INVESTIGATION OF THE AVERAGE NUCLEAR POTENTIAL PARAMETERS

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The parameters of the average nuclear potential have been found on the basis of data on the levels of nuclei with doubly-closed shells plus or minus one nucleon. It is shown that the potential parameters are the same for all nuclei lying on the nuclear stability curve. A formula has been derived for the depth of the potential for prescribed values of N and Z. An expansion of the nucleon functions in terms of spherical oscillator functions is considered.

THE wealth of experimental data leaves no doubt that there exists in the nucleus a self-consistent field that determines the approximately independent motion of the nucleons and the formation of nuclear shells. So far it has not been possible to derive the self-consistent field from the nucleon interactions. One is, therefore, limited to finding an average field close to the self-consistent field on the basis of the known experimental information.

A detailed knowledge of the form of the potential of this field is not very important for obtaining qualitative results. For example, to determine the level ordering, one may use an oscillator or square-well potential with the right choice of parameters. However, the correct level spacing cannot be obtained in this way.

On the other hand, it is known that the differential cross sections for the scattering of nucleons from nuclei agree with experiment only if the diffuseness of the potential at the boundary is taken into account.[1] The proton distribution in the nucleus has, according to the scattering experiments with fast electrons, a diffuse boundary, and so does the corresponding potential.[2] The analytic form of this potential, as proposed by Woods and Saxon,[3] is

$$V(r) = V_0/(1 + e^{\alpha(r-r_0)}), \qquad (1)$$

where $r_0 = R_0 A^{1/3} \times 10^{-13}$ cm is the nuclear radius; α is a parameter determining the diffuseness of the nuclear boundary (in units of 10^{13} cm^{-1}); V_0 is the depth of the potential at the center of the nucleus. To the potential (1) one must add a spin-orbit term of the form

$$-\lambda\left(\frac{\hbar}{2Mc}\right)^2 \frac{1}{r}\frac{\partial V(r)}{\partial r}(\mathbf{l}\cdot\mathbf{s}), \qquad (2)$$

where λ indicates the relative strengths of the nuclear spin-orbit interaction and the relativistic Thomas term. The potential, then, contains four parameters: α, λ, V_0, and R_0, which must be determined from the experimental data.

There already exists a series of papers devoted to the determination of the potential parameters from an analysis of various experiments.[4] It is, however, difficult to make the obtained parameter values agree with each other, since the dispersion of the results is so bad that it is even impossible to detect any regularity in it. In this connection, principal doubts have been raised as to the existence of a common potential for all nuclei or even for one and the same nucleus in different reactions.

We made the attempt to determine the average potential parameters for a number of nuclei distributed over the entire periodic system. If the parameters are subject to any variations, we give the physical reasons for these changes.

The determination of the parameters is based on the data on the ground and excited levels of nuclei. The lower levels are useful for our purposes, as their energies and spins are now known with good accuracy. We restrict ourselves to nuclei with doubly closed shells plus or minus one nucleon. In this way we are certain that the levels contain only one particle and that their energies are not altered by interactions between nucleons in unfilled shells nor by interactions with the core. To be sure, a nucleus with one odd nucleon will be weakly polarized; but this polarization will be so small that its effect on the position of the levels will not exceed 100 kev. It must also be kept in mind that the levels may deviate from their single particle position by a large measure for excitations of more than 2 Mev. We regard the excited states as real excitations of the last odd nucleon in the potential of type (1).

METHOD OF CALCULATION

The equation of motion for a neutron (or a "hole") in the potential $V(r)$ with spin-orbit coupling has the form

$$\left[\frac{\hat{p}^2}{2M} + V(r) - \lambda\left(\frac{\hbar}{2Mc}\right)^2 \frac{\mathbf{l}\cdot\mathbf{s}}{r}\frac{\partial V(r)}{\partial r}\right]\psi = E\psi. \quad (3)$$

In the case of a proton the additional Coulomb term

$$V_c = \begin{cases} [3/2 - 1/2\,(r/r_0)^2]\,(Z-1)\,e^2/r_0 & \text{for } r \leqslant r_0, \\ (Z-1)\,e^2/r & \text{for } r \geqslant r_0. \end{cases} \quad (4)$$

must be introduced. This choice of V_c is not entirely correct, since the charge radius is always smaller than the range of the nuclear forces. Furthermore, the proton density is not necessarily constant over the nucleus, but may fall off towards the center and may certainly have a diffuse boundary. The main effect of such refinements, which we shall not consider, would be a change in the depth of the proton potential by a few percent.

Among the four parameters to be determined, the radius R_0 is to be singled out as a constant. The linear substitution $R_0 \to R_0'$ has the following effect on Eq. (3) for neutrons: the equation remains invariant under the substitution

$$\alpha R_0 = \alpha' R_0', \quad (V_0 + E)\,R_0^2 = (V_0' + E')\,R_0'^2, \quad \lambda V_0 = \lambda' V_0', \quad (5)$$

For the eigenvalue E_i we have the following relation:

$$E_i(\alpha, \lambda, V_0, R_0) = E_i^{\text{exp}}, \quad (5a)$$

where E_i^{exp} is the experimental energy of the level.

The hypersurface (5a) is different for different levels of the same nucleus. The optimal parameters for all levels of a given element must be chosen such as to correspond to the point of intersection of all the hypersurfaces, if such exists. In the region under consideration, $1.2 \leq R_0 \leq 1.4$, these surfaces can be significantly different only if $|E_i/V_0| \sim 1$. In our case $|E_i/V_0| \ll 1$, so that the surfaces (5a) coincide within the limits of accuracy, and all R_0 are equivalent. Thus, for Pb^{207}, the parameter R_0 is varied along with the others, and a preference for any value of R_0 is impossible in the light of the foregoing considerations.

