RENORMALIZATION OF THE VERTEX PART IN PSEUDOSCALAR MESON THEORY

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The renormalization of the vertex part in pseudoscalar meson theory is investigated with the aid of the spectral representations of the vacuum average of the T-product of three Heisenberg operators proposed by Schwinger[1] and by Gribov.[2] The problem of the magnitude of the renormalization constants is discussed. An expression for Z_1 in terms of the spectral functions is obtained and the relation between these spectral functions and the spectral functions in the Källen-Lehmann representations for single particle Green's functions is established.

1. The spectral representations for the Green's functions proposed by Schwinger[1] and by the author[2] contain a number of the essential properties of these functions in a simple and clear form, and therefore they may turn out to be useful both for establishing different kinds of dispersion relations and for the further study of the structure of contemporary theory. In spite of the fact that the derivation of these representations based only on the conditions of causality and on the structure of the spectrum of the system has met with serious objections (Källen), the fact that they are valid in the case of perturbation theory suggests that they are actually correct (a more detailed discussion of this point will be given in a subsequent paper). However, it remains unclear as to what particular requirement must be imposed in order to limit the class of possible representations. One such requirement might be the requirement of renormalizability of the theory. In this paper it is shown that the representations referred to above satisfy these requirements.

With the aid of these representations it is possible to discuss in a simple way the problem of the magnitude of the renormalization constants, to obtain an expression for the constant Z_1 in terms of the spectral functions, and to establish the relation of these spectral functions with the spectral functions in the Källen-Lehmann representation for the single particle Green's functions.

2. We start from the following relation which is obtained by simple differentiation taking into account the renormalized field equations and the commutation relations:

$$\begin{gathered}\left(\gamma_\lambda \frac{\partial}{\partial x_{1\lambda}} + m\right)\left(\square_2 - \mu^2\right)\\ \times \langle 0 \mid T\psi(x_1)\,\varphi_i(x_2)\,\bar{\psi}(x_2) \mid 0\rangle \left(\gamma_\nu \frac{\overleftarrow{\partial}}{\partial x_{3\nu}} - m\right)\\ = \langle 0 \mid Tu(x_1)\,j_i(x_2)\,\bar{u}(x_3) \mid 0\rangle + gZ_1Z_2^{-1}Z_3^{-1}\tau_i\gamma_5\,\delta(x_1 - x_2)\\ \times \langle 0 \mid T\psi(x_1)\bar{\psi}(x_3) \mid 0\rangle \left(\gamma_\nu \frac{\overleftarrow{\partial}}{\partial x_{3\nu}} - m\right)\\ - gZ_1Z_2^{-1}Z_3^{-1}(\gamma_\lambda\partial/\partial x_{1\lambda} + m)\\ \times \langle 0 \mid T\psi(x_1)\bar{\psi}(x_3) \mid 0\rangle\,\tau_i\gamma_5\,\delta(x_2' - x_3)\\ + gZ_1Z_2^{-2}\gamma_5\tau_i(\square_2 - \mu^2)\langle 0 \mid T\varphi_j(x_1)\,\varphi_i(x_2) \mid 0\rangle\\ - 2ig\gamma_5\tau_i Z_1Z_2^{-2}Z_3^{-1}\delta(x_1 - x_2)\,\delta(x_2 - x_3);\\ u(x) = (\gamma_\mu\partial/\partial x_\mu + m)\,\psi(x); \qquad j_i(x) = (\square - \mu^2)\,\varphi_i(x). \quad (1)\end{gathered}$$

