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THE ROLE OF GROUP VELOCITY OF LIGHT EMITTED IN A REFRACTIVE MEDIUM

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The emission of light by a system of particles possessing an arbitrary natural frequency is considered for the case when the system moves uniformly in a refractive medium that is transparent to the emitted light. The role of the group velocity of the light emitted by such a source is investigated.

1. INTRODUCTION

THE Vavilov-Cerenkov effect has been extensively used recently in research in nuclear physics, particularly the physics of high-energy particles. At the present time one observes in the experiments either radiation from individual charged particles that move in a refractive medium or the total incoherent radiation produced by many charges that move independently of each other. However, this immediately raises the question of radiation by a moving system of particles.* A system of particles may have a natural frequency of oscillation. Therefore, an examination of the behavior of a radiator of arbitrary natural frequency moving in a medium may be not merely of fundamental but also of practical interest.

The purpose of this article is to clarify the role of the group velocity of light in radiation from such a source, moving uniformly in a refractive isotropic medium that is transparent to the radiated light.

It is known that the radiation from a moving source of light depends substantially on the ratio of the radiator velocity to the phase velocity of light at the radiated frequency. Thus, a particle producing a time-invariant electromagnetic field (electric charge, fixed magnetic dipole, etc.) i.e., a particle with a zero natural frequency, is subject to the Vavilov-Cerenkov effect if its velocity of motion reaches or exceeds the phase velocity of the light. Here the connection between the direction of the emission of light and the radiated frequency ω_θ is determined by the ratio of the velocity of motion v to the phase velocity of the light $u = c/n(\omega_\theta)$ for this frequency, namely

$$v \cos\theta / (c/n(\omega_\theta)) = 1, \quad n(\omega_\theta) = c/v\cos\theta. \quad (1.1)$$

If a particle (or a system of particles) produces a time-varying field of frequency ω_0' while moving in a medium (ω_0' is measured in the reference frame of the radiator), the radiated frequencies are subject to the Doppler effect. As in the case of the Vavilov-Cerenkov effect, the Doppler frequency is a function of the ratio $v/[c/n(\omega_\theta)] = \beta n(\omega_\theta)$. Actually, the radiated Doppler frequencies satisfy one of the following relations

$$\omega_\theta = \omega_0 / (1 - \beta n(\omega_\theta)\cos\theta), \quad \beta n(\omega_\theta)\cos\theta < 1; \quad (1.2)$$

$$\omega_\theta = \omega_0 / (\beta n(\omega_\theta)\cos\theta - 1), \quad \beta n(\omega_0)\cos\theta > 1. \quad (1.3)$$

Here $\omega_0 = \omega_0' \sqrt{1-\beta^2}$ is the natural frequency measured in the coordinate system in which the medium is at rest.

The first equation corresponds to the case when the projection of the radiator velocity on the direction of the beam, $v \cos\theta$, is less than the phase velocity of light $u = c/n(\omega_\theta)$ for the Doppler frequency (Doppler frequencies of the usual type). The second formula corresponds to the "faster than light" Doppler effect, i.e.,

$$v\cos\theta > c/n(\omega_\theta).$$

The group velocity of light does not enter the foregoing expressions explicitly. It is known, however, that it is precisely the group velocity of light that determines the rate of propagation of the radiated energy in the medium. It is therefore natural to raise the question of whether any singularities will arise in the radiation if the velocity of motion exceeds the group velocity, i.e., if the radiator overtakes the light energy it radiates. It will be shown later on that this is indeed so: Equality of the velocity of radiator motion to the group velocity of light is a necessary (although not sufficient) condition for the appearance of new radiation components.

*This statement of the problem arises, in particular, in connection with the so called coherent method of charged-particle acceleration, proposed by V. I. Veksler.

