RELATIVISTIC CORRECTIONS TO PHENOMENOLOGICAL HAMILTONIANS

Yu. M. SHIROKOV

P. N. Lebedev Physics Institute, Academy of Sciences, U.S.S.R.

Submitted to JETP editor June 14, 1958

J. Exptl. Theoret. Phys. (U.S.S.R.) **36**, 474-477 (February, 1959)

A general expression has been obtained for the relativistic corrections of order $(v/c)^2$ to a given phenomenological nonrelativistic Hamiltonian describing the interaction between particles of arbitrary mass and spin.

1. In view of the unsatisfactory state of meson theory, phenomenological Hamiltonians for the nucleon-nucleon and meson-nucleon interactions have found wide application at the present time. These Hamiltonians are usually chosen such as to agree with the experimental data on the corresponding two-body problem. The aim of the present paper is to calculate the relativistic corrections to these phenomenological potentials. It turns out that, starting from the general postulate of relativistic invariance, the phenomenological Hamiltonian can be corrected relativistically in a consistent fashion, at least up to terms of order $(v/c)^2$.

2. If the relativistic effects are neglected, the system of two particles with masses κ_1 and κ_2 and spins i and I is described by the Hamiltonian

$$H = \mathbf{p}_1^2/2\varkappa_1 + \mathbf{p}_2^2/2\varkappa_2 + H_{12}, \tag{1}$$

where H_{12} is the nonrelativistic interaction Hamiltonian. The general requirements on the Hamiltonian H_{12} have been analyzed in detail by Okubo and Marshak.[1] In particular, H_{12} must not depend on the total momentum $\mathbf{p}_1 + \mathbf{p}_2 = \mathbf{P}$ as a consequence of conservation law for the nonrelativistic center of intertia (Galilean invariance).

The inclusion of the relativistic effects leads to correction terms which do depend on $\mathbf{P}$.

3. To make the discussion sufficiently rigorous and general, we shall not make use of any special equations of motion. Instead, we start with group theoretical considerations and the invariance properties of the S matrix.

According to the theory of the representations of the inhomogeneous Lorentz group, the system of two free particles is described by the wave function[2,3]

$$\psi^{\varkappa_1\varkappa_2 iI}_{m_i m_I}(\mathbf{p}_1, \mathbf{p}_2), \tag{2}$$

which depends on the variables $\mathbf{p}_1$, $\mathbf{p}_2$ (the momenta) and m_i, m_I (the spin projections). We do not exhibit the other possible variables (isotopic spin, charge, etc.), since they are irrelevant for a discussion of the relativistic invariance. The scattering of two particles is described by the invariant S matrix

$$(\mathbf{p}_1\mathbf{p}_2 m_i m_I \mid S \mid \mathbf{p}_1'\mathbf{p}_2' m_i' m_I'), \tag{3}$$

which, owing to the conservation law for the 4-momentum, can also be written in the form

$$(\mathbf{p}_1\mathbf{p}_2 m_i m_I \mid S \mid \mathbf{p}_1'\mathbf{p}_2' m_i' m_I') = \delta(\mathbf{p}_1 + \mathbf{p}_2 - \mathbf{p}_1' - \mathbf{p}_2')$$
$$\times\, \delta(e_{p_1} + E_{p_2} - e_{p_1'} - E_{p_2'})\,(\mathbf{p}_1\mathbf{p}_2 m_i m_I \mid V \mid \mathbf{p}_1'\mathbf{p}_2' m_i' m_I'),$$
$$e_{p_1} = \sqrt{p_1^2 + \varkappa_1^2},\quad E_{p_2} = \sqrt{p_2^2 + \varkappa_2^2}. \tag{4}$$

The matrix V is invariant as a consequence of the invariance of the four-dimensional δ function. On the other hand, if perturbation theory is applicable, the S matrix is, in the interaction picture, expressed in terms of the Hamiltonian H_{12} by the relation

$$S = 1 + \frac{1}{i}\int dt H_{12} \exp\{-it(e_{p_1} + E_{p_2} - e_{p_1'} - E_{p_2'})\}$$
$$= 1 - 2\pi i H_{12}\delta(e_{p_1} + E_{p_2} - e_{p_1'} - E_{p_2'}). \tag{5}$$

Since the transformation properties of the Hamiltonian cannot depend on the applicability of perturbation theory to it, and since the expansion in powers of the coupling constant is invariant, it can be concluded from the comparison of (4) and (5) that the Hamiltonian has the form

$$(\mathbf{p}_1\mathbf{p}_2 m_i m_I \mid H_{12} \mid \mathbf{p}_1'\mathbf{p}_2' m_i' m_I')$$
$$= \delta(\mathbf{p}_1 + \mathbf{p}_2 - \mathbf{p}_1' - \mathbf{p}_2')(\mathbf{p}_1\mathbf{p}_2 m_i m_I \mid W \mid \mathbf{p}_1'\mathbf{p}_2' m_i' m_I'), \tag{6}$$

where $(\mathbf{p}_1\mathbf{p}_2 m_i m_I \mid H_{12} \mid \mathbf{p}_1'\mathbf{p}_2' m_i' m_I')$ is an invariant.