In the momentum representation this relation has the form:

$$\begin{gathered}ig\,(i\hat{p}_1 + m)\,G(\hat{p}_1)\,\tau_i\Gamma_5(\hat{p}_1, k, \hat{p}_3)\\ \times G(p_3)\,(i\hat{p}_3 + m)\,(k^2 + \mu^2)\,\Delta(k^2) = T_i(\hat{p}_1, k, \hat{p}_3)\\ + ig\,Z_1Z_2^{-1}Z_3^{-1}\gamma_5\tau_i\,G(\hat{p}_3)\,(i\hat{p}_3 + m) + ig\,Z_1Z_2^{-1}Z_3^{-1}\\ \times (i\hat{p}_1 + m)\,G(\hat{p}_1)\,\gamma_5\tau_i + ig\,Z_1Z_2^{-2}\gamma_5\tau_i(k^2 + \mu^2)\,\Delta(k^2)\\ - 2ig\,Z_1Z_2^{-2}Z_3^{-1}\gamma_5\tau_i; \qquad p_1 + k - p_3 = 0; \qquad (2)\end{gathered}$$

$G(\hat{p})$ and $\Delta(k^2)$ are the renormalized Green's functions; $T(\hat{p}_1, k_1, \hat{p}_3)$ is the Fourier component of the vacuum average of the T-product of the operators $u(x_1)\,j_i(x_2)\,\bar{u}(x_3)$. In reference 2 the following expression was obtained for this quantity

$$T_i(\hat{p}_1, k, \hat{p}_3)$$

$$= i\gamma_5 \tau_i \int_0^\infty d\varkappa_1^2\, d\varkappa_2^2\, d\varkappa_3^2 \int_0^1\int_0^1\int_0^1 d\alpha\, d\beta\, d\gamma\, \delta(\alpha + \beta + \gamma - 1)$$

$$\times \left\{ \frac{f_0(\varkappa_1^2, \varkappa_2^2, \varkappa_3^2) + (\gamma\hat{p}_1 + \alpha\hat{p}_3) f_1(\varkappa_1^2, \varkappa_2^2, \varkappa_3^2) + \beta(\hat{p}_1 + \hat{p}_3) f_2(\varkappa_1^2, \varkappa_2^2, \varkappa_3^2)}{p_1^2\beta\gamma + k^2\alpha\gamma + p_3^2\alpha\beta + \alpha\varkappa_1^2 + \beta\varkappa_2^2 + \gamma\varkappa_3^2 - i\varepsilon} \right.$$

$$\left. - \frac{(\beta/2i)[\hat{p}_1, \hat{p}_3] f_3(\varkappa_1^2, \varkappa_2^2, \varkappa_3^2)}{p_1^2\beta\gamma + k^2\alpha\gamma + p_3^2\alpha\beta + \alpha\varkappa_1^2 + \beta\varkappa_2^2 + \gamma\varkappa_3^2 - i\varepsilon} \right\}. \quad (3)$$

Here a small change in notation has been introduced, and an error made in reference 2 has been corrected. $f_i(\kappa_1^2, \kappa_2^2, \kappa_3^2)$ are real functions symmetric with respect to an interchange of κ_1^2 and κ_3^2, equal to zero if $\kappa_1 + \kappa_2 < m + \mu$ or $\kappa_2 + \kappa_3 < m + \mu$, or $\kappa_1 + \kappa_3 < 3\mu$. The integral over κ_1^2, κ_2^2, and κ_3^2 in (3) may be either convergent or divergent depending on whether the combinations of the renormalization constants appearing in (2) are finite or infinite, but the functions $f_i(\kappa_1^2, \kappa_2^2, \kappa_3^2)$ are necessarily finite. This follows from the fact that (cf. reference 2) they are simply related to the Fourier-component $\rho_{uj\bar{u}}(\hat{p}_1, \hat{p}_3)$ of the unordered product of the operators $u(x_1)$, $j_i(x_2)$, $\bar{u}(x_3)$. The latter differs from the Fourier component $\rho_{\psi\varphi\bar{\psi}}(\hat{p}_1, \hat{p}_3)$ of the average product of the operators $\psi(x_1)$, $\varphi_i(x_2)$, $\bar{\psi}(x_3)$, which is finite, only by the factor

$$\rho_{uj\bar{u}}(\hat{p}_1, \hat{p}_3)$$

$$= (i\hat{p}_1 + m)\rho_{\psi\varphi\bar{\psi}}(\hat{p}_1, \hat{p}_3)(i\hat{p}_3 + m)[(p_1 - p_3)^2 + \mu^2]. \quad (4)$$

