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DISPERSION RELATIONS FOR THE ELECTROMAGNETIC FORM FACTOR OF THE PION

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Dispersion relations are derived for the electromagnetic form factor of a charged π meson. By including only the contribution to the imaginary part from a state with two π mesons, an equation is obtained which gives the form-factor in terms of the phase shift for scattering of π mesons by π mesons.

FOR the electromagnetic form factor of the π meson one can derive dispersion relations that relate it to the imaginary part of the annihilation amplitude for two π mesons. These may be compared with the relations of the same type for nucleons, which have been considered by Bernstein and Goldberger;[1] the dispersion relations for the π meson do not involve the nonphysical region, and can be simply derived with complete rigor.

We shall consider the analytic properties of the matrix element of the electromagnetic current operator $j_\mu(x)|_{x=0}$ $[\Box A_\mu(x) = -j_\mu(x)]$ between π-meson states with momenta p' and p and isotopic indices i and k — the element $\langle p', i | j_\mu(0) | p, k\rangle$. Such a matrix element enters directly into the expression for the scattering amplitude of π-meson-electron collisions in the lowest approximation in the electromagnetic charge e. From the relativistic and isotopic invariances it follows that the gauge-invariant part of this matrix element can be written in the form

$$\langle p'i | j_\mu(0) | pk\rangle = e(p'+p)_\mu [a_S(q^2) + a_V(q^2) T_3]_{ik}, \quad (1)$$

where $q = p' - p$, T_3 is the operator for the isotopic spin component for $T = 1$, and

$$ea_V(q^2) = \frac{(p'+p)_\mu \langle p' | j_\mu^V(0) | p\rangle}{(p+p')^2} 2\sqrt{\omega_p \omega_{p'}} \quad (2)$$

is the form-factor of the π meson (it will be shown that the factor $a_S(q^2)$ defined by the analogous formula with $j_\mu^S(0)$ is equal to zero). $j_\mu^S(0)$ and $j_\mu^V(0)$ are the isotopic-scalar and the isotopic-vector parts of the current, respectively:

$$j_\mu(0) = j_\mu^S(0) + j_\mu^V(0). \quad (3)$$

It follows from Eq. (1) that the electromagnetic form-factor of the π^0 meson is equal to $a_S(q^2)$, but in virtue of the charge parity of the π^0 meson we have the matrix element

$$\langle p'\pi^0 | j_\mu(0) | p\pi^0\rangle = 0, \text{ i.e., } a_S(q^2) = 0. \quad (4)$$

Consequently in the lowest approximation in e there remains only the one form-factor $a_V(q^2)$ for the charged mesons.

Taking the complex conjugate of Eq. (1) and using the fact that for one-particle states $\langle p|^* = |p\rangle$, we show that $a_V(q^2)$ is a real function for real momenta of the particles $(q^2 > 0)$. In addition, $a_V(q^2) \to 1$ for $q^2 \to 0$.

By means of reduction formulas[2] we can write $a_V(q^2) = a(q^2)$ in the form

$$a(q^2) = i\frac{\sqrt{2\omega_p}}{(p+p')^2}\int e^{-ip'x}dx \times \{\langle 0 | T(j(x), j_e^V(0)) + [\dot{\Phi}(x), j_e^V(0)]\,\delta(x_0) | p\rangle\}. \quad (5)$$

$j_e^V(0) = j_\mu^V(0)(p+p')_\mu$; $\dot{\Phi}(x)$ is the time derivative of the meson field $\Phi(x)$ $[(\Box - \mu^2)\Phi(x) = -j(x)]$.

In the physical region $\omega_{p'} \geq \mu$ (μ is the mass of the π meson) the integral of the T product in Eq. (5) is the same as the analogous integrals of the retarded and advanced commutators. Applying the technique of Goldgerger[3] and Bogolyubov[4] we construct a function $F(\omega_{p'})$ analytic in the entire complex plane of $\omega_{p'}$ ($\mathbf{p}' = (\omega_{p'}^2 - \mu^2)^{1/2}\mathbf{e}$, with $\mathbf{e}$ the unit vector in the direction of $\mathbf{p}'$ and fixed vectors $\mathbf{p}$ and $\mathbf{e}$) except on the negative part of the real axis from $-\mu$ to $-\infty$, where it has branch points.