The most consistent way to obtain R_0 would be the following: from Eq. (3) with the potential (1) we could determine all the nucleon wave functions in the given nucleus; we then could calculate the density distribution of the nucleons, and determine R_0 by comparing the latter with experiment. Since we only calculated the wave function for one nucleon, we had to choose a different method. As usual, we assumed that $R_0 = 1.3$.*

The determination of the energy levels of the excited states of the nucleon in a given potential reduces mathematically to finding the eigenvalues of the second order differential equation

$$-\frac{\hbar^2}{2M}\frac{1}{r^2}\frac{d}{dr}\left[r^2\frac{dR(r)}{dr}\right] + \left[V + V_c - E + \frac{\hbar^2}{2M}\frac{l(l+1)}{r^2} - \lambda\frac{\hbar^2}{4M^2c^2}\frac{\mathbf{l}\cdot\mathbf{s}}{r}\frac{dV}{dr}\right]R(r) = 0, \quad (6)$$

where $R(r)$ is the radial part of the wave function.

For neutrons with $l = 0$, Eq. (6) reduces to the hypergeometric equation. The eigenvalues E are determined by a certain transcendental equation. In the general case it is not possible to express the solutions to (6) through special functions. The eigenfunctions and eigenvalues must then be found by numerical integration.

We chose the Runge-Kutta method for our calculations.[5] We used this method for the integration in the whole interval, except at $r \to 0$, where the potential is singular. Ordinary power expansions were used for the integration in the region $r \approx 0$. The eigenvalues were found in the following order. For a tentative energy value E the corresponding wave function was determined. If the function had k zeros in the interval $(0 \div +\infty)$, after which it grew beyond all limits, then $E_k < E < E_{k+1}$. Changing E gradually, we found the interval in which E lies. The error in E did not exceed 0.03 Mev. The eigenfunctions were, at each point, determined with a relative accuracy $\epsilon \leq 10^{-5}$; they were computed up to values of r at which $|R(r)| \leq 5 \times 10^{-3}$ of the last extremum.

By varying the potential parameters α, λ, and V_0 we obtained a series of eigenvalues for each level with given n, l. If three or more levels are known in the nucleus under consideration, it is possible to find the optimal values for all three parameters, for which the eigenvalues give the best agreement with the experimental data. The calculations showed that the level energy E_{nl} depends linearly on λ and V_0 with good accuracy. We therefore list, in Table I, only the values of the derivatives $\partial E/\partial\lambda$, $\partial E/\partial V_0$ for each nucleus. To determine the dependence of E_{nl} on α, we had to compute a sufficient number of points for the construction of the diagram (see, for example, Fig. 2). As a result of the interpolations, the accuracy of the calculations of the levels dropped to 0.1 and possibly 0.2 Mev for the levels with large

*If necessary, one can always convert the neutron levels to a different R_0 by making use of the relations (5).

TABLE I. Determination of the potential parameters from the experimental data on the energy levels*

$_{82}Pb^{207}_{125}$

α	1.05	1.15	1.4	1.15	Δ	E_b	$\frac{\partial E}{\partial V_0}$	$\frac{\partial E}{\partial \lambda}$
λ	39	39	39	28				
V_0, Mev	41.8	41.8	41.8	41.8				
$3p^1/_2$	—6.8	—6.8	—6.8	—7.0	—0.2	—6.8	—0.62	0.02
$2f^5/_2$	—7.0	—7.1	—7.4	—7.5	—0.1	—7.42	—0.66	0.04
$3p^3/_2$	—7.75	—7.75	—7.80	—7.6	0.2	—7.75	—0.66	—0.01
$1i^{13}/_2$	—6.9	—7.3	—8.4	—6.5	1.9	—8.41	—0.75	—0.07
$2f^7/_2$	—9.4	—9.6	—9.9	—9.3	—0.2	—9.13	—0.75	—0.025

$_{8}O^{15}_{7}$

α	1.00	1.15	1.25	1.15	Δ	E_b	$\frac{\partial E}{\partial V_0}$	$\frac{\partial E}{\partial \lambda}$
λ	44	47	47	33				
V_0, Mev	54.1	54.1	54.1	54.1				
$1p^1/_2$	—13.3	—13.3	—13.4	—13.3	—1.00	—13.3	—0.52	0.075
$1d^5/_2$	—8.0	— 8.0	— 7.9	—7.00	0.9	— 8.03	—0.44	—0.06
$1p^3/_2$	—18.8	—19.4	—19.7	—19.0	0.4	—19.44	—0.58	—0.038

$_{7}N^{15}_{8}$

α	1.00	1.125	1.25	1.15	Δ	E_b	$\frac{\partial E}{\partial V_0}$	$\frac{\partial E}{\partial \lambda}$
λ	41	42	47	34				
V_0, Mev	53.6	53.6	53.6	53.6				
$1p^1/_2$	—10.2	—10.2	—10.2	—10.9	—0.7	—10.2	—0.50	0.075
$1d^5/_2$	—5.05	— 4.9	— 4.9	— 4.4	0.5	— 4.9	—0.42	—0.055
$1p^3/_2$	—15.8	—16.2	—16.5	—16	0.6	—16.53	—0.60	—0.038

$_{82}Pb^{209}_{127}$

α	1.05	1.15	1.4	E_b	$\frac{\partial E}{\partial V_0}$	$\frac{\partial E}{\partial \lambda}$
λ	28	28	28			
V_0, Mev	42.6	42.6	42.6			
$2g^9/_2$	—3.8	—3.9	—4.1	—3.87	—0.66	—0.031
$2i^{13}/_2$	—2.7	—3.1	—4.2	—3.12	—0.66	0.075
$3d^5/_2$	—2.7	—2.5	—2.2	—2.31	—0.53	—0.011
$2g^7/_2$	—1.7	—1.6	—1.4	—1.85	—0.53	0.042
$3d^3/_2$	—1.6	—1.5	—1.2	—1.33	—0.46	0.015

*The first three columns give the values of the level energies and the corresponding parameters. The fourth column for the elements Pb^{207}, O^{15}, and N^{15} gives the level energies computed with parameter values α and λ, which are optimal for Pb^{209}, O^{17}, and F^{17}, respectively. The fifth column gives the difference, Δ, between these energies and the experimental values. The last three columns give E_b, the binding energy of the level, and $\partial E/\partial \lambda$ and $\partial E/\partial V_0$, the partial derivatives of the energy with respect to the parameters λ and V_0. The experimental decay schemes are shown in Fig. 1.