If the normalizing constants are finite then the integral (3) must be convergent. From its convergence it follows (cf. the analogous discussion in reference 3) that, for example, for $\hat{p}_1 = im$, $\hat{p}_3 = im$ and $k^2 \to \infty$, $T_i(im, k^2, im) \to 0$. We then obtain under the conditions stated above

$$\Gamma_5(im, k^2, im) \to \gamma_5 Z_1 Z_2^{-1}(2 - Z_2^{-1}). \quad (5)$$

Having this asymptotic expression for Γ_5, it is easy to show by utilizing the Källen-Lehmann expression for Z_3^{-1}, that $Z_3^{-1} \to \infty$. This represents the substance of Källen's proof[3] that one of the renormalization constants is infinite. We shall not discuss the special case $Z_2^{-1} = 2$ (cf. reference 3).

If the renormalization constants themselves are infinite, but their combinations appearing in (2) are finite, then, since in this case $\Delta(k^2)$ falls off slower than $1/k^2$, we can neglect all the terms in (2) with the exception of the penultimate one, and obtain:

$$\Gamma_5(im, k^2, im) \to \gamma_5 Z_1 Z_2^{-2} \quad (6)$$

and, similarly,

$$\Gamma_5(\hat{p}_1, -\mu^2, im) \to \gamma_5 Z_1 Z_2^{-1} Z_3^{-1}. \quad (6a)$$

But, as has been shown by Lehmann, Symanzik, and Zimmerman,[4] in a consistent theory $\Gamma_5(im, k^2, im) \to 0$, $\Gamma_5(\hat{p}, -\mu^2, im) \to 0$ and, consequently, $Z_1Z_2^{-2} = 0$, $Z_1Z_2^{-1}Z_3^{-1} = 0$. But in this case those terms of the field equations which describe the interaction are equal to zero. Therefore, such a case is not very probable. On the other hand that case is most probable when the combinations of the constants appearing in (2) are infinite. In this case the integral (3) must diverge, and it is necessary to obtain a finite expression for the vertex part. To do this we first determine the constant Z_1 in terms of the functions $f_i(\kappa_1^2, \kappa_2^2, \kappa_3^2)$. We define Z_1 by the condition

$$\Gamma_5(im, -\mu^2, im) = \gamma_5. \quad (7)$$

This condition corresponds to the determination of the coupling constant by means of the dispersion relations. The fact that the conditions $\hat{p}_1 = \hat{p}_3 = im$, $k^2 = -\mu^2$ cannot be satisfied for real momenta does not lead to any difficulties since we have an explicitly analytic expression for $T_i(\hat{p}_1, k^2, \hat{p}_3)$.

It follows from (7) that

$$T_i(im, -\mu^2, im)$$

$$= ig\gamma_5\tau_i[1 - 2Z_1Z_2^{-1}Z_3^{-1} - Z_1Z_2^{-2} + 2Z_1Z_2^{-2}Z_3^{-1}]. \quad (8)$$

By utilizing this relation we can write the right hand side of (2) in the form:

$$T_i'(\hat{p}_1, k^2, \hat{p}_3) + ig Z_1 Z_2^{-1} Z_3^{-1}\gamma_5\tau_i[(i\hat{p}_1 + m)G(\hat{p}_1) - 1]$$
$$+ ig Z_1 Z_2^{-1} Z_3^{-1}[G(\hat{p}_3)(i\hat{p}_3 + m) - 1]\gamma_5\tau_i$$
$$+ ig Z_1 Z_2^{-2}\gamma_5\tau_i[(k^2 + \mu^2)\Delta(k^2) - 1] + ig\gamma_5\tau_i.$$

$$T_i'(\hat{p}_1, k^2, \hat{p}_3) = T_i(\hat{p}_1, k^2, \hat{p}_3) - T_i(im, -\mu^2, im). \quad (9)$$

