CONVERGING CYLINDRICAL DETONATION WAVE

Ya. B. ZEL' DOVICH

Submitted to JETP editor September 13, 1958

J. Exptl. Theoret. Phys. (U.S.S.R.) **36**, 782-792 (March, 1959)

The properties of detonation waves close to the normal detonation wave are considered. A theory is set up for the amplification of cylindrical converging detonation waves, which describes exactly the amplification in the initial stages of the process. By comparison with numerical calculations, it is shown that the theory remains satisfactory even for small radii and appreciable amplification of the wave.

1. INTRODUCTION

A converging detonation wave was first considered by L. D. Landau and K. P. Stanyukovich in 1944 (this research was reported in the well-known monograph of Stanyukovich,[1] pp. 567-574). The authors found the asymptotic law of the increase in pressure for the approximation of a spherical wave at the center or for a cylindrical wave on the axis. If the adiabatic exponent is $n = 3$, the pressure in this case increases as $r^{-1.13}$ in the spherical case and as $r^{-0.47}$ in the cylindrical case.

This research was noteworthy not only in its physical results but also in its methodology as an example of a power solution in which the exponent of the power was determined by the singular points of an ordinary differential equation and not by considerations of dimensionality. In the case of an unrestricted increase in the pressure, it is possible to neglect the chemical energy of the explosive material in comparison with the action of the pressure. Therefore, the theory developed by the authors is the same for converging detonation waves and for converging shock waves in a chemically inert substance.*

Neglect of the chemical energy is not only a consequence but also a premise of the theory: from dimensionality it follows that only in this approximation does the power solution satisfy the equations and we can go from the equation in partial derivatives to ordinary differential equations. Neglect of the chemical energy, which is valid in the last stage of convergence, is quite unsuitable at the beginning of the detonation of the charge of explosive material: at the beginning of the process, the pressure is entirely determined precisely by the chemical energy which is released upon the explosion.

The purpose of the present research was the approximate (asymptotic) treatment of just this initial stage of the process of convergence of the detonation wave. At the starting moment, in the excitation of the explosion on the external surface of the sphere or cylinder, a normal detonation wave appears which does not differ from a plane detonation wave. The normal detonation wave is essentially different from a shock wave primarily because the amplitude of the detonation wave is determined by the properties of the explosive substance and does not depend upon the method of generation, while the amplitude of the shock wave is entirely determined by the external action which the phenomenon of the shock wave brings about.

This independence of the normal shock wave of the external reaction is associated with the fundamental properties of the state achieved in a detonation wave; these properties are considered in Sec. 2.

Along with the convergence there begins an increase in the pressure at the front of the wave and the normal detonation wave is replaced by the so-called compressional detonation wave, in which the pressure is higher than the normal; the state in the compressional wave depends not only on the chemical energy but also on the state of the products of the explosion which are found at the front of the detonation. Experimentally, the existence of the compressional wave in gaseous detonation was first shown by B. V. Aĭvazov and the author.[3]

The reasons for the increase in pressure is the center- (or axially) directed motion of the products of the explosion behind the detonation wave front: the products of the explosion move

*The work of Stanyukovich and Landau was completed independently of similar work by Guderley[2] on the theory of converging shock waves in air.

in the direction of decreasing radius as though they were compressed and their compression is transferred to the wave front. In the limit of high pressure, the compressional detonation wave does not differ from a shock wave. However, we shall use the opposite limiting case, that is, we shall make use of the circumstance that at the beginning of the process the compressional wave differs slightly from the normal detonation wave. This circumstance, together with the fundamental properties of the normal detonation wave, allow us to develop a very effective approximate method of calculation of the pressure on the wave front, which is given in Sec. 3. This method is applicable, inasmuch as the amplitude is not too large; since the pressure in a cylindrical wave grows more slowly than in a spherical wave, the calculations are given for the cylindrical case where the range of applicability of the method is the greater. In Sec. 4 we give the numerical results of a calculation according to this method.

Finally, in Sec. 5, we compare our approximate method with the results of the numerical solution of the partial differential equation. The region of applicability is shown to be greater and the accuracy better than could be expected a priori: differences in the pressure, velocity, density do not exceed 10 per cent for convergence of the wave up to a radius equal to $1/25$ of the original, when the pressure itself increases three times in comparison with the original. All the calculations are carried out for the simplest case of an equation of state with $n = 3$; however, in reality, the assumed approximate method is applicable for any equation of state for the products of the explosion, since it is based upon very general properties of the detonation wave.