TABLE I (continued)

$_{20}Ca^{41}_{21}$

α	1.05	1.15	1.23	E_b	$\frac{\partial E}{\partial V_0}$	$\frac{\partial E}{\partial \lambda}$
λ	34	27	20			
V_0, Mev	49.3	49.6	50.0			
$1f_{7/2}$	—8.5	—8.5	—8.5	—8.4	—0.62	—0.08
$2p_{3/2}$	—6.5	—6.5	—6.5	—6.45	—0.48	—0.016
$2p_{1/2}$	—4.3	—4.5	—4.7		—0.39	0.028

$_{39}Y^{89}_{50}$

α	1.125	1.25	1.40	E_b	$\frac{\partial E}{\partial V_0}$	$\frac{\partial E}{\partial \lambda}$
λ	43	40	38			
V_0, Mev	54.5	53.9	53.2			
$2p_{1/2}$	—7.3	—7.3	—7.3	—7.3	—0.70	0.033
$1g_{9/2}$	—6.4	—6.4	—6.4	—6.39	—0.75	—0.09
$2p_{3/2}$	—9.7	—9.5	—9.4	—8.83	—0.75	—0.017

$_{21}Sc^{49}_{28}$

α	1.05	1.15	1.25	E_b	$\frac{\partial E}{\partial V_0}$	$\frac{\partial E}{\partial \lambda}$
λ	36	28	21			
V_0, Mev	56,4	56.6	56.8			
$1f_{7/2}$	—9.5	—9.5	—9.5	—9.5	—0.68	—0.09
$2p_{3/2}$	—6.4	—6.4	—6.4	—6.43	—0.58	—0.023
$1f_{5/2}$	—1.9	—3.4	—4.7		—0.58	0.13

$_{83}Bi^{209}_{126}$

α	1.0	1.15	1.25	E_b	$\frac{\partial E}{\partial V_0}$	$\frac{\partial E}{\partial \lambda}$
λ	18,5	25	28			
V_0, Mev	57.8	56.9	56.4			
$1h_{9/2}$	—3.72	—3.72	—3.72	—3.72	—0.77	0,075
$2f_{7/2}$	—2,81	—2,81	—2.81	—2,82	—0.77	—0,035

$_{8}O^{17}_{9}$

α	1.00	1.125	1.25	E_b	$\frac{\partial E}{\partial V_0}$	$\frac{\partial E}{\partial \lambda}$
λ	45	35	26			
V_0, Mev	43.0	44.2	45.0			
$1d_{5/2}$	—4.14	—4.14	—4.14	—4.14	—0,46	—0.0625
$2s_{1/2}$	—3.27	—3.27	—3.27	—3.27	—0.36	0

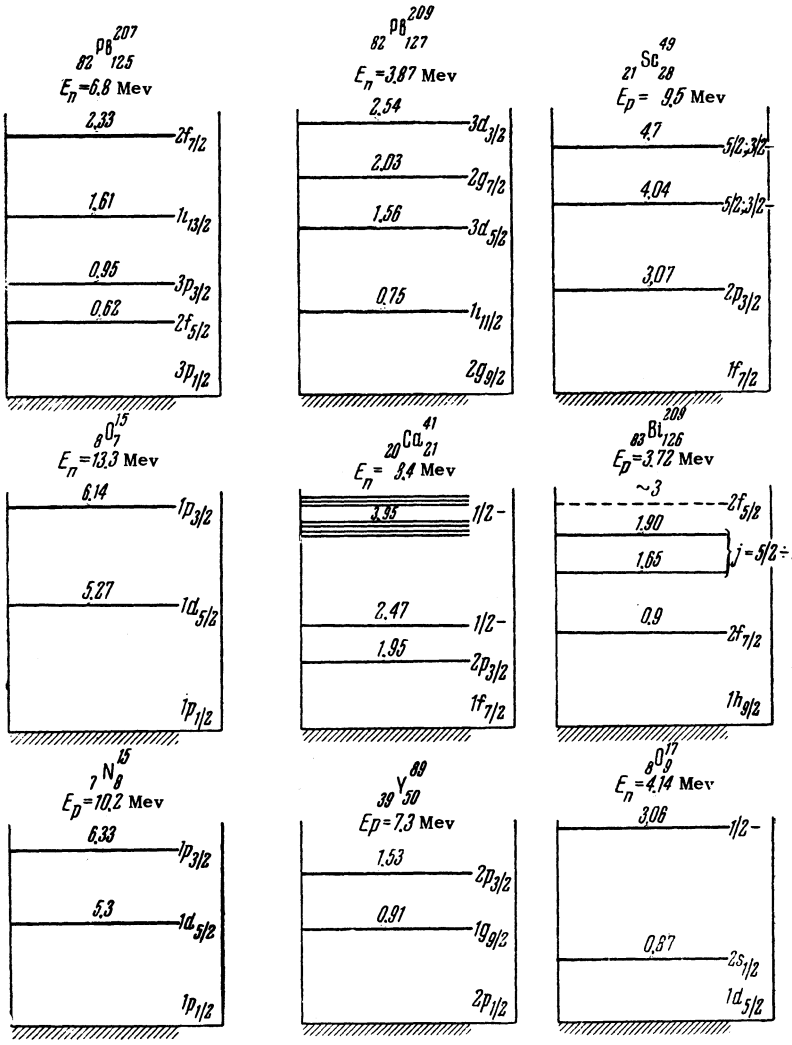

FIG. 1. Nuclear Decay Schemes (cf. Table I).

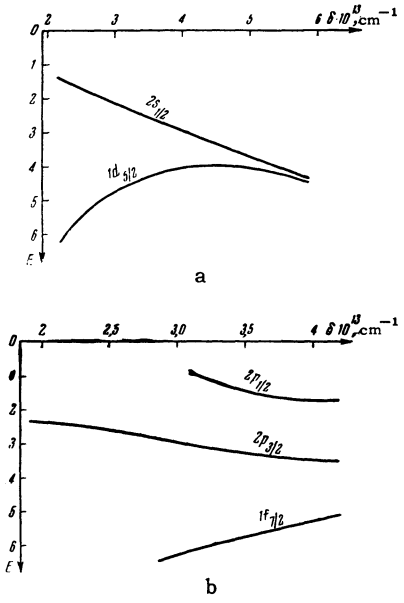

FIG. 2. Dependence of the level energy on the surface thickness $\delta = (2/\alpha)\ln g/\alpha$. (a) Levels of O^{17}: $2s_{1/2}$ and $1d_{5/2}$, (b) levels of Ca^{41}: $1f_{7/2}$, $2p_{3/2}$, $2p_{1/2}$. $V_0 = 42.8$ Mev, $\lambda = 42$, $R_0 = 1.3$.

l. We therefore determined the optimal choice of potential parameters for each nucleus separately.