The requirement that the operator W be invariant implies that it must commute with the operator $\mathbf{N}$ which generates the infinitesimal Lorentz transformation:

$$[\mathbf{N}, W]_- = 0. \tag{7}$$

The square brackets with a minus sign denote the commutator. The operator N has, according to reference 2, the form

$$\mathbf{N} = i\sqrt{e_{p_1}}\,\frac{\partial}{\partial \mathbf{p}_1}\sqrt{e_{p_1}} - \frac{[\mathbf{i}\times\mathbf{p}_1]}{e_{p_1}+\varkappa_1} + i\sqrt{E_{p_2}}\,\frac{\partial}{\partial \mathbf{p}_2}\sqrt{E_{p_2}} - \frac{[\mathbf{I}\times\mathbf{p}_2]}{E_{p_2}+\varkappa_2}. \quad (8)$$

The square brackets without a minus sign denote the vector product. The slight difference in the appearance of the operator N in (8) and in (11) of reference 2 (cf. also reference 3) is due to the fact that, in the present paper, we use a normalization integral without the weight factor $1/e_{p_1}E_{p_2}$.

4. In the nonrelativistic approximation, the operators W, N can be expanded in powers of $\mathbf{p}_1/\kappa_1$ and $\mathbf{p}_2/\kappa_2$:

$$W = W_0 + W_1, \quad (9)$$

$$\mathbf{N} = \mathbf{N}_0 + \mathbf{N}_1, \quad (10)$$

$$\mathbf{N}_0 = i\varkappa_1\partial/\partial\mathbf{p}_1 + i\varkappa_2\partial/\partial\mathbf{p}_2 \quad (11)$$

$$\mathbf{N}_1 = i\frac{\mathbf{p}_1}{2\varkappa_1} + i\frac{\mathbf{p}_1^2}{2\varkappa_1}\frac{\partial}{\partial\mathbf{p}_1} - \frac{[\mathbf{i}\times\mathbf{p}_1]}{2\varkappa_1} + i\frac{\mathbf{p}_2}{2\varkappa_2} + i\frac{\mathbf{p}_2^2}{2\varkappa_2}\frac{\partial}{\partial\mathbf{p}_2} - \frac{[\mathbf{I}\times\mathbf{p}_2]}{2\varkappa_2}. \quad (12)$$

Substituting (9) and (10) in (7), we obtain, in zeroth order, the condition of Galilean invariance for the interaction Hamiltonian:

$$[\mathbf{N}_0,\ W_0]_- = 0. \quad (13)$$

We now subject the momenta $\mathbf{p}_1$, $\mathbf{p}_2$ to the Jacobi transformation:

$$\mathbf{p}_1 + \mathbf{p}_2 = \mathbf{P},\ \ \mathbf{p}_1\varkappa_2/(\varkappa_1+\varkappa_2) - \mathbf{p}_2\varkappa_1/(\varkappa_1+\varkappa_2) = \mathbf{p}, \quad (14)$$

and similarly for the momenta $\mathbf{p}_1'$, $\mathbf{p}_2'$. On account of the δ function in (6), one may regard the operator W as independent of $\mathbf{P}'$. In this case, relation (13) takes the form

$$\partial W_0(\mathbf{p}_1, \mathbf{p}', \mathbf{P})/\partial\mathbf{P} = 0. \quad (15)$$

In the zeroth approximation the Hamiltonian does, therefore, not depend on the total momentum, i.e., the requirement of Galilean invariance is satisfied, as was to be expected. In the next nonvanishing approximation, condition (7) has the form

$$\mathbf{N}_0W_1 - W_1\mathbf{N}_0 = W_0\mathbf{N}_1 - \mathbf{N}_1W_0. \quad (16)$$