2. PROPERTIES OF A NORMAL DETONATION WAVE

It is well-known that for a detonation wave of explosive materials, the equations for the conservation of mass, momentum and energy can be satisfied for any pressure exceeding the pressure which is developed in the chemical reaction in the material at rest. To each value of the pressure there corresponds the definite density (the corresponding line in the plane p, $v = 1/\rho$ is known as the Hugoniot adiabatic), a definite velocity of motion of the products of the explosion, etc.

Of this series of states in the normal detonation there exists only one completely determined state (the corresponding point on the Hugoniot adiabat is known as the Jouguets point[4]). The fundamental property of this state is the condition

$$D_0 = u_0 + c_0, \tag{1}$$

where D is the propagation velocity of the detonation wave, u is the velocity of motion of the material, c is the velocity of sound. The 0 indicates the quantities are evaluated at the Jouguets point.

With the aid of thermodynamical relations, we can obtain the following properties of the Jouguets point and the properties of the Hugoniot adiabatic near the Jouguets point from Eq. (1):

$$(dD/dp)_{H,0} = 0. \tag{2}$$

The symbol H indicates that the derivative is taken along the Hugoniot adiabatic, the 0 indicates the Jouguets point; at this point D has a minimum. Further,

$$(dS/dp)_{H,0} = 0. \tag{3}$$

In Eq. (3), S is the entropy of the products of the explosion; the entropy also has a minimum at the Jouguets point on the Hugoniot adiabatic. Finally,

$$(du/dp)_{H,0} = 1/\rho c. \tag{4}$$

This latter property will be resolved in Sec. 3.

In the propagation of a plane detonation wave from the free boundary of explosive material there enters into the products of the explosion the so-called central rarefaction wave, in which the pressure and all other quantities depend upon the ratio x/t, where x is the coordinate measured from the original position of the boundary of the explosive material, t is the time from the beginning of the detonation (A. A. Grib[9], see also refs. 4, 5, and 13).

At the coordinates x, t for which the characteristics are determined, i.e., the lines on which

$$\alpha)\ dx/dt = u + c, \quad \beta)\ dx/dt = u - c, \tag{5}$$

the plane detonation wave corresponds to the picture of Fig.1: the α lines (solid) represent the set of straight lines diverging from the coordinate origin; the β lines (dash) are, in the general case the set of similar curves with the center of similarity at the origin of coordinates. In the special case where the adiabatic exponent is $n = 3$,

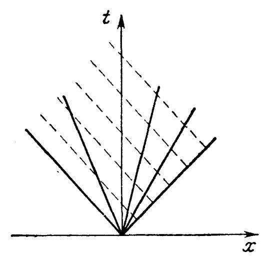

FIG. 1

the β lines form a family of parallel straight lines. This picture of the hydrodynamic dispersal of the explosive products behind the front of the detonation wave leads to the condition

$$D \geqslant u + c \tag{6}$$

for the detonation wave itself.

On the other hand, consideration of the chemical reaction of the conversion of the explosive material to products of the explosion leads to the condition

$$D \leqslant u + c \tag{7}$$

(this condition was first obtained in this manner by the authors [6] and later (independently) by Döring[7] and von Neumann[8]; see also the researches of Grib[9] and Abramovich and Vulis[10]). In the aggregate, (6) and (7) lead to the condition (1).

In what follows we shall consider the width of the zone of the chemical reaction to be very small and shall use only condition (7), which does not depend upon the absolute value of the reaction rate and the width of the zone. The impulse of pressure which produces (initiating) the detonation on the surface both above in the consideration of the non-trivial rarefaction wave and also below in Sec. 3, we shall consider small and shall not take into account.

3. CONVERGING CYLINDRICAL WAVE, EQUATIONS, AND PRINCIPLES OF ITS APPROXIMATE SOLUTION

We consider the detonation and the movement of products of the explosion soon after the time when the detonation was initiated at the moment $t = 0$, simultaneously on both lateral surfaces of a long cylindrical charge, $r = r_0$. Initially, although, for the path described by the wave, $Dt \ll r_0$, the process is evidently not different from the propagation of a plane wave and we shall deal with the normal detonation wave and a central rarefaction wave adjoining it (for further details see Fig. 2). In comparison with Fig.1, the only purely formal difference lies in the fact that instead of $x = 0$ we must set $r = r_0$ and reverse the direction: the detonation is propagated inside, in the direction of the decreasing r.