For $\omega_{p'} \geq \mu$

$$F(\omega_{p'}) = a(q^2), \quad q^2 = (p-p')^2 > 0,$$

$$F(-\omega_{p'}) = b_1(q^2) = \frac{1}{e}\frac{((p-p')_\mu \langle 0 | j_\nu^V(0) | pp'\rangle)^*}{(p-p')^2} 2\sqrt{\omega_p\omega_{p'}} \quad (6)$$

on the upper edge of the cut and

$$F(-\omega_{p'}) = b_2(q^2) = \frac{1}{e}\frac{(p'-p)_\mu \langle pp' | j_\mu^V(0) | 0\rangle}{(p-p')^2} 2\sqrt{\omega_p\omega_{p'}} \quad (7)$$

on the lower edge $(q^2 = (p+p')^2 \leq -4\mu^2)$; $b_1(q^2)$ is the "form-factor" for the annihilation of a pair of π mesons by the electromagnetic interaction, and $b_2(q^2)$ is the corresponding "form-factor" for the production of a pair of π mesons.

Since $F(\omega_{p'})$ is an analytic function in the cut $\omega_{p'}$ plane and is real on the part of the real axis from μ to ∞, we have by the principle of symmetry for analytic functions

$$b_1(\omega_{p'}) = b_2^*(\omega_{p'}) = b(\omega_{p'}). \quad (8)$$

In the coordinate system in which $\mathbf{p} = 0$, $q^2 = 2\mu q_0 = 2\mu(\omega_{p'} - \mu)$; that is, $F(\omega_{p'})$ is also an analytic function with respect to the variable q^2. In this system we can write the dispersion relations in terms of q^2, and because of relativistic invariance they will be independent of the choice of coordinate system.

Nothing definite can be said about the behavior of $a(q^2)$ and $b(q^2)$ for $|q^2| \to \infty$. It can be hoped that at high energies the cross-sections fall off with increasing q^2 more rapidly than q^{-2}. Then $a(q^2)$ and $b(q^2)$ fall off for $|q^2| \to \infty$, and

$$a(q^2) = -\frac{1}{\pi}\int_{4\mu^2}^{\infty} \frac{\operatorname{Im} b(-\xi^2)\,d\xi^2}{\xi^2 + q^2}, \quad q^2 > 0, \quad (9)$$

$$\operatorname{Re} b(q^2) = -\frac{1}{\pi} P\int_{4\mu^2}^{\infty} \frac{\operatorname{Im} b(-\xi^2)\,d\xi^2}{\xi^2 + q^2}, \quad q^2 < -4\mu^2. \quad (10)$$

If $a(q^2)$ and $b(q^2)$ approach constant values or increase for $|q^2| \to \infty$, dispersion relations can be written if we divide a and b by a certain power of q^2. If we confine ourselves to the contribution to $\operatorname{Im} b(q^2)$ from two π mesons only, we get from the unitarity relations for the S matrix

$$\operatorname{Im} b(q^2) = \pm |b(q^2)| \sin \delta(q^2). \quad (11)$$

δ is the pion-pion scattering phase shift for the state with angular momentum $l = 1$ and isotopic spin $T = 1$ (the $\pm$ sign remains undetermined.

The formula (11) is an exact relations for $q^2 \le -16\mu^2$. If $|b(q^2)|$ falls off rapidly with increasing q^2 we can substitute Eq. (11) in the right members of Eqs. (9) and (10); this gives the relation

$$a(q^2) = \pm\frac{1}{\pi}\int_{4\mu^2}^{\infty} \frac{|b(-\xi^2)| \sin \delta(\sqrt{\xi^2 - 4\mu^2})\,d\xi^2}{\xi^2 + q^2}. \quad (12)$$

between physical quantities.

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[1] J. Bernstein and M. L. Goldberger, Report at the Stanford Conference on the Sizes of Nuclei, 1957.

[2] Lehmann, Symanzik, and Zimmermann, Nuovo cimento **1**, 205 (1955).

[3] M. L. Goldberger, Phys. Rev. **99**, 979 (1955).

[4] N. N. Bogolyubov and D. V. Shirkov, **Введение в теорию квантованных полей** (Introduction to the Theory of Quantized Fields), GITTL 1957, pp. 405-407.

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