case from the condition $l \ll \delta$ follows always that $\tau_{tr}\Delta \ll 1$. The Pippard superconductors are thus described by Eq. (40) for $l \ll \delta$. In the case when the mean free path $l \gg \delta$ the theory developed in reference 1 is clearly correct for Pippard superconductors,

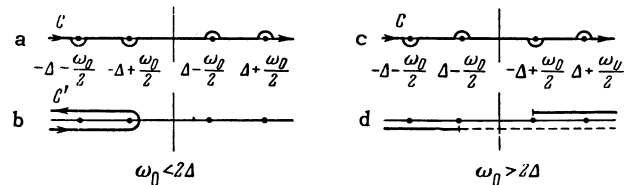


FIG. 7

independent of the relation between l and v/Δ .
From this theory we obtain

$$Q(k, 0) = \frac{3\pi^2}{4mc} \frac{Ne^2\Delta}{vk}. \quad (41)$$

Comparison of (40) and (41) shows that Eq. (40) must be used, if

$$\delta/l_{tr} \gg 3\pi/4. \quad (42)$$

We now proceed to the case of an alternating field. It is of interest to follow the transition from a superconductor to a normal metal as the frequency ω_0 increases from zero to $\omega_0 \gg \Delta$. We consider only the case $\tau_{tr}\Delta \ll 1$. We subtract from the expression under the integral sign its value at $\omega_0 = 0$. The remainder can then be integrated first over ξ . In the case when $\omega_0 < 2\Delta$ we go over to the contour C' (Fig. 7b) for the integration over ω . We get

$$Q(\omega_0) = \frac{Ne^2}{mc} \left\{ \int_{\Delta+\omega_0/2}^{\infty} d\omega \left[1 - \frac{\omega_+\omega_- + \Delta^2}{\sqrt{\omega_+^2 - \Delta^2} \sqrt{\omega_-^2 - \Delta^2}} \right] \right. \\ \times \frac{V\omega_+^2 - \Delta^2 + V\omega_-^2 - \Delta^2}{(V\omega_+^2 - \Delta^2 + V\omega_-^2 - \Delta^2)^2 + 1/\tau_{tr}^2} \\ \left. + \tau_{tr} \int_{\Delta-\omega_0/2}^{\Delta+\omega_0/2} \frac{\omega_+\omega_- + \Delta^2}{\sqrt{\omega_+^2 - \Delta^2} \sqrt{\omega_-^2 - \Delta^2}} \right\}.$$

The contribution from the first integral is of the order $(\tau\Delta)^2$ and we can neglect it. The second integral leads to a complete elliptic integral.⁷ The final expression has the following form for the case $\omega_0 < 2\Delta$

$$Q(\omega_0) = \frac{2\sigma\Delta}{c} E\left(\frac{\omega_0}{2\Delta}\right). \quad (43)$$

In particular, we get for $\omega_0 \ll 2\Delta$

$$Q(\omega_0) = \frac{\pi\sigma\Delta}{c} \left[1 - \left(\frac{\omega_0}{4\Delta}\right)^2 \right]. \quad (44)$$

In the case $\omega_0 > 2\Delta$ it is convenient to complete the integral to go from $-\infty$ to $+\infty$ in the lower half-plane (Fig. 7c). This part can then be discarded, since there are no poles in the expression under the integral sign. This makes it possible to simplify the expressions considerably. As a result, Q can be expressed as the sum of elliptic integrals, if $\tau\Delta \ll 1$:

$$Q(\omega_0) = \frac{\sigma}{c} \left\{ \omega_0 E(\kappa) - \frac{\omega_0^2 - (2\Delta)^2}{\omega_0} K(\kappa) \right. \\ \left. - i \left[\omega_0 E(\kappa_1) - \frac{(2\Delta)^2}{\omega_0} K(\kappa_1) \right] \right\}, \quad (45)$$

where $\kappa = 2\Delta/\omega_0$; $\kappa_1 = \sqrt{1 - \kappa^2}$.