Along with the propagation of the detonation wave, a compressional wave develops; however, we limit ourselves to that region in which we can consider the wave as differing but slightly from the normal. Then, from the properties of (2), we conclude that we can regard

$$D = D_0 = \text{const.} \tag{8}$$

It follows from the property (3) that

$$S = S_0 = \text{const.} \tag{9}$$

Finally, we get from the property (4) that (on the wave front)

$$-u = -u_0 + \int_{p_0}^{p} \frac{dp}{\rho c}; \quad u + \int^{p} \frac{dp}{\rho c} = u_0 + \int^{p_0} \frac{dp}{\rho c} \tag{10}$$

(with account of the fact that the wave now moves from right to left; we change the side of the velocity on the front $u < 0$).

We now consider the motion of explosion products behind the front and shall make clear in what manner the approximations (8) — (10) simplify the calculation of the process.

The equations of motion, for cylindrical symmetry, take the form

$$\frac{du}{dt} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} = -\frac{1}{\rho}\frac{\partial p}{\partial r}; \quad \frac{dS}{dt} = \frac{\partial S}{\partial t} + u\frac{\partial S}{\partial r} = 0;$$

$$\frac{d\rho}{dt} = \rho\frac{\partial u}{\partial t} + u\frac{\partial \rho}{\partial t} = -\frac{\rho}{r}\frac{\partial}{\partial r} ru. \tag{11}$$

We must add to these the thermodynamical equation of state $p = p(\rho, s)$.

Since the entropy is conserved in each particle in the course of time, the entropy at the moment of completion of the chemical reaction in the approximation considered is also the same, according to (3) and (9), consequently the entire motion as a whole is isentropic, $S = S_0$ everywhere, and the pressure can be regarded as a function of the density alone.

It is known that Eqs. (11) can be written in terms of characteristics

$$\left.\frac{d\alpha}{dt}\right|_{dx=(u+c)\,dt} = -\frac{uc}{r}; \quad \left.\frac{d\beta}{dt}\right|_{dx=(u-c)\,dt} = \frac{uc}{r}, \tag{11'}$$

where

$$d\alpha = du + \frac{1}{\rho c}dp; \quad d\beta = du - \frac{1}{\rho c}dp. \tag{12}$$

Since the entropy is constant, we can then regard ρ and c as functions of a single pressure, and introduce

$$\varphi = \varphi(p) = \int \frac{dp}{\rho c}; \quad \alpha = u + \varphi; \quad \beta = u - \varphi. \tag{13}$$

it is appropriate to express all quantities in terms of α and β. For this purpose we must transform the dependence of $\varphi(p)$. We obtain

$$u = \frac{\alpha+\beta}{2}; \quad \varphi = \frac{\alpha-\beta}{2};$$
$$p = p(\varphi); \quad c = c(p) = c\left(\frac{\alpha-\beta}{2}\right). \tag{14}$$

For the known dependence of $p(\rho)$, the function $c(\alpha-\beta)/2$ is found to be elementary. As was noted by Stanyukovich,[11] an especially simple dependence is obtained for $p = \text{const}\,\rho^3$: in this case,

$$\varphi = c; \quad c = (\alpha - \beta)/2; \quad u + c = \alpha; \quad u - c = \beta. \qquad (15)$$

This case will be considered below in concrete calculations; however, we can also make some steps toward a general form for arbitrary dependence of $c(\varphi)$.

The picture of the motion under consideration in the plane r, t is shown in Fig. 2. The broad line represents the front of the detonation wave; its equation is $r = r_0 - Dt = r_0 - D_0 t$ in our approximation (D is the absolute value of the speed of the detonation). In comparison with Fig. 1, the roles of α- and β-characteristics are reversed.

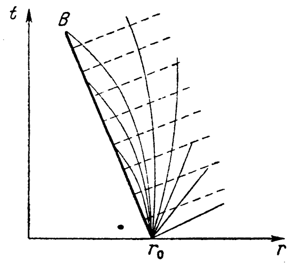

FIG. 2

The thin solid lines in Fig. 2 are the β-characteristics ($dr/dt = u - c$), while the dashed lines are the α-characteristics ($dr/dt = u + c$); in the region around the point $t = 0$, $r = r_0$, the motion is not different from a plane wave; we have a central wave and from this point there go out the rays of the β-characteristics. The α-characteristics go out from the wave front.

Equations (4) and (10) show that, close to the Jouguets point, when the wave has become compressed, in our approximation,

$$\alpha = u + \int dp/\rho c = \text{const} = \alpha_0; \qquad (16)$$

the invariant α remains constant on the wave front.