This result is elementary in the case of the Vavilov-Cerenkov effect. In fact, the least velocity at which Vavilov-Cerenkov radiation occurs at at angle θ is, according to (1.1)

$$v_{\min} \cos \theta = c / n_{\max}, \tag{1.4}$$

where n_{max} is the maximum value that the index of refraction can assume in the given medium (we denote the corresponding limiting frequency by ω_{lim}). Since the group velocity w is determined by the relation

$$\frac{1}{w} = \frac{1}{c}\frac{d}{d\omega}(\omega n) = \frac{1}{c}\left(n + \omega\frac{dn}{d\omega}\right) \tag{1.5}$$

and $[dn/d\omega]_{\omega=\omega_{lim}} = 0$ for the maximum index of refraction, we get

$$w(\omega_{\mathrm{lim}}) = c / n_{\max}. \tag{1.6}$$

Comparing with (1.4) we obtain

$$v \cos \theta / w(\omega_{\mathrm{lim}}) = 1. \tag{1.7}$$

Thus, the condition of the threshold is that $v \cos \theta$ be equal to the group velocity of the light for the frequency that first appears in the spectrum of the Vavilov-Cerenkov radiation. In other words, the threshold is determined by the fulfillment of conditions (1.1) and (1.7) for a frequency that is the the same in both cases, $\omega_\theta = \omega_{lim}$.

Equation (1.7) has remained unnoticed apparently because the problem is usually formulated in a different manner when the radiation threshold is determined. Usually the radiation frequency is assumed specified, say in the optical region of the spectrum, and the threshold velocity is determined from the known index of refraction for this frequency.

A condition like (1.7) is significant also for other types of radiation in a medium. To explain this we must turn to an examination of the question of the splitting of the radiated frequencies, to the so called "complex" radiation effects.[1]

It is seen from (1.1) — (1.3) that in the case a dispersing medium these equations are not linear in ω_θ, since the index of refraction $n(\omega_\theta)$ may depend on the frequency in a complicated manner. As a result, for specified ω_0, v, and θ there are several possible frequencies ω_θ satisfying the Doppler condition[1] (complex Doppler effect). Mandel'shtam was the first to indicate in his lectures that the occurrence of the complex Doppler effect is related to the value of the group velocity of light.[2*] Actually, it was found that the previously-derived condition for the occurrence of the complex Doppler effect[1] is tantamount to stating that the ratio of $v \cos \theta$ to the group velocity of light exceeds unity for one of the Doppler frequencies.[3]

It can be shown that Eq. (1.7) is a condition for the threshold of the complex radiation effect, meaning also the threshold for the appearance of new components of the spectrum, under all circumstances.

An examination of this group of phenomena leads also to other conclusions. Thus, it can be stated that, in any refractive medium, a high-speed radiator of arbitrary natural frequency cannot overtake fully the light it radiates, no matter what its velocity of motion.

2. CONDITIONS FOR THE OCCURRENCE OF RADIATION OF COMPLEX COMPOSITION

For a more detailed examination of the problem of the spectrum of radiated frequencies, we used a graphical method, somewhat modified from the one used previously for this same purpose.[1]

We write Eqs. (1.1) — (1.3) in such a form that the left side contains the quantity $\omega_0 n(\omega_\theta)/c$ (in the case of Eq. (1.1) we multiply the left and right halves by ω_θ):

$$\frac{1}{c}\omega_\theta n(\omega_\theta) = \frac{\omega_\theta - \omega_0}{v \cos \theta}, \quad n(\omega_\theta)\beta \cos \theta < 1; \tag{2.1}$$

$$\frac{1}{c}\omega_\theta n(\omega_\theta) = \frac{\omega_\theta + \omega_0}{v \cos \theta}, \quad n(\omega_\theta)\beta \cos \theta > 1; \tag{2.2}$$

$$\frac{1}{c}\omega_\theta n(\omega_\theta) = \frac{\omega_0}{v \cos \theta}, \quad n(\omega_\theta)\beta \cos \theta = 1. \tag{2.3}$$