However, T_i' is not yet, generally speaking, a convergent expression, since the subtraction of $T_i(im, -\mu^2, im)$ does not yet regularize even the first term in the integral of (3). Indeed, let us consider this first term in T_i', which contains the function $f_0(\kappa_1^2, \kappa_2^2, \kappa_3^2)$. It has the form:

$$\int d\varkappa_1^2\, d\varkappa_2^2\, d\varkappa_3^2\, f_0(\varkappa_1^2, \varkappa_2^2, \varkappa_3^2) \int d\alpha\, d\beta\, d\gamma$$

$$\times \frac{\delta(\alpha + \beta + \gamma - 1)[\alpha\beta(p_3^2 + m^2) + \alpha\gamma(k^2 + \mu^2) + \beta\gamma(p_1^2 + m^2)]}{[\alpha\beta p_3^2 + \beta\gamma p_1^2 + \alpha\gamma k^2 + \varkappa^2][-\alpha\beta m^2 - \alpha\gamma\mu^2 - \beta\gamma m^2 + \varkappa^2]},$$

$$\varkappa^2 = \alpha\varkappa_1^2 + \beta\varkappa_2^2 + \gamma\varkappa_3^2. \quad (10)$$

At first sight it might appear that the integral (10) cannot be divergent, since if the integral diverges then only large values of κ_1^2, κ_2^2 and κ_3^2 are of importance in it, and the terms containing p_1^2, p_3^2, and k^2 in the denominator may be neglected. Then the infinite part of this integral must be of the form

$$A_1(p_1^2 + m^2) + A_2(k^2 + \mu^2) + A_3(p_3^2 + m^2), \quad (11)$$

where A_1, A_2, and A_3 are infinite. But these terms could not cancel against other infinite terms in (9) since $\Delta(k^2)$, $G(\hat{p}_1)$, and $G(\hat{p}_3)$ fall off for large momenta. However, this conclusion is not correct since, for example, for large values of κ_2^2 in the integral over β the region of small $\beta \sim 1/\kappa_2^2$ is important. In this region the denominators are not large. But the size of the region is of order $1/\kappa_2^2$, so that the whole integral over β behaves like $1/\kappa_2^2$ for large values of κ_2^2. It is also important that for $\beta \sim 1/\kappa_2^2$ the integrand in (10) becomes dependent only on the momentum k^2.

However, from these arguments it follows that the divergence of the integral (10) may be due only to regions of correspondingly small β, α, and γ and that in order to regularize it, it is sufficient to subtract from the integrand its limiting values in these regions, viz. to subtract from the integrand the expression

$$\frac{\alpha\beta\,(p_3^2+m^2)}{(\alpha\beta p_3^2+\varkappa^2)(-\alpha\beta m^2+\varkappa^2)} + \frac{\alpha\gamma\,(k^2+\mu^2)}{(\alpha\gamma k^2+\varkappa^2)(-\alpha\gamma\mu^2+\varkappa^2)} + \frac{\beta\gamma\,(p_1^2+m^2)}{(\beta\gamma p_1^2+\varkappa^2)(-\beta\gamma m^2+\varkappa^2)}. \tag{12}$$

After this subtraction the integral (10) will be convergent, and it now remains only to discuss the divergences produced by the integration of expression (12). Since each of the terms of (12) depends on only one of the momenta there exists a possibility for its infinite parts to cancel against the infinite parts contained in the remaining terms of (9).

If we believe in the renormalizability of the theory, then we must conclude that such a cancellation does indeed take place. In principle we could terminate at this point our investigation of the representation (10), since after subtraction we obtain a representation of the contribution to the vertex part made by $f_0(\kappa_1^2, \kappa_2^2, \kappa_3^2)$ in the form of a convergent integral, plus finite terms which have the same analytic properties as the original integral. However, it is of interest to establish the requirements which are imposed on the function $f_0(\kappa_1^2, \kappa_2^2, \kappa_3^2)$ by the condition of renormalizability.