As a consequence of the fact that the motion is not plane but axially symmetric, there is a right hand part to Eq. (11′), and along the α-lines, the value of α does not remain constant; also the value of β is not a constant along the β-characteristic. Close to the front, $|\beta|$ increases with time, in accord with (11′) ($u < 0$, $\beta = u - \varphi < 0$, $d\beta/dt < 0$); therefore the velocity of propagation of the β-characteristic increases, and the β-lines, which were initially close to r_0 are propagated somewhat more slowly than the detonation wave; they then overtake it as is shown in Fig. 2. The growth law of the amplitude of the detonation wave depends on the β-characteristics intersecting it. We note that the fact of intersection is itself connected with the fact that for the compressed wave, $D < |u| + c$, which is quite permissible from the chemical point of view. Equation (1) which applies to the normal (non-compressional) wave cannot be differentiated along the Hugoniot adiabatic.

The value of β on the characteristic at the point of intersection of the β line with the front is determined by the previous motion of the product of the explosion. The value of β, together with the equations of conservation (the Hugoniot adiabatic), completely determines the parameters of the detonation wave. In this case, for $|\beta| > |_0$, we obtain a compressed detonation wave (the value of β_0 corresponds to the Jouguets point).

On the other hand the β line on which $|\beta| < |\beta_0|$ does not overtake, and does not intersect, the line on the detonation wave so that the states below the Jouguets point do not exist (for more details on this subject see reference 12).

As is seen from Fig. 2, the pressure on the wave front of interest to us is entirely determined in the r, t plane by the motion in a narrow segment bounded on the left by a detonation wave emanating from the point $t = 0$, r_0 at the point B, and the right, by the β-characteristic which joins these two points. The assumed approximation is that in this segment we have

$$\alpha = \alpha_0. \qquad (17)$$

As has already been pointed out, inside this segment, $d\alpha/dt \neq 0$ along the α-characteristic. However, $\alpha = \alpha_0$, $d\alpha/dt = 0$ along the detonation wave.

Since the segment is narrow, then the value of α for a finite derivative changes only slightly inside the segment.

For a given curvature of the β line, the width of the segment is proportional to $(r_0 - r)^2$, so that for small $r_0 - r$, for investigation of the initial period of convergence of a cylindrical wave, the assumption (17) appears asymptotically accurate, the terms thrown away in this case being of much higher order in smallness of $(r_0 - r)$ in comparison with those retained. Actually, thanks to a favorable numerical factor, the approximation (17) is applicable in practice for any r.

Because of (17), Eq. (11′) for the β-characteristics becomes an ordinary differential equation and is solved in elementary fashion. Actually, we can write it in the form

$$\left.\frac{d\beta}{dt}\right|_{dr=(u-c)dt} = (u - c)\left.\frac{d\beta}{dr}\right|_{dt=dr/(u-c)} = \frac{uc}{r}. \qquad (18)$$

In this equation u and c can always be expressed in terms of β and α. Moreover if, in accord with (17), α is constant, then we get

$$d\beta / dr = F(\beta, \alpha_0)/r; \tag{19}$$

The variables are separable and the equation can be integrated in elementary fashion.

In integrating the equations, we must put in the initial conditions; a whole pencil of β-characteristics emanate from the point $r = r_0$. We select one of these, giving the definite initial value $\beta = \beta_1$, at the point $r = r_0$, $t = 0$. We shall take α_0 to be constant in what follows.

Let us find $r(\beta, \beta_1)$ and then the equation of the β-characteristic in the plane $r-t$. Knowing the expressions u and c in terms of β (for a constant $\alpha = \alpha_0$), we write $1/(u-c) = \psi(\beta)$. Then

$$\frac{dt}{dr} = \frac{1}{u-c} = \psi(\beta);$$

$$dt = \psi(\beta)\,dr = \psi(\beta)\,\frac{dr(\beta, \beta_1)}{d\beta}\,d\beta. \tag{20}$$

Integrating the latter expression, we find $t = t(\beta, \beta_1)$. Thus the family of β characteristics is given by two parameters:

$$r = r(\beta, \beta_1); \quad t = t(\beta, \beta_1). \tag{21}$$

Varying β_1, we change from one characteristic to another; varying β for constant β_1, we move along the characteristic.

Now the amplitude of the detonation wave at a given radius r is found in simple manner: we substitute in (21) the value of r and the instant of time corresponding to it on the detonation wave, $t = (r_0 - r)/D_0$. Solving the two equations (21) for β_1 and β, we find the value of β on the wave front at the given radius. The value of β, together with the Hugoniot equations, determine the amplitude of the wave. The numerical side of the work is given in the following section.