RESULTS OF THE CALCULATIONS

$_{82}Pb^{209}_{127}$. The spectrum of Pb^{209} may serve as an example for a single particle neutron excitation. The levels of Pb^{209} were investigated in the reaction (d, p).[6] The level ordering found is in agreement with the shell model. The magnitude of the spin-orbit splitting conforms to the semi-empirical formula of Leventov.[7] Five levels were computed. The final results are listed in Table I. The best agreement with the experimental data is obtained for $\alpha = 1.15$, $\lambda = 28$, $V_0 = 42.6$ Mev. The deviation of the last level by 0.3 Mev may prove to be fictitious, as the position of this level is not known exactly. Even if the level energy is confirmed, one should keep in mind that possible deviations from the single particle character of the excitation may show up more strongly for such a high level.

$_{82}Pb^{207}_{125}$. The spectrum of Pb^{207} (reference 6) may serve as an example for a neutron "hole" excitation. The excited levels are the result of the transition of a neutron from a filled lower level to the ground level, which earlier contained one neutron. The energy of such an excited state will be given by the energy difference between the ground state and the lower level from which the neutron made its transition. With the experimental energies and spins of the excited and ground states it is possible to determine the energies and spins of the filled states. We must, however, assume that the binding energies of nucleon pairs is the same for all levels. Below we shall return to the question of a possible discrepancy in the pair energy and make estimates for it. The three best choices for the parameters are listed in Table I. For the first and second choice the level $1\,i_{13/2}$ should lie considerably higher. For the third set of parameters the first four levels agree well with experiment; only the last level, $2\,f_{7/2}$, lies below

the experimental value by 0.8 Mev. As in the case of Pb^{209}, the discrepancy in the last level may be explained by a deviation from the single particle state.

$_{20}Ca^{41}_{21}$. In contrast to the other nuclei with a single nucleon above the closed shells, Ca^{41} has a large group of levels following after the first excited level $2\,p_{3/2}$ (references 8, 9). The spins and measured capture cross sections of these levels indicate that they cannot be single particle levels, with the exception, perhaps, of the level $E = 3.95$ Mev, which may be identified with the $2\,p_{1/2}$ level. Using these three levels for the determination of the parameters, we obtain $\alpha = 1.15$, $\lambda = 27$, and $V_0 = 49.6$ (see Table I). If the level $E = 2.47$ Mev is taken to be the $2\,p_{1/2}$ state, we obtain the absurd value $\lambda \sim 5$. The value of V_0 for Ca^{41} is remarkably high: it is larger than in the Pb^{209} case by 7 Mev. For an explanation we consider the levels of Ca^{49}. Since 28 neutrons also form a shell (which, however, is less tight than the 20-neutron shell), one may expect that the levels of Ca^{49} are single particle levels. Two levels of Ca^{49} are known: the ground level with the neutron binding energy ~ 5 Mev and an excited level 2 Mev above the former. Using these data and setting $\alpha = 1.15$, we obtain for the two other parameters the values $V_0 = 43.4$ Mev and $\lambda = 22$. Since the number of neutrons per proton is higher in Ca^{49} than in Ca^{41}, there is reason for the assumption that the depth of the neutron potential depends on the relative number of protons in the nucleus.

$_{8}O^{17}_{9}$. The nucleus O^{17} has only two single particle levels, $1\,d_{5/2}$ and $2\,s_{1/2}$. The third level $(-\frac{1}{2})$ cannot be a single particle level, even if it would correspond to a hole. From the two levels we find $V_0 = 44.5$ and $\lambda = 33$, with α chosen to be $\alpha = 1.15$.

$_{8}O^{15}_{7}$. The first three levels were used for the determination of the parameters. The levels were identified on the basis of references 8 and 9. We found $\alpha = 1.15$, $\lambda = 47$, and $V_0 = 54.1$ Mev.

We now turn to the analysis of the single particle proton levels.

$_{83}Bi^{209}_{126}$ (see Table I). The first two levels are $1\,h_{9/2}$ and $2\,f_{7/2}$. The spins and the parities of the following levels are not exactly known. Therefore only the first two levels were used for the determination of the potential parameters. Taking $\alpha = 1.15$, we found $\lambda = 25$, $V_0 = 56.9$. A larger value for V_0 results if the Coulomb potential is taken into account, which has a height of about 15 Mev for Bi^{209}. With the above-mentioned parameters the energy of the level $2\,f_{5/2}$ can be computed; it is found to be 3 Mev (the levels $E = 1.65$ and $E = 1.9$ Mev should rather correspond to $1\,i_{13/2}$ and $3\,p_{3/2}$). It is very important to obtain more detailed information on the levels of Bi^{209}, before any conclusive statements can be made about the potential or even the single particle character of the levels themselves. We recall that the quadrupole moment, which is twice as large as in the single particle case, can be consistently explained by the weak polarization of the core. The anomalous magnetic moment can, however, not be interpreted in this fashion, which casts some doubt on the pure single particle character of even the ground level.

$_{81}Tl^{207}_{126}$. The ground level is $3\,s_{1/2}$, the excited level, representing a hole, is $2\,d_{3/2}$. The binding energy has not been measured exactly. The optimal parameters, for $\alpha = 1.375$, are $\lambda = 38$ and $V_0 = 55.8$ Mev.

$_{21}Sc^{49}_{28}$. As in the case of Ca^{49}, we may assume that the 28 neutrons form a closed shell and that the levels of Sc^{49} are possibly single particle levels. The first two levels are known to be $1\,f_{7/2}$ and $2\,p_{3/2}$. The spins of the other two levels are not known exactly. From the two levels we obtained the parameters $V_0 = 56.6$ and $\lambda = 29$. The third level, $1\,f_{5/2}$, will then have the binding energy $E_b = 4.7$ Mev, which almost coincides with the excited level $E = 4.7$ Mev $(\frac{5}{2}, \frac{3}{2}^-)$.