3. We consider the second term in (12) and carry out the integration over β and γ. We obtain

$$\int d\varkappa_1^2 d\varkappa_2^2 d\varkappa_3^2 f_0(\varkappa_1^2, \varkappa_2^2, \varkappa_3^2) \int_0^1 (1-\alpha)\frac{d\alpha}{\varkappa_2^2(1-\alpha)+\varkappa_1^2\alpha} \times \left\{ \ln\frac{\varkappa_1^2\alpha+\varkappa_3^2(1-\alpha)+\alpha(1-\alpha)k^2}{\varkappa_1^2\alpha+\varkappa_3^2(1-\alpha)-\alpha(1-\alpha)\mu^2} + \text{terms of order } \varkappa_2^{-4}\ln\varkappa_2^2 \right\}. \tag{13}$$

This integral diverges when the integration over κ_2^2 is carried out. This means that $f_0(\kappa_1^2, \kappa_2^2, \kappa_3^2)$ does not fall off sufficiently rapidly as $\kappa_2^2 \to \infty$. However, it cannot increase faster than κ^{2q}, where $q < 1$, since in this case the integral (10) would diverge when the integration over κ_2^2 is carried out not only for small values of β, and this would contradict the previous conclusions.

For the sake of simplicity we consider the case $f_0(\kappa_1^2, \kappa_2^2, \kappa_3^2) \to \varphi_0(\kappa_1^2, \kappa_3^2)$ as $\kappa_2^2 \to \infty$. It is simple to make generalizations to other possible cases. We consider that the integral containing $f_0(\kappa_1^2, \kappa_2^2, \kappa_3^2) - \varphi_0(\kappa_1^2, \kappa_3^2)$ in place of $f_0(\kappa_1^2, \kappa_2^2, \kappa_3^2)$ converges. Then after integrating over κ_2^2, and after introducing the cut-off limit λ, we shall obtain for the divergent part (13) the following expression:

$$\int d\varkappa_1^2 \int d\varkappa_3^2 \varphi_0(\varkappa_1^2, \varkappa_3^2) \times \int \ln\frac{\alpha\varkappa_1^2+(1-\alpha)\varkappa_3^2+\alpha(1-\alpha)k^2}{\alpha\varkappa_1^2+(1-\alpha)\varkappa_3^2-\alpha(1-\alpha)\mu^2} \ln\left(\frac{\lambda}{\varkappa_3^2}\,\frac{1-\alpha}{\alpha}\right) d\alpha. \tag{14}$$

In order that the infinite part of (14) should cancel against the fourth term in (9), it is necessary to have

$$\int d\varkappa_1^2 d\varkappa_3^2 \varphi_0(\varkappa_1^2, \varkappa_3^2) \int_0^1 d\alpha \ln\frac{\alpha\varkappa_1^2+(1-\alpha)\varkappa_3^2+\alpha(1-\alpha)k^2}{\alpha\varkappa_1^2+(1-\alpha)\varkappa_3^2-\alpha(1-\alpha)\mu^2} = C\,[(k^2+\mu^2)\,\Delta(k^2)-1], \tag{15}$$

where C is a finite constant.

In order that (15) should hold it is necessary, first, that its left hand side should increase slower than the first power of k^2. If $\varphi(\kappa_1^2, \kappa_3^2)$ is such that this holds, then by equating the imaginary parts on the right and on the left hand sides we easily obtain the relation between the function $\varphi_0(\kappa_1^2, \kappa_3^2)$ and the function $\kappa(\sigma^2)$ in the Kallen-Lehmann representation for $\Delta(k^2)$. If

$$\Delta(k^2) = \frac{1}{k^2+\mu^2} + \int_{9\mu^2}^{\infty} \frac{\sigma(\varkappa^2)\,d\varkappa^2}{k^2+\varkappa^2-i\varepsilon},$$

then after integration over α we get from (15)

$$C(\varkappa^2-\mu^2)\,\sigma(\varkappa^2) = \frac{1}{\varkappa^2}\int d\varkappa_1^2 d\varkappa_3^2\, \vartheta(\varkappa-\varkappa_1-\varkappa_3)\, S(\varkappa, \varkappa_1, \varkappa_3)\, \varphi_0(\varkappa_1^2, \varkappa_3^2), \tag{16}$$

where

$$S(\varkappa, \varkappa_1, \varkappa_3) = [(\varkappa-\varkappa_1-\varkappa_3)(\varkappa-\varkappa_1+\varkappa_3)(\varkappa+\varkappa_1-\varkappa_3)(\varkappa+\varkappa_1+\varkappa_3)]^{1/2},$$

$$\vartheta(x) = 1 \text{ for } x>0 \text{ and } \vartheta(x)=0 \text{ for } x<0. \tag{17}$$