On the theoretical side the treatment is connected with the singular acoustical properties of the normal detonation wave: the characteristic bearing the corresponding value of β is such that $(|\beta| < |\beta_0|)$ does not reach the detonation wave and then the wave "yields" the value $\alpha = \alpha_0$ and the entropy $S = S_0$, which depend only on the properties of the explosion material. The characteristic possessing such a β that $|\beta| > |\beta_0|$, reaches the detonation wave, transforms the normal wave into a compressional wave. In principle now, α and S, created by the wave, depend on the incoming β, but in first approximation (in the value of $p - p_0$ or $\beta - \beta_0$), the excitation carried by the β characteristic is not reflected from it as though completely absorbed by the detonation wave. This leads to the fact that in first approximation the values of S and α going out from the surface of the detonation wave do not change with β. In the following approximation it can be shown that $S - S_0$ and $\alpha - \alpha_0$ are proportional to $(\beta - \beta_0)^2$, i.e., are of a higher order of smallness.

4. NUMERICAL RESULTS OF THE APPROXIMATE METHOD

Let us consider the case of the equation of state

$$p = A\rho^3, \quad c = \sqrt{\partial p/\partial\rho} = \rho\sqrt{3A}, \quad \alpha = u + c, \quad \beta = u - c.$$

In this case, as is well-known,[11]

$$\rho_0 = {}^4/_3\,\rho_{00}, \quad D_0 = {}^4/_3\,c_0 = {}^4/_3\sqrt{3A}\,\rho_0,$$

$$c_0 = {}^3/_4\,D_0, \quad u_0 = {}^1/_4\,D_0, \quad \alpha_0 = {}^1/_2\,D_0, \quad \beta_0 = -D_0. \tag{22}$$

Here the initial density of the exploded material before the detonation is designated by ρ_{00}, in contrast to ρ_0 — the density of the products of the explosion at the Jouguets point. The equation in characteristics (11′) has the form

$$\left.\frac{d\beta}{dt}\right|_{dr/dt=\beta} = \beta \left.\frac{d\beta}{dr}\right|_{dt/dr=1/\beta} = \frac{\alpha^2 - \beta^2}{4r}. \tag{23}$$

Substituting $\alpha = \alpha_0$ = constant and the initial conditions $\beta = \beta_1$, $r = r_0$, we find

$$r = r_0\,(\beta_1^2 - \alpha_0^2)/(\beta^2 - \alpha_0^2), \tag{24}$$

$$t = -\frac{r_0(\beta_1^2 - \alpha_0^2)}{4\alpha_0^5}\,[\chi(\beta) - \chi(\beta_1)],$$

$$\chi = \frac{4\alpha_0^3\beta}{(\beta^2 - \alpha_0^2)^2} - \frac{6\alpha_0\beta}{\beta^2 - \alpha_0^2} + 3\ln\frac{\alpha_0 + \beta}{\alpha_0 - \beta}. \tag{25}$$

After $\beta'(r)$ on the front is found from (24), (25), and $t = (r_0 - r)/D$, we find, within the framework of our approximation,

$$u = (\alpha_0 + \beta')/2, \quad c = (\alpha_0 - \beta')/2,$$

$$\rho = \text{const}\cdot c, \quad p = \text{const}\cdot c^3 \tag{26}$$

(the numerical values of β'/D_0 and p'/p_0 as functions of r/r_0 are given in Table I)

TABLE I

$\frac{r}{r_0}$	$-\frac{\beta'}{D_0}$	$\frac{p'}{p_0}$	$\frac{r}{r_0}$	$-\frac{\beta'}{D_0}$
1	1.000	1.000	0.02	1.908
0.8	1.023	1.047	0.01	2.238
0.6	1.055	1.115		
0.4	1.108	1.230		
0.2	1.224	1.514		
0.15	1.283			
0.10	1.376			
0.08	1.435			
0.06	1.516			
0.04	1.646			

However, the condition $\alpha = \alpha_0$ on the front of the compressed wave is only approximate; therefore such p, ρ satisfy the conservation equations only with accuracy up to terms $(\beta' - \beta_0)^2$. To obtain internal consistency of the value u, ρ, p on the front, we must take the exact Hugoniot adiabatic, find $u(p)$, $\rho(p)$, $c(p)$, on it, construct $\beta(p) = u(p) - c(p)$ (we remember that $u < 0$, $\beta < 0$ for the case of motion toward the center) and for a given β' find p, ρ, u which satisfy the conservation equations exactly.

At the radius 0.2, even if the calculation of β' does not contain a large error, in each case it is no longer possible to make use of Eqs. (26) for the determination of u, c, ρ, p for a given β' (see Sec. 5).