In particular if $\omega_0 \gg 2\Delta$

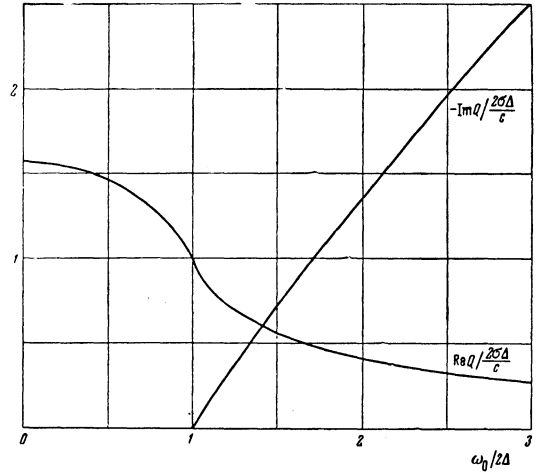


FIG. 8

$$Q(\omega_0) = -\frac{\sigma\omega_0}{c} \left\{ i \left[1 - \frac{2\Delta^2}{\omega_0^2} \left(\ln \frac{2\omega_0}{\Delta} + \frac{1}{2} \right) \right] - \frac{\pi\Delta^2}{\omega_0^2} \right\}. \quad (46)$$

From Eq. (46) it is clear that if $\omega_0 \gg \Delta$ the superconducting properties disappear; the usual formula for the normal skin effect turns up. It is of interest that the imaginary part of the current corresponding to absorption occurs only for $\omega_0 > 2\Delta$; the connection of this threshold energy and the binding energy of the pairs is evident. Near the threshold $|\omega_0 - 2\Delta| \ll \Delta$ we have:

$$Q = \frac{\sigma}{c} \left[2\Delta + \left(\Delta - \frac{\omega_0}{2} \right) \left(\ln \frac{16\Delta}{2\Delta - \omega_0} - 1 \right) \right]; \quad \omega_0 < 2\Delta, \quad (47)$$

$$Q = \frac{\sigma}{c} \left[2\Delta - \left(\frac{\omega_0}{2} - \Delta \right) \left(\ln \frac{16\Delta}{\omega_0 - 2\Delta} - 1 + i\pi \right) \right]; \quad \omega_0 > 2\Delta.$$

In Fig. 8 we have given the complete curves of the dependence of the real and imaginary parts of Q on ω_0 . Along the ordinate axis are put $\text{Re} \left[Q / \frac{2\Delta\sigma}{c} \right]$ and $-\text{Im} \left[Q / \frac{2\Delta\sigma}{c} \right]$ in their dependence on $\omega_0/2\Delta$. The impedance of a bulk specimen with a plane surface is obtained with the aid of Eq. (4) and is of the form

$$Z = -\frac{i\omega_0}{c} \sqrt{\frac{4\pi}{cQ(\omega_0)}}; \quad (48)$$

If Q is complex the radical means the analytical continuation of that quantity which is positive for real Q .

The Eqs. (39), (40), and (43) to (48) describe completely the behavior of alloys in a constant or alternating weak electromagnetic field at $T = 0$. Apart from alloys, these expressions can also be applied to films of a pure substance with a thickness less than the penetration depth. Such films are as a rule polycrystalline and the dimensions of the crystallites are not more than the film thickness. The presence of boundaries between the

crystallites will lead to a break-up of the correlation. The local character of the connection in the case $\delta \gg l$ makes it possible to apply the results obtained above to films without any changes. It may then turn out to be convenient to use the complex dielectric susceptibility which is connected with Q through the relation

$$\varepsilon = -4\pi cQ/\omega^2 \quad (49)$$

In conclusion the authors use this opportunity to express their gratitude to Academician L. D. Landau for his constant attention and valuable remarks.