These equations have a simple physical meaning: they result from the law of conservation of momentum in the emission of a photon. This becomes obvious immediately if the right and left sides of all equations are multiplied by $\hbar \cos \theta$. We then obtain on the left sides of all equations $(n\hbar\omega_\theta/c) \cos \theta$, i.e., the projection of the momentum of the emitted photon on the direction of motion of the radiator. As to the quantity on the right side of the equation, it is easy to verify that it equals the momentum given up by the moving system to radiation. Actually, in all cases this is the quantity ΔE, consumed in radiation from the kinetic energy of motion, divided by the velocity v. For example, in (2.1) we have on the right side $(\hbar\omega_\theta - \hbar\omega_0)/v$, i.e., $1/v$ times the difference between the energy of the radiated photon and the energy of transition to the excited state. Obviously, this difference comes from the kinetic energy. As to Eq. (2.2) the distinguishing feature of the faster-than-light Doppler effect is that the emission of the photon is accompanied by the excitation of the

*The published text of the Mandel'shtam lectures[2] covers this briefly and not clearly enough. See also reference 3.

system.[3] This is why the kinetic energy loses an amount equal to $\hbar\omega_\theta + \hbar\omega_0$. The right half of Eq. (2.3) is obvious and requires no explanation.

The change in momentum accompanying a small change in kinetic energy, as is well known, satisfies the relation

$$\Delta p = \Delta E / v. \tag{2.4}$$

Thus, Eqs. (2.1) — (2.3) really follow from the law of conservation of energy and momentum, while (1.1) — (1.3) can be considered as their consequences.

Let us note also that Eq. (2.3) is a particular case of Eqs. (2.1) and (2.2). It corresponds, as expected, to the case of a radiator field that is stationary with time, i.e., $\omega_0 = 0$. It is obvious that (2.1) and (2.2) can be satisfied for $\omega_0 = 0$ only if the inequality sign on the right side is replaced by an equal sign. The difference between (2.1) and (2.2) is also not of fundamental nature: one equation is derivable from the other by reversing the sign of ω_0 or ω_θ, which, from the wave point of view, means merely reversal of the phase of the oscillation. Thus, Eqs. (2.1) — (2.3) can be considered as different cases of the same relation.

The field of any radiator moving uniformly in the medium can be expanded into a spectrum of natural frequencies $\omega_0' = \omega_0/\sqrt{1-\beta^2}$. Therefore, the corrolaries of Eqs. (2.1) — (2.3), which remain valid for arbitrary ω_0', are also correct for any spectrum of frequencies ω_0', i.e., they include the general case of radiation during uniform motion in a medium.*

Figure 1 shows graphically the magnitude of the wave vector

$$k(\omega) = \omega n(\omega) / c \tag{2.5}$$

as a function of ω. The curve $k(\omega)$ divides (provided $n(\omega) > 0$) the portion of the plane bounded by the positive coordinate axis, into two regions: region I below the curve $k(\omega)$, and region II above it.

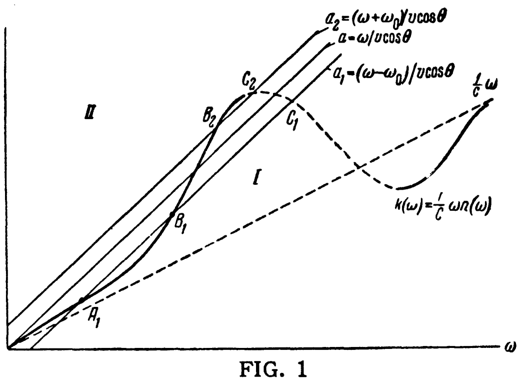

FIG. 1

*In particular, the path along which the radiation occurs may be bounded. This corresponds to a frequency spectrum at which the field vanishes outside a specified time interval.