For r/r_0 close to unity at the beginning of the process, it is not difficult to obtain expansion of the solution in a series. We find

$$-\frac{\beta'(r)}{D_0} = 1 + \frac{3}{32}\left(1 - \frac{r}{r_0}\right);$$

$$\frac{p'(r)}{p_0} = 1 + \frac{3}{16}\left(1 - \frac{r}{r_0}\right) = \left(\frac{r}{r_0}\right)^{-3/16}.$$

Thus, the rate of growth of the pressure at the beginning of the process close to the free surface of the charge is approximately twice as slow as the asymptotic rate, close to the axis, found by Landau and Stanyukovich ($r^{-0.188}$ instead of $r^{-0.47}$).

In the solution of Landau and Stanyukovich, the pressure at each moment has a maximum at a certain distance from the surface of the detonation wave (Fig. 3a); it can be thought that this maximum passes on its excess pressure to the wave front and is the reason and the necessary condition for the growth of pressure on the wave front in connection with the propagation.

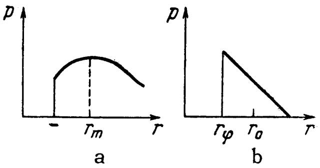

FIG. 3

In our solution, the pressure distribution at the beginning of propagation (Fig. 3b) differs but little from a plane wave; the pressure is maximum on the front and has a finite negative partial derivative with respect to the coordinates in the vicinity of the wave front. Nevertheless, as is seen from the result obtained above, the pressure on the front begins to increase rapidly along with the convergence of the wave, beginning with $r = r_0$, the increase is determined by the arrival at the front of the β-characteristics, which yield a value of β which increases in absolute magnitude; the quantity β depends not only on the pressure but also on the velocity of the material. Any estimate of the variation of the wave derived from a single distribution of pressure is invalid.

5. EXTRAPOLATION OF THE SOLUTION AND THE EQUATION OF STATE

It must be expected that the solution remains numerically satisfactory even there where it is impossible to consider the change of pressure on the front of the detonation wave ("compression") as small, and therefore the value of the entropy and the invariant α wil differ significantly from S_0 and α_0 at the Jouguets point.

The basis for such optimism lies in the fact that the approximation is contained in the replacement of α by α_0 in the expression $(\beta^2 - \alpha^2)/ur$. In this case, β^2 is initially four times α^2 and then even greater; in such a case, a significant relative change of α changes $\beta^2 - \alpha^2$ but slightly.

However, in the final stage of the calculation, in computing the pressure and other quantities corresponding to a given β, it is no longer possible on the wave front to make use of the simple relations (26), which refer only to the vicinity of the Jouguets point. One must construct an exact adiabat. Here there arises a question on the equation of state of the products of the explosion. Following Landau and Stanyukovich, we have assumed $p = A\rho^3$, where the quantity A is easily found from the velocity of a normal detonation.

Up to the present time, although isentropic processes of the expansion of the products of the explosion (slight supercompression) have been considered, nothing more has been required for the calculation of the hydrodynamic side of the problem, inasmuch as we are not especially interested in the temperature of the explosion products. However, for strong supercompressions that reach (in the approximation of the wave) to the axis of symmetry, is necessary for the hydrodynamical calculation to know not only the pressure but also the energy of the explosion products, in their dependence on density and entropy.

We can write the equation of state in the form

$$E = {}^1/_2 A_0 \rho^2 + E_{\text{therm}}; \quad p = A_0 \rho^3 + \omega E_{\text{therm}}\, \rho, \qquad (27)$$

where $A_0\rho^3$ is the elastic pressure, $A_0\rho^2/2$ is the corresponding elastic energy; we can calculate these as well as the pressure and energy at absolute zero. (Actually at absolute zero the pressure vanishes for a density ρ_C of the condensed products of the explosion (solid hydrocarbons, ice, etc.) which is close to 1 gm/gm^3, and it would be more accurate to write $\rho^3 - \rho_C^3$ in place of ρ^3. Inasmuch as we are interested in densities of 2 — 3 gm/cm³ and higher, we do not need to consider this correction). The second term E_{therm} represents the thermal energy, ω is a dimensionless constant.

It follows from thermodynamics that

$$p_{therm} = -\frac{\partial E_{therm}(\rho, S)}{\partial (1/\rho)} = \rho E_{therm} \frac{\partial \ln E_{therm}(\rho, S)}{\partial \ln \rho}, \quad (28)$$

in which the meaning of the dimensionless constant ω of (27) is determined. For a power-law dependence of E_{therm} on ρ, ω is constant.

For ideal gasses and constant heat capacity, $\omega = \kappa - 1$ where $\kappa = c_p/c_v$ is the adiabatic exponent. In our case, ω cannot be constant. Here we shall not enter into the physical theory of ω. In the consideration of the detonation, two limiting assumptions are possible, each of which simplifies the calculation considerably.