$_{21}Sc^{41}_{20}$. This nucleus is radioactive. The binding energy of the proton is small (1.62 Mev). The great number of relatively close lying levels points to their non-single particle character. If one can speak of a single-particle level at all, it is only of the ground level. The ground level has led only to the determination of the depth $V_0 = 49$ Mev.

$_{9}F^{17}_{8}$. Only the two bound states $1\,d_{5/2}$ and $2\,s_{1/2}$ were taken into account. Taking $\alpha = 1.15$, we obtained $\lambda = 34$ and $V_0 = 53.6$ Mev.

$_{7}N^{15}_{8}$. In analogy to the case of O^{15}, we obtained the optimal values for all three parameters from the first three levels: $\alpha = 1.25$, $\lambda = 44$, $V_0 = 53.6$ Mev.

To investigate the proton shell "40", we scanned the levels of the nuclei $_{39}Y^{89}_{50}$, $_{40}Zr^{89}_{49}$, $_{40}Zr^{91}_{51}$, and $_{41}Nb^{91}_{50}$. The ground and first excited levels of Y^{89} led to reasonable parameters for the potential (with $\alpha = 1.40$): $\lambda = 38$, $V_0 = 53.2$ Mev. The next excited level, however, does not fit in any more. All excited levels of Nb^{91}, Zr^{89}, and Zr^{91} must be assumed to have lost their single-particle character. Otherwise anomalous parameters would be obtained (for example, $\lambda \sim 60$).

Considering only the ground states to be single particle states and fixing α and λ, we obtain

TABLE II. Optimal parameters α, λ of the potential V(r) for which the energies of the ground and low excited single particle levels coincide with the experimental values

Nucleus with an "extra" nucleon			Nucleus with a "hole"		
	α	λ		α	λ
$_8O_9^{17}$	1.15	33	$_7N_8^{15}$	1.25	44
$_9F_8^{17}$	1.15	34	$_8O_7^{15}$	1.15	47
$_{20}Ca_{21}^{41}$	1.15	27	$_{40}Zr_{49}^{89}$	1.375	38
$_{21}Sc_{20}^{41}$	1.15	27	$_{39}Y_{50}^{89}$	1.40	38
$_{20}Ca_{29}^{49}$	1.15	22	$_{81}Tl_{126}^{207}$	1.375	38
$_{21}Sc_{28}^{49}$	1.15	28	$_{82}Pb_{125}^{207}$	1.4	39
$_{41}Zr_{50}^{91}$	1.15	27			
$_{41}Nb_{50}^{91}$	1.15	27			
$_{82}Pb_{127}^{209}$	1.15	28			
$_{83}Bi_{126}^{209}$	1.15	25			

$Nb^{91}: V_0 = 53.6$; $Zr^{91}: V_0 = 46.2$; $Zr^{89}: V_0 = 43.2$.

With all these data, 40 nucleons do not form a shell, but a subshell, which is sufficiently tight only for the ground state of the odd nucleon.

DISCUSSION OF THE RESULTS OF THE CALCULATION OF THE NUCLEAR POTENTIAL PARAMETERS

In Table II we collect the results of the computation of the parameters α and λ for nuclei with closed shells plus or minus a single nucleon. We call attention to the fact that these parameters remain constant for the elements from O^{15} to Pb^{209}. The average value of α is 1.15, the average value of λ is 28. The somewhat larger fluctuations of λ are, in part, due to the fact that we fixed α in a number of cases, so that λ also includes any possible fluctuations in α. Consequently, the observed deviations of α and λ from the average do not exceed 10%.

The foregoing refers to nuclei with one extra nucleon. For nuclei with one missing nucleon, we have, on the average, $\alpha = 1.4$ and $\lambda = 40$. To investigate the reasons for the increase in the values of the parameters, we computed the energies of the hole levels, say for Pb^{207}, using the values of α and λ taken from Pb^{209} (cf. the fifth column in Table I). The resulting deviations of the calculated levels from the experimental ones do not exceed 0.2 Mev, where the deviations go in both directions. An exception is the level $1\,i_{13/2}$, for which the discrepancy reaches 2 Mev. We recall that, in the calculation of the hole levels, we have assumed that the pair binding energy is the same for different levels. In reality this is, of course, not the case, and the discrepancies can therefore be viewed as a criterion for the validity of this assumption. In the cases of O^{15} and N^{15} the difference in the pair binding energy for different levels, Δ, is greater than or equal to ~ 0.6 Mev. This increase was to be expected, for the pair energy, as well as the fluctuations in it, is larger for light nuclei than for heavy nuclei.

TABLE III. Depth of the potential V_0

Nucleus with an odd neutron				Nucleus with an odd proton				
Element	ΔZ	V_0, calculated	V_0 according to (8)	Element	ΔN	V_c	V_0, calculated	V_0 according to (8)
$_8O_7^{15}$	2.0	54.1	54.6	$_7N_8^{15}$	1.0	2.7	53.6	52.0
$_8O_9^{17}$	0	44.6	44	$_9F_8^{17}$	—1.0	3.6	43.4	42.9
$_{20}Ca_{21}^{41}$	2.2	49.6	48.3	$_{21}Sc_{28}^{49}$	2.6	6.0	56.6	55.3
$_{40}Zr_{49}^{89}$	2.2	43.2	46.0	$_{39}Y_{50}^{89}$	—0.2	9.4	53.2	53.2
$_{40}Zr_{51}^{91}$	0.6	46.2	44.5	$_{41}Nb_{50}^{91}$	—2.6	9.8	53.6	52
$_{82}Pb_{125}^{207}$	—0.8	41.8	43.7	$_{81}Tl_{126}^{207}$	2.8	14.9	55.8	60.0
$_{82}Pb_{127}^{209}$	2.2	42.6	43.2	$_{83}Bi_{126}^{209}$	0.2	15.3	56.9	59.3

We may thus consider the parameters α and λ to be identical for all nuclei, with the values $\alpha = 1.15$ and $\lambda = 28$, whereas the observed discrepancies for the hole levels are to be regarded as a numerical measure of the difference in the pair energies at different levels.