The same diagram shows the straight lines that satisfy the following equations

$$a_1 = (\omega - \omega_0) / v \cos\theta, \tag{2.6}$$

$$a_2 = (\omega + \omega_0) / v \cos\theta, \tag{2.7}$$

$$a = \omega / v \cos\theta. \tag{2.8}$$

The intersections of these lines with the curve $k(\omega)$ obviously yield values of ω_θ satisfying (2.1), (2.2), and (2.3) respectively, i.e., yielding the Doppler frequencies in the medium under consideration for given κ_0, v, and θ, and yielding the frequency of the Vavilov-Cerenkov radiation for given v and θ.

For convenience in the analysis, the same values of v and θ are used in all three cases of Fig. 1. Therefore all three straight lines are parallel and inclined to the ω axis at an angle $\alpha = \tan\varphi = 1/v\cos\theta$ (Fig. 1 shows the case $0 \le \theta \le \pi/2$, i.e., $\cos\theta > 0$).

The character of the intersection of the lines a with the curve $k(\omega)$ may vary. If one moves along a straight line in the direction of increasing ω, the line may go at the point of intersection from region I into region II, provided the slope of the tangent to the curve $k(\omega)$ i.e., $dk/d\omega$ is less than $\alpha = 1/v\cos\theta$. To the contrary, if $dk/d\omega > 1/v \times \cos\theta$, a transition from II into I occurs at the intersection point. Finally, $dk/d\omega = 1/v\cos\theta$ corresponds, obviously, to tangency (from the inner or outer side).

The slope of the tangent to the curve $k(\omega)$ is, as is obvious from (2.5), the reciprocal of the group velocity of light for the frequency ω:

$$\frac{d}{d\omega} k(\omega) = \frac{1}{c}\frac{d}{d\omega}\omega n(\omega) = \frac{1}{w(\omega)}. \tag{2.9}$$

Thus, we have three cases:

$v\cos\theta / w(\omega_\theta) < 1$	transition from I to II	(2.10)
$v\cos\theta / w(\omega_\theta) > 1$	transition from II to I	(2.11)
$v\cos\theta / w(\omega_\theta) = 1$	tangency	(2.12)

A transition from I to II or from II into I can occur also at the point of tangency (tangency at a point of inflection). In principle, therefore, the equal sign is also possible in Eqs. (2.10) and (2.11). Actually, however, the angle θ is always specified with accuracy to certain very small but finite $\Delta\theta$, which defines a certain frequency interval $\Delta\omega$, for which the quantity w is indeed determined. If the tangency occurs at a point of inflection, then at neighboring points, on the right or on the left, the quantity w satisfies the particular inequality sign that characterizes the corresponding intersection. Thus, when averaged over the angular interval $\Delta\theta$, the inequality assumes a definite sign and the equal sign can be discarded.

For the same reason, Eq. (2.12) implies that the tangency occurs without intersection. In this case when going through the point of tangency the inequality reverses its sign, and consequently, averaged over a small $\Delta\theta$, we can consider the equal sign to hold.

Part of the curve $k(\omega)$ is shown dotted in Fig. 1 (and in the analogous Fig. 2). In this frequency region it is incorrect to identify the derivative to the curve $k(\omega)$ with the reciprocal of the group velocity. This region, which includes also the region of negative w (if it exists), corresponds to the frequency interval at which the absorption of light cannot be neglected. In this case, as is known, the quantity defined by (1.5) cannot be given the meaning of a group velocity of light.