APPENDIX

In the case of the superconducting alloy the equations replacing (14) are easily found by drawing the corresponding diagrams. They are of the form

$$G(\mathbf{p}, \mathbf{p}', \omega) = G_0(p) \delta(\mathbf{p} - \mathbf{p}') + \frac{1}{(2\pi)^3} \left\{ G_0(p) \int u(\mathbf{p} - \mathbf{p}'') \sum_a e^{i(\mathbf{p}-\mathbf{p}'') \cdot \mathbf{x}_a} G(\mathbf{p}'', \mathbf{p}', \omega) d^3 \mathbf{p}'' + F_0(p) \int u(\mathbf{p} - \mathbf{p}'') \sum_a e^{i(\mathbf{p}-\mathbf{p}'') \cdot \mathbf{x}_a} F^+(\mathbf{p}'', \mathbf{p}', \omega) d^3 \mathbf{p}'' \right\};$$

$$F^+(\mathbf{p}, \mathbf{p}', \omega) = F_0^+(p) \delta(\mathbf{p} - \mathbf{p}') \quad (A.1)$$

$$+ \frac{1}{(2\pi)^3} \left\{ F_0(p) \int u(\mathbf{p} - \mathbf{p}'') \sum_a e^{i(\mathbf{p}-\mathbf{p}'') \cdot \mathbf{x}_a} G(\mathbf{p}'', \mathbf{p}', \omega) d^3 \mathbf{p}'' + G_0(-p) \int u(\mathbf{p} - \mathbf{p}'') \sum_a e^{i(\mathbf{p}-\mathbf{p}'') \cdot \mathbf{x}_a} F^+(\mathbf{p}'', \mathbf{p}', \omega) d^3 \mathbf{p}'' \right\}$$

In principle it is necessary to consider also the equations for the function $-\hat{I}_{\alpha\beta} F(\mathbf{x}, \mathbf{x}') = \langle T(\psi_\alpha(\mathbf{x}) \psi_\beta(\mathbf{x}')) \rangle$.

For a pure superconductor $F(\mathbf{x}, \mathbf{x}') = F^+(\mathbf{x}, \mathbf{x}')$. This property will be true also for alloys after Eq. (A.1) is averaged over the positions of the impurity atoms.

The averaging is performed exactly as in Sec. 1. Using the equations⁴ that are satisfied by the functions $G_0(p)$ and $F_0(p)$, this set can be put into a very simple form:

$$\begin{aligned} (\omega - \xi - \bar{G}_\omega) G(p) + (i\Delta - \bar{F}_\omega^+) F^+(p) &= 1, \\ (\omega + \xi - \bar{G}_\omega) F^+(p) - (i\Delta - \bar{F}_\omega^+) G(p) &= 0. \end{aligned}$$

The solution of this set is

$$\begin{aligned} G(p) &= \frac{\omega - \bar{G}_\omega + \xi}{(\omega - \bar{G}_\omega)^2 + (i\Delta - \bar{F}_\omega^+)^2 - \xi^2}; \\ F^+(p) &= \frac{i\Delta - \bar{F}_\omega^+}{(\omega - \bar{G}_\omega)^2 + (i\Delta - \bar{F}_\omega^+)^2 - \xi^2}. \end{aligned} \quad (A.2)$$

Remembering the definition of \bar{G}_ω we see that as before \bar{G}_ω contains a constant term which means a renormalization of the chemical potential and which arises from the integration far from the Fermi surface:

$$\delta\mu = \frac{n}{(2\pi)^3} \int \frac{|u(\mathbf{p} - \mathbf{p}')|^2 \xi d\xi}{(\omega - G_\omega)^2 + (i\Delta - F_\omega^+)^2 - \xi^2}.$$

After subtracting this term, the remaining part of \bar{G}_ω is apart from a coefficient determined by the same integral as \bar{F}_ω^+ , as is clear from (A.2). In this way

$$\bar{G}_\omega / \omega = \bar{F}_\omega^+ / \Delta.$$

After this it is easy to find that the solution of the set (A.2) gives Eqs. (30) and (32).

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