For the depth of the potential, V_0, the picture is more complicated. In Table III we list the results of the calculation of V_0 for 16 nuclei. First of all, we note the approximate equality of V_0 for O^{17}, F^{17}, and Pb^{209}. For O^{15} and N^{15}, on the other hand, the potential increases by ~ 10 Mev, and for Ca^{41}, by 6 Mev. This increase in V_0 cannot possibly be explained by a variation in the radius, for then one would have to set $R_0 \approx 1.0$ instead of $R_0 = 1.3$, for the case of O^{15}. Better founded is the assumption that the neutron potential depends on the relative number of protons, and the proton potential depends on the relative number of neutrons in the nucleus. The depth of the potential is the same for all those nuclei in which the number of neutrons and the number of protons are in statistical equilibrium. These nuclei lie on the stability curve, which may be obtained from the semi-empirical mass formula[10] by setting $(\partial M/\partial Z)_A = 0$:

$$A = 2Z + 0.0146\, A^{2/3} Z. \qquad (7)$$

The values of N and Z for the nuclei on the stability curve correspond to the stable neutron and proton numbers. We denote them by N_{st} and Z_{st}. A deviation of N or Z from their stable values

leads to a change in the depth of the potential V_0. The following semi-empirical formula may be written down for V_0:

$$V_0^n = V_0^{stab} + (a/A)(Z - Z_{st}). \quad (8)$$

The proton potential will be deeper than the neutron potential by the Coulomb energy:

$$V_0^p = V_0^{stab} + (a/A)(N - N_{st}) + V_c. \quad (8a)$$

In Table III we list the values of $\Delta Z = Z - Z_{St}$ and $\Delta N = N - N_{St}$ obtained with formula (7). The experimental values of ΔZ and ΔN were used only for the lightest nuclei, $A = 15$ and $A = 17$, since it is known that (7) yields incorrect values for these nuclei. In the fourth and last columns of Table III we list the values of V_0 computed from (8) and (8a), with $V_0^{stab} = 44$ Mev and $a = 80$ Mev for both V_0^n and V_0^p. The agreement between the two values for V_0 is better for neutrons. The value of V_C entering into the proton potential was taken to be the value of the Coulomb energy of the last proton at the nuclear boundary $r = r_0$. To obtain $V_0 - V_C = 44$ Mev (with V_0 from (8a)) for Bi^{209}, as it should be, the value of V_C must be lowered by 3 Mev. On the other hand, we obtain a somewhat larger value for the average Coulomb energy, if we use the wave functions obtained for the ground state of Bi^{209}. As mentioned earlier, we did not treat the Coulomb energy in an entirely accurate way. Furthermore, it is not quite clear how one should properly compute the Coulomb energy from V_0. Despite these inaccuracies in V_C, formula (8a) gives satisfactory agreement within 4%.

The value $a = 80$ Mev indicates that each nucleon of a different kind above or below the stable number gives a contribution of $(80/A)$ Mev to the potential. This is twice as large as the average value for one nucleon $(44/A)$. This result is not surprising, since the nucleons interact with each other mainly over small distances, and the number of interacting states for nucleons of different kind is, on account of the Pauli principle, twice as large.

The second term in (7) indicates the excess of neutrons necessary for the compensation of the Coulomb potential in the nucleus with Z protons. The neutron excess is equal to the ratio $V_C/(a/A)$. Taking $V_C = (Z-1)e^2/r_0$ and $a = 80$ Mev, we obtain $V_C/(a/A) \approx 0.014 A^{2/3}(Z-1)$, which is in good agreement with the semi-empirical formula (7).

This analysis of the nuclear potential supports the hypothesis of the charge independence of nuclear forces.

TABLE IV. Depth of the neutron potential V_0 for the isotopes of Ca

Element	ΔZ	V_0,* calculated	V_0,** according to (8)
$_{20}Ca^{41}_{21}$	2.2	49.6	48.3
$_{20}Ca^{43}_{23}$	0.6	46.6	45.1
$_{20}Ca^{45}_{25}$	—1.2	45.0	41.8
$_{20}Ca^{47}_{27}$	—3	42.5	38.9
$_{20}Ca^{49}_{29}$	—4.6	42.5	36.5

*Computed with the experimental binding energies of the last neutron.

**The depths V_0, calculated with formula (8).

Another point deserves attention. The depth of the potential for Pb^{207} was calculated to be less than for Pb^{209}, despite the fact that Pb^{207} lies closer to the equilibrium isotope Pb^{206}. The reason for this is to be sought in the difference between the experimental binding energy and the depth of the ground level of the odd nucleon. Usually these two quantities are equal. However, the binding energy also contains part of the energy of the other nucleons, which is given off as a result of the removal of the particle. In particular, the contraction of the nuclear radius $(A^{1/3} \rightarrow (A-1)^{1/3})$ leads to a raising of the levels of all other nucleons. The energy contributed by the other nucleons will be larger for the nucleus Pb^{207}, which has an unfilled neutron shell, than for the nucleus Pb^{209}, which has a tight core.

We have determined the effective potential for one-nucleon motions in the nucleus. However, if the nucleus contains several nucleons above the closed shells, the last nucleon will not be in a single-particle state, owing to the remaining interactions. This is clearly evident from the calculation of V_0 for the ground state of the isotopes of Ca. As the number of nucleons above the shell increases, the potential deviates more and more from the single particle potential of formula (8) in that it becomes deeper (cf. Table IV).