We note that formula (16) is self consistent in the sense that in virtue of the condition $f_0(\kappa_1^2, \kappa_2^2, \kappa_3^2) = 0$ for $\kappa_1 + \kappa_3 < 3\mu$ the right hand side differs from zero only for $\kappa^2 > 9\mu^2$.

To obtain a similar relation between $f_i(\kappa_1^2, \kappa_2^2, \kappa_3^2)$ and the spectral functions in the representation for $G(\hat{p})$ it is necessary to investigate the remaining terms in (3) containing f_1, f_2, and f_3. These terms cannot give a contribution to (16) since they depend on the Dirac matrices, but they can give a contribution to the relation for $G(\hat{p})$.

Actually it is easy to see, by taking into account the fact that each of the infinite terms in (9) depends on only one of the momenta and increases slower than $\hat{p}_1$, $\hat{p}_3$, and k^2, that the integrals containing f_2 and f_3 cannot diverge at all, while the integral containing f_1 can diverge only as a result of integration over κ_1^2, or over κ_3^2. We write out the contribution to $T_i'(\hat{p}_1, k^2, \hat{p}_3)$ of the terms containing f_1 in a more detailed manner. It has the form:

$$\int d\varkappa_1^2\, d\varkappa_2^2\, d\varkappa_3^2\, f_1(\varkappa_1^2, \varkappa_2^2, \varkappa_3^2)$$
$$\times \int \frac{d\alpha\, d\beta\, d\gamma\, \delta(\alpha+\beta+\gamma-1)}{(\alpha\beta p_3^2 + \alpha\gamma k^2 + \beta\gamma p_1^2 + \varkappa^2)(-\alpha\beta m^2 - \alpha\gamma\mu^2 - \beta\gamma m^2 + \varkappa^2)}$$
$$\times \{[\gamma(i\hat{p}_1 + m) + \alpha(i\hat{p}_3 + m)]\varkappa^2 + m(\alpha+\gamma)(\alpha\beta p_3^2 + \alpha\gamma k^2 + \beta\gamma p_1^2)$$
$$-(\alpha\beta m^2 + \alpha\gamma\mu^2 + \beta\gamma m^2)(i\alpha\hat{p}_3 + i\gamma\hat{p}_1)\}. \quad (18)$$

Owing to the presence of κ^2 in the numerator the regions of small values of α, γ and of the values of α, γ of order of magnitude of unity give contributions, generally speaking, of the same order of magnitude to the divergent part of $T_i'(\hat{p}_1, k^2, \hat{p}_3)$. However, the contribution from the region in which α, γ are of the order of magnitude of unity behaves for large $\hat{p}_1$ or $\hat{p}_3$ as the first power of $\hat{p}_1$ or $\hat{p}_3$, and consequently cannot cancel against the remaining terms of (9) due to the falling off of $G(\hat{p})$ for large values of $\hat{p}$. Therefore the function $f_1(\kappa_1^2, \kappa_2^2, \kappa_3^2)$ must have such properties that the contribution of this region to the divergent part will be equal to zero. If again we make the simplest assumption that as $\kappa_1^2 \to \infty$ $f_0(\kappa_1^2, \kappa_2^2, \kappa_3^2) \to \varphi_0(\kappa_2^2\, \kappa_3^2)$. while $f_1(\kappa_1^2, \kappa_2^2, \kappa_3^2) \to \varphi_1(\kappa_2^2, \kappa_3^2)$, then the divergent part of $T_i'(\hat{p}_1, k^2, \hat{p}_3)$ which must cancel the second term in (9) may be written in the form