The first assumption $\omega = 0$:

$$E = A_0\rho^2/2 + F(S); \quad p = A_0\rho^3 \quad (29)$$

corresponds to a representation of a purely elastic pressure. The elastic term in the energy $A_0\rho^2/2$ corresponds to the elastic pressure; moreover, there is also a thermal term $F(S)$ in the energy; however the thermal pressure is equal to zero, since $\partial F/\partial \rho = 0$.

The second assumption, $\omega = 2$, leads to the equations

$$E = A(S)\rho^2/2; \quad p = A(S)\rho^3 = 2E\rho. \quad (30)$$

In this case there are elastic and thermal pressures, but the value of ω is specially chosen so that the adiabatics of the heated explosion products $(S \neq 0)$ have the same form functionally, $A \sim \rho^3$, as does the elastic pressure, $p = A(0)\rho^3$. In this case,

$$E_{therm} = {}^1/_2 [A(S) - A(0)]\rho^2;$$

$$p_{therm} = [A(S) - A(0)]\rho^3. \quad (31)$$

Estimates based on statistical mechanics show that the probable value of ω lies between zero and 2, so that the actual results lie comewhere in the region between the two assumptions (29) and (30).

It is easy for each case to construct the Hugoniot adiabatic of the detonation process. It is convenient to be given the magnitude of ρ; then, in the first case (29),

$$p = A_0\rho^3; \quad c = \rho\sqrt{3A_0}; \quad u = -\rho(A_0(\rho - \rho_{00})/\rho_{00})^{1/2},$$
$$D = \rho^2(A_0/\rho_{00}(\rho - \rho_{00}))^{1/2}. \quad (32)$$

the sign of the velocity u corresponds to the motion of the detonation toward the center.

From the energy equation we obtain

$$F(S) = Q - A_0\rho^2 + A_0\rho^3/2\rho_{00}, \quad (33)$$

where Q is the chemical energy of the exploded material.

The Jouguets point corresponds to $\rho_0 = {}^4/_3\rho_{00}$, all its properties are easily verified directly from Eqs. (32) — (33). Precisely in this case, the velocity of the normal detonation is directly proportional to the initial density

$$D_0 = \frac{16}{9}\rho_{00}\sqrt{3A_0}. \quad (34)$$

D and D_0 do not depend on the chemical energy, only the entropy S depends upon Q. The regime is possible only for $Q > {}^{16}/_{27} A_0\rho_{00}^2$. In the compression wave, the density can increase without limit. With the aid of (32) we find $\beta = u - c = \beta(\rho)$. In the previous section, $\beta(r)$ was found (Table I). From this it is easy to find $\rho(r)$ and $p(r)$, i.e., to bring to conclusion the approximate solution of the problem of the pressure of a converging cylindrical detonation wave without assumptions on the smallness of the compression. The results are tabulated in Table II. The pressure is given in column 2 and is found from our approximate theory. The pressure plotted in column 3 was found under the same assumptions on the equation of state by numerical integration in differences of the partial differential equations; integration was carried out with an electronic computer.

TABLE II

$\frac{r}{r_0}$	$\frac{p}{p_0}$	$\left(\frac{p}{p_0}\right)_{exact}$
1	2	3
1	1	1
0.8	1.05	1.05
0.6	1.11	1.11
0.4	1.23	1.24
0.2	1.51	1.54
0.15	1.67	1.71
0.1	1.94	2.01
0.08	2.12	2.20
0.06	2.40	2.49
0.04	2.86	3.00
0.02	3.96	4.18

The limiting law for $r \to 0$ was obtained in the following fashion: in accord with (24), for large β,

$$r \sim (\beta^2 - \alpha_0^2)^{-2} \sim \beta^{-4}; \quad \beta \sim r^{-1/4}.$$

On the other hand from (32) for large ρ and p,

$$u \sim \rho^{3/2} \sim \sqrt{p}; \quad c \sim \rho \sim p^{1/3}; \quad u \gg c; \quad \beta \approx u \sim \sqrt{p}.$$

Hence

$$r^{-1/4} \sim \sqrt{p}; \quad p \sim r^{-1/2}. \quad (35)$$

We now carry out a similar analysis of the second assumption with regard to the equations of state (30). In this case we must make direct use of all the conservation equations, eliminating the energy equation. We obtain

$$p = 2\rho_{00}\rho Q / (2\rho_{00} - \rho);$$

$$c = \sqrt{3p/\rho}; \quad u = -[p(\rho - \rho_{00}) / \rho_{00}\rho]^{1/2};$$

$$D = [p\rho / \rho_{00}(\rho - \rho_{00})]^{1/2}, \quad A(S) = 2\rho_{00}Q / \rho^2 (2\rho_{00} - \rho). \quad (36)$$