Substituting (11) and (12) in (16) and using (15), we obtain

$$-i(\varkappa_1+\varkappa_2)\frac{\partial}{\partial\mathbf{P}}W_1(\mathbf{P},\mathbf{p},\mathbf{p}') = \Big\{i\frac{\mathbf{p}_1}{2\varkappa_1} + i\frac{\mathbf{p}_2}{2\varkappa_2} + i\frac{\mathbf{p}_1'}{2\varkappa_1} + i\frac{\mathbf{p}_2'}{2\varkappa_2} + i\frac{\mathbf{p}_1^2}{2\varkappa_1}\frac{\partial}{\partial\mathbf{p}_1} + i\frac{\mathbf{p}_2^2}{2\varkappa_2}\frac{\partial}{\partial\mathbf{p}_2} + i\frac{\mathbf{p}_1'^2}{2\varkappa_1}\frac{\partial}{\partial\mathbf{p}_1'} + i\frac{\mathbf{p}_2'^2}{2\varkappa_2}\frac{\partial}{\partial\mathbf{p}_2'} - \frac{[\mathbf{I}\times\mathbf{p}_1]}{2\varkappa_1} - \frac{[\mathbf{I}\times\mathbf{p}_2]}{2\varkappa_2}\Big\}W_0(\mathbf{p},\mathbf{p}') + W_0(\mathbf{p},\mathbf{p}')\Big\{\frac{[\mathbf{i}\times\mathbf{p}_1']}{2\varkappa_1} + \frac{[\mathbf{I}\times\mathbf{p}_2']}{2\varkappa_2}\Big\}. \quad (17)$$

Equation (17) must be written in terms of the $\mathbf{P}$, $\mathbf{p}$, and $\mathbf{p}'$ alone. The operator W_1 can be split up into two parts:

$$W_1 = W_1' + W_1'', \quad (18)$$

where W_1' does not depend on the momentum $\mathbf{P}$:

$$\partial W_1'/\partial\mathbf{P} = 0, \qquad W_1' = W_1'(\mathbf{p},\mathbf{p}'), \quad (19)$$

and W_1'' reduces to zero for $\mathbf{P} = 0$. The operator $W_1'(\mathbf{p},\mathbf{p}')$ is not uniquely determined by Eq. (17). This operator does, however, satisfy the nonrelativistic condition (15), and it can always be considered to be included in the operator W_0 of the nonrelativistic approximation. The structure of the phenomenological Hamiltonian H_{12} is significantly affected only by the operator W_1'', which is, however, indeed uniquely defined by equation (17).

Since we are only interested in corrections of order $(v/c)^2$, we write the operator $W_1''(\mathbf{P},\mathbf{p},\mathbf{p}'')$ as a polynomial of no higher than second degree in $\mathbf{P}$:

$$W_1'' = P_iA_i + P_iP_jB_{ij}. \quad (20)$$

Substituting (20) in (17), we can uniquely determine the operators A_i and B_{ij} by using the fact that the operator $(\mathbf{p}_1\mathbf{p}_2m_im_I\,|\,W_1''\,|\,\mathbf{p}_1'\mathbf{p}_2'm_i'm_I')$ is invariant in three-dimensional space, i.e., that it commutes with the momentum operator $\mathbf{M}$:

$$\mathbf{M} = -i[\mathbf{p}_1\partial/\partial\mathbf{p}_1] - i[\mathbf{p}_2\partial/\partial\mathbf{p}_2] + \mathbf{i} + \mathbf{I}. \quad (21)$$

A simple calculation leads to the following expression for W_1'':

$$W_1'' = -\frac{\mathbf{P}^2W_0}{(\varkappa_1+\varkappa_2)^2} - \frac{(\mathbf{pP})(\mathbf{P}\partial/\partial\mathbf{p}) + (\mathbf{p}'\mathbf{P})(\mathbf{P}\partial/\partial\mathbf{p}')}{2(\varkappa_1+\varkappa_2)^2}W_0 - \frac{(\varkappa_2-\varkappa_1)(\mathbf{p}+\mathbf{p}')\mathbf{P}W_0}{2\varkappa_1\varkappa_2(\varkappa_1+\varkappa_2)} + \frac{i}{2(\varkappa_1+\varkappa_2)}\Big\{W_0[\mathbf{P}\times\mathbf{p}']\Big(\frac{\mathbf{i}}{\varkappa_1} - \frac{\mathbf{I}}{\varkappa_2}\Big) - [\mathbf{P}\times\mathbf{p}]\Big(\frac{\mathbf{i}}{\varkappa_1} - \frac{\mathbf{I}}{\varkappa_2}\Big)W_0\Big\}. \quad (22)$$