$$(i\hat{p}_1 + m)\int d\varkappa_1^2\, d\varkappa_2^2\, d\varkappa_3^2 \int \gamma\, d\alpha\, d\beta\, d\gamma\, \delta(\alpha+\beta+\gamma-1)$$
$$\times \frac{\beta(m - i\hat{p}_1)\varphi_0(\varkappa_2^2, \varkappa_3^2) + (\varkappa^2 - \beta\gamma i\hat{p}_1 m)\varphi_1(\varkappa_2^2, \varkappa_3^2)}{(\beta\gamma\hat{p}_1 + \varkappa^2)(-\beta\gamma m^2 + \varkappa^2)}. \quad (19)$$

Carrying out the integration in (19) first over α, β and then over κ_1^2 with the cut-off limit λ, we shall obtain for its infinite part the expression:

$$\int d\varkappa_2^2\, d\varkappa_3^2 \int_0^1 d\gamma \left\{ \left[\varphi_0(\varkappa_2^2, \varkappa_3^2) - \frac{i\hat{p}_1 m}{m - i\hat{p}_1}\varphi_1(\varkappa_2^2, \varkappa_3^2)\right] \ln\frac{\varkappa_2^2(1-\gamma) + \varkappa_3^2\gamma + p_1^2\gamma(1-\gamma)}{\varkappa_2^2(1-\gamma) + \varkappa_3^2\gamma - m^2\gamma(1-\gamma)} \cdot \ln\lambda + \varphi_1(\varkappa_2^2, \varkappa_3^2)\gamma(i\hat{p}_1 + m) \right.$$
$$\left. \times \left[\frac{1}{2}\ln^2\lambda(1-\gamma) - \frac{m^2\ln(\varkappa_2^2(1-\gamma) + \varkappa_3^2\gamma - m^2\gamma(1-\gamma)) + p_1^2\ln(\varkappa_2^2(1-\gamma) + \varkappa_3^2\gamma + p_1^2\gamma(1-\gamma))}{m^2 + p_1^2} \cdot \ln\lambda\right]\right\} \quad (20)$$

The condition given above for the slowness of increase for large values of $\hat{p}_1$ leads to the requirement

$$\int d\varkappa_2^2\, d\varkappa_3^2\, \varphi_1(\varkappa_2^2, \varkappa_3^2) = 0. \quad (21)$$

When (21) holds it is easy to obtain the relation between $\varphi_0(\kappa_2^2, \kappa_3^2)$ and $\varphi_1(\kappa_2^2, \kappa_3^2)$ and the functions $\sigma_1(\kappa^2)$ and $\sigma_2(\kappa^2)$ in the representation for $G(\hat{p})$. If

$$G(\hat{p}) = \frac{1}{i\hat{p} + m} + \int_{(m+\mu)^2}^{\infty} d\varkappa^2\, \frac{-i\hat{p}_1\sigma_1(\varkappa^2) + \sigma_2(\varkappa^2)}{p^2 + \varkappa^2 - i\varepsilon}, \quad (22)$$

then

$$\sigma_1(\varkappa^2) = \frac{1}{2\varkappa^4}\int d\varkappa_2^2\, d\varkappa_3^2\, \vartheta(\varkappa - \varkappa_2 - \varkappa_3)$$
$$S(\varkappa, \varkappa_2, \varkappa_3)(\varkappa_2^2 - \varkappa_3^2 - \varkappa^2)\varphi_1(\varkappa_2^2, \varkappa_3^2), \quad (23)$$

$$\sigma_2(\varkappa^2) = \frac{1}{2\varkappa^2}\int d\varkappa_2^2\, d\varkappa_3^2\, \vartheta(\varkappa - \varkappa_2 - \varkappa_3)$$
$$S(\varkappa, \varkappa_2, \varkappa_3)\varphi_0(\varkappa_2^2, \varkappa_3^2). \quad (24)$$

These relations are also self consistent in the sense that the right hand side differs from zero only for $\kappa^2 > (m+\mu)^2$.

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91