Again the Jouguets point corresponds to $\rho_0 = 4\rho_{00}/3$. The velocity of the normal detonation

$$D_0 = 4Q \quad (37)$$

does not depend upon the density as is the case for ideal gasses. Actually, the velocity of the detonation of given explosive material with given Q increases appreciably, although somewhat more slowly than ρ_{00}, with increase in the initial density; it is therefore evident that the second variant, Eq. (30) certainly does not exist in pure form, $\omega < 2$. On the other hand, $\omega = 0$ would denote the complete absence of thermal expansion, absence of the effect of temperature on pressure. In the same way, it is shown that (29) and (30) are actually limiting assumptions; the truth actually lies in between.

Returning to the second variant [equation of state (30)], we note that, in accord with (36), the density in the compression wave never exceeds $2\rho_{00}$. Landau and Stanyukovich in the theory of a converging cylindrical detonation wave considered just this case.

In the second variant, when the entropy is a variable, strictly speaking, the fundamental equation (11′) also changes, and the very definition $\beta = u - c$ becomes inaccurate, since the density ρ now depends not only on p but also on S. Neglecting these corrections, we find the value of $u - c$ from (36) in its dependence on ρ and with the help of the known $\beta'(r)$ we finally obtain p(r) as in the first variant. We have positive results in Table III. In column 2 are given the results of the approximate calculation described above. In column 3 are given the results obtained on an electronic computer.

TABLE III

$\frac{r}{r_0}$	$\frac{p}{p_0}$	$\left(\frac{p}{p_0}\right)_{exact}$
1	2	3
1	1	1
0.8	1.05	1.05
0.6	1.11	1.12
0.4	1.23	1.25
0.2	1.51	1.57
0.15	1.67	1.76
0.1	1.93	2.08
0.08	2.10	2.30
0.06	2.36	2.63
0.04	2.93	3.21
0.02	3.80	

The asymptote of the second variant does not differ from the asymptote of the first: $p = r^{-1/2}$. It is significantly close to the result of Landau and Stanyukovich: $p \sim r^{-0.47}$ for this case. We note that the opposite limiting case of a cylindric acoustical (weak) case also yields $p \sim p_0 \sim r^{-1/2}$.

As a whole, as is seen from a comparison with numerical calculations and the asymptote, the region of applicability of the approximate theory is much broader than could be presumed previously.

We note that it follows from the very nature of the derivations that the approximate theory gives only a description of the conditions on the wave front, but is neither intended nor suitable for a description of the motion of the entire mass of the explosion products.

I take this opportunity to express my thanks for discussion of the work and help in the calculations to S. B. Aratskin, E. I. Zababakhin, Ya. M. Kazhdan, A. S. Kompaneets, K. A. Semendyaev, and K. P. Stanyukovich.

[1] K. P. Stanyukovich, Неустановившееся движение сплошной среды (Unsteady Motion of a Continuous Medium) Gostekhizdat, 1955.

[2] Guderley, Luftfahrtforschung **19**, No. 9 (1943).

[3] B. V. Aĭvasov and Ya. B. Zel'dovich, J. Exptl. Theoret. Phys. (U.S.S.R.) **17**, 889 (1947).

[4] E. Jouguets, Mécanique des Explosifs (Paris, 1917).

[5] R. Courant and K. Friedrichs, Supersonic Flow and Shock Waves (Interscience, N. Y., 1948).

[6] Ya. B. Zel'dovich, J. Exptl. Theoret. Phys. (U.S.S.R.) **10**, 542 (1940).

[7] W. Döring, Ann. Physik **43**, 421 (1943).

[8] J. von Neumann, OSRD Report 549 (1942).

[9] A. A. Grib, Прикл. матем. мех. (Appl. Math. and Mech.) **8**, 169 (1944).

[10] G. N. Abramovich and L. A. Vulis, Dokl. Akad, Nauk SSSR **80**, [sic] 525 (1946).

[11] K. P. Stanyukovich, Dokl. Akad, Nauk SSSR **60**, [sic] 628 (1947).

[12] Ya. B. Zel'dovich and A. S. Kompaneets, Теория детонации (Theory of Detonations) Moscow, GTTI, 1955.

[13] Ya. B. Zel'dovich, J. Exptl. Theoret. Phys. (U.S.S.R.) **12**, 389 (1942).

Translated by R. T. Beyer
142