5. The operator W_1'' from (22) does not satisfy the nonrelativistic conservation law for the center of inertia and contains all relativistic corrections needed for the phenomenological formulation of the problem[1] (apart from trivial corrections to the kinetic energy). Expression (22) refers to a Hamiltonian which is written in the form of an integral operator. If, as is ordinarily the case, the Hamiltonian is given as a function of the coordinates and the momenta, the correction H_{12}' (derived from the operator W_1'') to the Hamiltonian H_{12} has the form

$$H'_{12} = -\frac{\mathbf{P}^2 H_{12}}{2(\varkappa_1+\varkappa_2)^2} - \frac{\mathbf{P}\mathbf{p}}{2(\varkappa_1+\varkappa_2)^2}\left(\mathbf{P}\frac{\partial H_{12}}{\partial \mathbf{p}}\right) + \frac{i}{2(\varkappa_1+\varkappa_2)^2}\left(\mathbf{P}\frac{\partial H_{12}}{\partial \mathbf{x}}\right)\left(\mathbf{P}\frac{\partial}{\partial \mathbf{p}}\right) - \frac{1}{2(\varkappa_1+\varkappa_2)}\left(\frac{1}{\varkappa_1}-\frac{1}{\varkappa_2}\right)\left[\frac{\partial H_{12}}{\partial \mathbf{x}}\mathbf{P}\right] - \frac{i}{2(\varkappa_1+\varkappa_2)}\left(\frac{1}{\varkappa_1}-\frac{1}{\varkappa_2}\right)H_{12}[\mathbf{p}\times\mathbf{P}]$$
$$+\frac{iH_{12}}{2(\varkappa_1+\varkappa_2)}\left(\frac{1}{\varkappa_1}-\frac{1}{\varkappa_2}\right)[\mathbf{p}\times\mathbf{P}] + \frac{\varkappa_2-\varkappa_1}{\varkappa_1\varkappa_2(\varkappa_1+\varkappa_2)}\left\{\frac{i\mathbf{P}}{2}\frac{\partial H_{12}}{\partial \mathbf{x}} - \frac{1}{2}\mathbf{p}^2\left(\mathbf{P}\frac{\partial H_{12}}{\partial \mathbf{p}}\right) + \frac{1}{2}\frac{\partial^2 H_{12}}{\partial \mathbf{x}^2}\left(\mathbf{P}\frac{\partial}{\partial \mathbf{p}}\right) + i\left(\frac{\partial H_{12}}{\partial \mathbf{x}}\mathbf{p}\right)\left(\mathbf{P}\frac{\partial}{\partial \mathbf{p}}\right)\right\}. \quad (23)$$

The operator $\mathbf{x}$ in (23) denotes the coordinate difference

$$\mathbf{x} = \mathbf{x}_1 - \mathbf{x}_2 = i\partial/\partial\mathbf{p}_1 - i\partial/\partial\mathbf{p}_2. \quad (24)$$

Including the corrections to the kinetic energy, we arrive at the final result: the relativistic Hamiltonian for the two-body problem has, up to terms of order $(v/c)^2$, the form

$$H = \mathbf{p}_1^2/2\varkappa_1 + \mathbf{p}_2^2/2\varkappa_2 - \mathbf{p}_1^4/8\varkappa_1^3 - \mathbf{p}_2^4/8\varkappa_2^3 + H_{12} + H'_{12}, \quad (25)$$

The correction term H'_{12} reduces, according to (24), to zero in the center of inertia system. In the two-body problem the relativistic corrections can, therefore, be avoided by going to the center of inertia system. The corrections thus become significant only in the many-body problem.

We emphasize that, according to (18) and (19), the correction term H'_{12} does not contain all corrections, but only those which violate the Galilean invariance of the Hamiltonian. However, these are just the essential corrections to the phenomenological Hamiltonian.

The Hamiltonian (25) can be used for the investigation of the interaction of mesons and nucleons of high energy with nuclei. Besides that, this Hamiltonian is useful for an estimate of the effect of relativistic corrections on the structure of the nucleus.

[1] S. Okubo and R. E. Marshak, Ann. of Phys. (in press).

[2] Yu. M. Shirokov, J. Exptl. Theoret. Phys. (U.S.S.R.) **33**, 1196 (1957); Soviet Phys. JETP **6**, 919 (1958).

[3] Yu. M. Shirokov, J. Exptl. Theoret. Phys. (U.S.S.R.) **35**, 1005 (1958); Soviet Phys. JETP **8**, 703 (1959).

Translated by R. Lipperheide
79