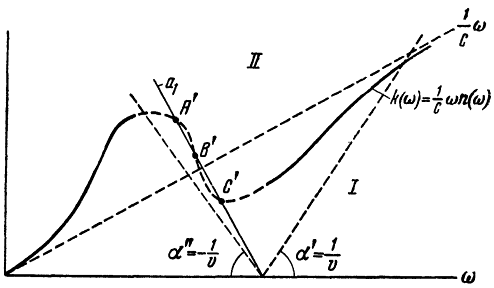

FIG. 2

The foregoing pertains only to the physical meaning of the derivative of $k(\omega)$. As to the curve itself, it retains its meaning in the entire region of variation of ω, provided n is taken to mean its real part and its intersections with the straight lines, a, give the values of the radiated frequencies. We shall not exclude from consideration the region of negative w, since in principle such values are possible, although in an isotropic refractive medium this case is not realizable outside the strong-absorption region.

Let us discuss the consequences that result from an examination of Fig. 1 and Eqs. (2.10) — (2.12). At sufficiently large ω, $n(\omega)$ approaches unity and, consequently, the slope of the curve $k(\omega) = \omega n(\omega)/c$ is equal to $[1/w]_{\omega\to\infty} = 1/c < 1/v\cos\theta$ for $\theta \leq \pi/2$. Consequently, at sufficiently large values of ω, all the three straight lines a lie in region II. The line a_1, the intersection of which with $k(\omega)$ determines the frequencies of the ordinary Doppler effect [Eqs. (1.2) or (2.1)], intercepts the abscissa axis at $\omega = \omega_0$, and consequently lies in region I. Therefore the line a_1 must have at least one intersection accompanied by a transition from region I to region II (for example A_1 or C_1 in Fig. 1). This is true not only for $\cos\theta \geq 0$, i.e., $\theta \leq \pi/2$, shown in Fig. 1, but for all $0 \leq \theta \leq \pi$. When θ increases from 0 to π, the slope of the line a_1 ranges from $\alpha' = +1/v$ to $\alpha'' = -1/v$ (see dotted lines, Fig. 2). It is seen from Fig. 2 that when $\cos\theta < 0$ the line a_1 must intersect the curve $k(\omega)$.

Thus, for any velocity v, for any angle θ, and for any natural frequency ω_0 there must be at least one Doppler frequency satisfying (1.2) and satisfying, in addition, (2.10).

The Doppler effect may be complex, since this frequency may be accompanied by several other frequencies (crossings B_1 and C_1, Fig. 1 or B' and C', Fig. 2). The number of such additional frequencies must be even, since they should be due to an equal number of crossings from regions I into II and from II into I (in the limiting case of tangency, they may become equal to each other). Thus, the complex Doppler effect is known to occur if there is at least one crossing from II into I or a tangency of the line a_1 to the curve $k(\omega)$ from the outside or from the inside. From this we get that the condition for the occurence of the complex Doppler effect is the presence of a Doppler frequency for which, according to (2.11) or (2.12)

$$v\cos\theta / w(\omega_\theta) \geqslant 1. \tag{2.13}$$

From an examination of Fig. 2 we can conclude that all the above pertains to the case of obtuse angles θ. For the complex Doppler effect to occur in this case, Eq. (2.13) must be satisfied. Since $\cos\theta < 0$ when $\theta > \pi/2$, this is possible only for a group velocity that has a negative absolute magnitude, less than $|v\cos\theta|$. In this case, if the frequency splits up, the larger frequency satisfies inequality (2.10).

A complex Doppler effect may also result from the appearance of faster-than-light Doppler components. The line a_2 which determines these frequencies at small values of ω, as well as at large ω, lies in region II and therefore cannot in general cross the curve $k(\omega)$. This is understandable, since unlike ordinary frequencies, which occur always, faster-than-light Doppler frequencies are supplementary and occur only under certain conditions, and naturally, only for acute angles θ. However, if a_2 does cross $k(\omega)$, it does so in at least two points (in general as even number), since every time the line a_2 passes from II into I, it crosses back from I into II at large frequencies. The former transitions are satisfied by (2.11) while the transition corresponding to the higher frequency obeys (2.10). In the limiting cases the intersection is replaced by tangency, i.e., a pair of such frequencies becomes one. Thus, the threshold for the occurence of the faster-than-light Doppler effect is (1.7). For each given θ, the faster-than-light Doppler effect is always complex, and contains a component that satisfies (2.13).