THE NUCLEON WAVE FUNCTIONS

In this section we consider the nucleon wave functions calculated with the average nuclear potential. These functions are expanded into a series in terms of spherical oscillator functions, in order to simplify the further computations. The wave equation for the wave functions with an average diffuse proton potential has the form

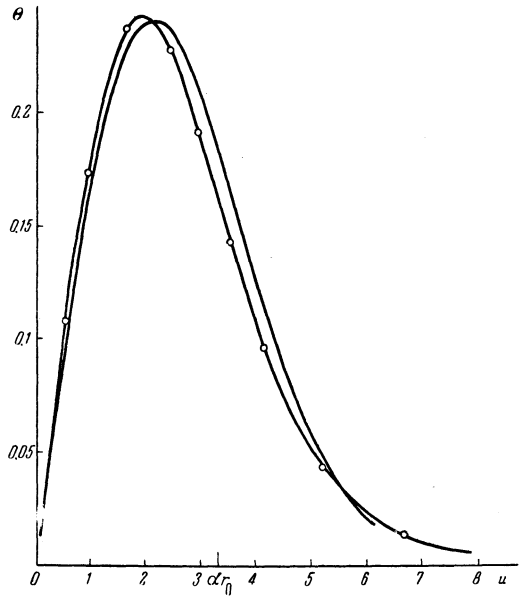

FIG. 3. Normalized neutron function for the $1p_{1/2}$ level of O^{15} (curve with points; $\alpha = 1 \times 10^{13}$ cm^{-1}, $\lambda = 25$, $V_0 = 43.8$ Mev, $R_0 = 1.3$) and normalized oscillator function $\Theta_1^1(u/u_0)$ ($u_0 = 1.5$).

$$-\frac{\hbar^2}{2M}\frac{1}{u^2}\frac{d}{du}\left[u^2\frac{dR^l(u)}{du}\right]+\left[\frac{V+V_c-E}{\alpha^2}+\frac{\hbar^2}{2M}\frac{l(l+1)}{u^2}\right.$$
$$\left.-\lambda\left(\frac{\hbar}{2Mc}\right)^2\frac{\mathbf{l}\cdot\mathbf{s}}{u}\frac{dV}{du}\right]R^l=0, \qquad (9)$$

where

$$V=-V_0/(1+e^{u-b}); \quad u=\alpha r;$$

$$b=\alpha r_0=1.3\alpha A^{1/3}\cdot 10^{-13}\ \text{cm}.$$

For the protons we have

$$V_c=\begin{cases}[3/2-1/2(u/b)^2]\,\alpha(Z-1)/137\,b & \text{for } u\leqslant b,\\ \alpha(Z-1)/137\,u & \text{for } u\geqslant b.\end{cases} \qquad (10)$$

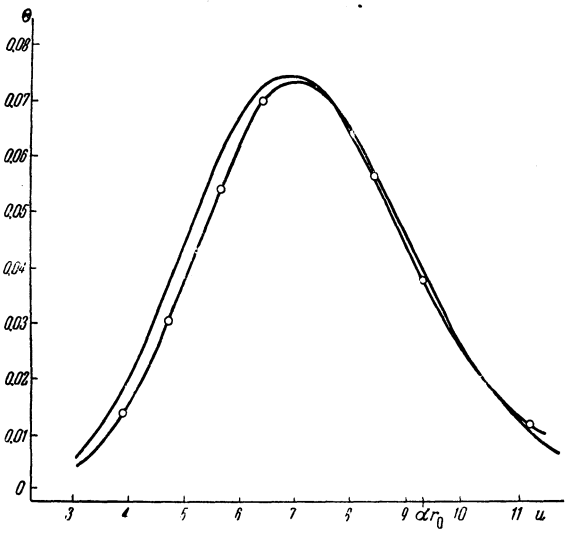

FIG. 4. Normalized neutron function for the $1i_{13/2}$ level of Pb^{209} (curve with points; $\alpha = 1.15 \times 10^{13}$ cm^{-1}, $\lambda = 28$, $V_0 = 42.6$ Mev, $R_0 = 1.3$) and normalized oscillator function $\Theta_1^6(u/u_0)$ ($u_0 = 2$).

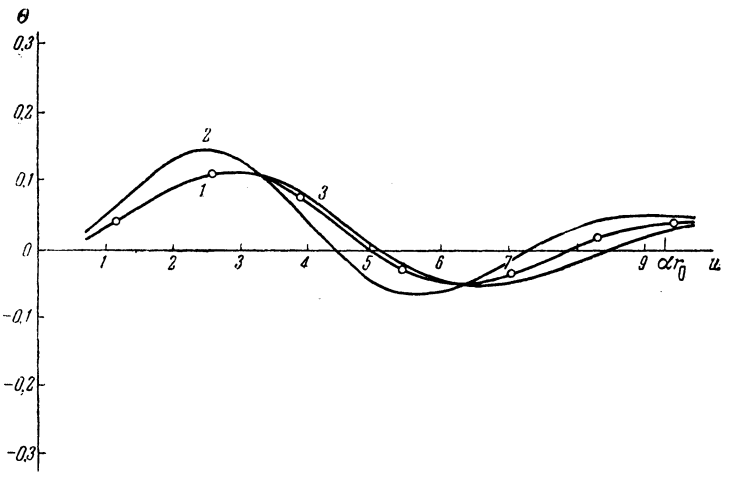

FIG. 5. Normalized function for the $3d_{5/2}$ level of Pb^{209} (curve 1; $\alpha = 1.15 \times 10^{13}$ cm^{-1}, $\lambda = 28$, $V_0 = 42.6$ Mev, $R_0 = 1.3$) and normalized oscillator function $\Theta_3^2(u/u_0)$ for two values of u_0 (curve 2: $u_0 = 2$, curve 3: $u_0 = 2.32$).

For neutrons, $V_C = 0$.

The normalized oscillator functions are

$$\Theta_n^l\left(\frac{u}{u_0}\right)=\frac{\exp[-1/2(u/u_0\sqrt{2})^2](u/u_0\sqrt{2})^l}{[u_0^3\sqrt{2}\,\Gamma(n)\,\Gamma(n+l+1/2)]\,2^{1/2}}L_{n-1}^{l+1/2}\left[\left(\frac{u}{u_0\sqrt{2}}\right)^2\right],$$

$$\int_0^\infty\Theta_n^l\left(\frac{u}{u_0}\right)\Theta_m^l\left(\frac{u}{u_0}\right)u^2du=\delta_{n,m},$$

$$n=1,\ 2,\ 3\ldots; \quad u_0=\alpha\sqrt{\hbar/2M\omega}. \qquad (11)$$

$L_n^p(x)$ is the Laguerre polynomial, which satisfies the relation

$$L_n^p(x)=x^{-p}e^x\frac{d^n}{dx^n}(e^{-x}x^{n+p}), \qquad (12)$$

$\hbar\omega$ is the distance between two neighboring levels. The normalized solution, $R^l(u)$, of Eq. (1), corresponding to the momentum l and k zeros in the interval $(0 < u < \infty)$, is expanded in terms of the corresponding functions $\Theta_n^l(u/u_0)$:

$$R^l(u)=\sum_{n=1}^{\infty}C_n\Theta_n^l(u/u_0), \qquad (13a)$$

$$C_n=\int_0^\infty R^l(u)\,\Theta_n^l(u/u_0)\,u^2\,du, \qquad (13b)$$

$$1=\int_0^\infty[R^l(u)]^2u^2\,du=\sum_{n=1}^{\infty}C_n^2. \qquad (13c)$$

The system of functions (11) is complete for arbitrary finite u_0, but the convergence properties of the series (13) depend on the choice of u_0. We chose u_0 such as to make the contribution of the function $\Theta_{k+1}^l(u/u_0)$ (which has k zeroes) to the normalization integral (13c) maximal. This means, graphically, that we superpose the first maxima of the functions $R^l(u)$ and $\Theta_{k+1}^l(u/u_0)$.

In this way the optimal scale will be different for different levels of the same nucleus. In Figs.

TABLE V. Dependence of the expansion coefficients C_i of the normalized neutron function for the 1 $p_{1/2}$ level of O^{15} on the scale u_0 (parameter values: $\alpha = 1.00$, $\lambda = 25$, $V_0 = 43.9$; energy $E = -9.4$)

u_0	1.0	1.5	2.0	5.0	10
C_1	0.8456	0.9832	0.9055	0.2821	0.130
C_2	—0.4416	0.0206	0.3573	0.3478	0.190
C_3	0.2352	0.0866	0.2019	0.3648	0.2350
C_4	—0.1376	—0.0133	0.0954	0.3600	0.2700
C_5	0.0866	0.0111	0.0488	0.3429	0.3000
C_6	—0.0555	—0.0067	0.0244	0.3210	0.320
C_7	0.0377	0.0020	0.0113	0.2967	0.335
C_8	—0.0266	—0.0016	0.0051	0.2724	0.350
C_9	0.0178	—0.0002	0.0024	0.2505	0.355
C_{10}	—0.0111	0.0005	0.0009	0.2286	0.365
C_{11}	0.0089	—0.0005	—0.0002	0.2067	0.370

3 and 4 we plot the functions corresponding to the levels 1 $p_{1/2}$ (O^{15}) and 1 $i_{11/2}$ (Pd^{209}), as calculated on the "Strela" computer, and the corresponding oscillator functions. The approximation turns out to be satisfactory. A slight change in u_0 would result in even better agreement. The vertical mark on the abscissa indicates the nuclear boundary r_0 in the given scale. Figure 5 shows the curve for 3 $d_{5/2}$ (Pb^{209}), as computed on the machine, and the oscillator function for two values of u_0. Here the agreement is already worse, especially in the "tail." This indicates that, independently of the choice of u_0, one cannot restrict oneself to only a single term in the expansion of functions $R^l(u)$ with several zeros according to formula (13a).

Table V shows the dependence of the coefficients C_n^l on u_0 for the 1 $p_{1/2}$ level of O^{15}.

CONCLUSIONS

The above calculations of the ground and low excited levels of 16 nuclei with doubly closed shells plus or minus one nucleon show that:

1. The average nuclear potential of the form (1) with the appropriate choice of parameters gives the correct level ordering, and, with good accuracy, the correct spacing.

2. The parameters α and λ, which determine the diffuseness of the boundary of the potential and the magnitude of the spin-orbit interaction, are approximately equal for all nuclei. This result is in agreement with the known data on the scattering of fast electrons from nuclei with spin-orbit splitting of the levels.

3. The depth of the potential, V_0, is the same for all nuclei lying on the stability curve. It is equal to ~ 44 Mev. For the other nuclei, V_0 is determined by formulas (8) and (8a).

4. The presence of several nucleons above closed shells leads to an increase in the depth of the potential over that given by (8) and (8a), owing to the residual interaction between the nucleons.

5. The investigation of the properties of nuclei with a proton shell $Z = 40$ shows that the excited levels of these nuclei are not single particle levels. This indicates that the shell "40" is less tight than the shells "20", "50", and others.

6. The wave function of the last odd nucleon is close to the corresponding oscillator function only in the first quantum states (1p, 1i, and similar levels). For these, a good approximation can be obtained with an appropriate choice of the scale u_0. Wave functions with several zeros (2p, 3p, and similar levels) cannot be satisfactorily approximated by a single oscillator function.

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[1] B. L. Cohen and R. V. Neidigh, Phys. Rev. **93**, 282 (1954), Beyster, Walt, and Salmi, Phys. Rev. **104**, 1319 (1956).

[2] Hahn, Ravenhall, and Hofstadter, Phys. Rev. **101**, 1131 (1956).

[3] R. D. Woods and D. S. Saxon, Phys. Rev. **95**, 577 (1954).

[4] R. M. Sternheimer, Phys. Rev. **97**, 1314 (1955). Ross, Mark, and Lawson, Phys. Rev. **102**, 1613 (1956) and **104**, 401 (1956). P. É. Nemirovskiĭ, J. Exptl. Theoret. Phys. (U.S.S.R.) **30**, 551 (1956) and **33**, 746 (1957), Soviet Phys. JETP **3**, 484 (1956) and **6**, 573 (1958).

[5] L. Collatz, Numerical Methods for the Solution of Differential Equations (Russ. Transl.), IIL, 1953.

[6] J. A. Harvey, Canad. J. Phys. **31**, 278 (1953).

[7] I. I. Levintov, Dokl. Akad. Nauk SSSR **107**, 240 (1956), Soviet Phys. "Doklady" **1**, 175 (1956). Doctoral Dissertation, 1957.

[8] R. U. Nussbaum, Revs. Modern Phys. **28**, 423 (1956).

[9] L. V. Groshev and A. M. Demidov, Атомная энергия (Atomic Energy) **3**, 91 (1957).

[10] C. F. v. Weizsaecker, Z. Physik **96**, 431 (1953).

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