It follows, therefore, that satisfaction of condition (2.13) for any Doppler frequency is a necessary and sufficient condition for the complex Doppler effect to occur. This indeed is the previously derived condition for the occurence of the complex Doppler effect,[1] expressed in a somewhat more general form.

The foregoing can be readily generalized to include Vavilov-Cerenkov radiation. For large ω, as indicated, the line a lies in region II. At small ω (excluding from consideration the point $\omega = 0$), the line may lie either above or below the curve $k(\omega)$. In the former case there must be an intersection with a crossing from I to II, i.e., light is radiated in the direction θ with a frequency for which condition (2.10) holds. In addition to this frequency, there can exist an even number of supplementary frequencies, determined by the equal number of crossings from I to II and from II to I.

In the second case there may be no crossing, i.e., the condition for the occurence of the Vavilov-Cerenkov effect for a given θ is not satisfied for any of the frequencies. If crossings do occur, there are at least two (in general an even number), and after each crossing from II into I there occurs, at large θ, a crossing from I into II.

The threshold of the Vavilov-Cerenkov effect or of new additional radiation components remains, obviously, the tangency of the line a to the curve $k(\omega)$, i.e., (2.12) or, what is the same, (1.7). The complex effect is certain to occur if a component exists for which condition (2.11) is satisfied. In the limiting case, when the inequality goes into the equality (2.12), these components merge into a dual frequency.

We note that it is precisely the complex Vavilov-Cerenkov effect that is observed in the experiments. Actually, the Vavilov-Cerenkov effect is practically always observed in the visible or near-ultraviolet region of the spectrum. This is a region of normal dispersion, in which $w < c/n$. Since, according to (1.1), we have in the case of the Vavilov-Cerenkov radiation $c/n = v \cos\theta$, consequently, for the radiated frequency, $w(\omega) < v \cos\theta$. The frequencies paired with the observed one lie in the ultraviolet portion of the spectrum, in the region of anomalous dispersion, and are not observed because of the strong absorption.

Thus, the foregoing relations between the value of the group velocity and the velocity of motion are common for all radiators moving uniformly in space. One must not forget here that these relations are meaningful only when the group velocity satisfies the corresponding Doppler condition (1.2) — (1.3) or condition (1.1) for the Vavilov-Cerenkov effect. These conditions may be expressed in terms of the magnitude of the group velocity if they are represented in the form of an integral equation. Actually, Eqs. (2.1) — (2.3) can obviously be written

$$\int_0^{\omega_\theta} \left(\frac{1}{w(\omega)} - \frac{1}{v\cos\theta}\right) d\omega = \pm \frac{\omega_0}{v\cos\theta}. \tag{2.14}$$

Here $w(\omega)$ means a value of $d\omega/dk$ equal to the group velocity in the absence of absorption. The minus sign corresponds to the ordinary Doppler effect, the plus sign to the faster-than-light effect, and $\omega_0 = 0$ corresponds to the Vavilov-Cerenkov effect.

If the Doppler effect is complex, then for any pair of Doppler frequencies ω'_θ and ω''_θ of the same type, i.e., ordinary or faster-than-light frequencies, we obtain directly from (2.14)

$$\int_{\omega''_\theta}^{\omega'_\theta} \left(\frac{v\cos\theta}{w(\omega)} - 1\right) d\omega = 0, \tag{2.15}$$

or

$$\left(\overline{\frac{1}{w}}\right) = \frac{1}{\omega'_\theta - \omega''_\theta} \int_{\omega''_\theta}^{\omega'_\theta} \frac{1}{w(\omega)} d\omega = \frac{1}{v\cos\theta}. \tag{2.16}$$

Thus, the mean value $1/w$ over the interval of ω, contained between the Doppler frequencies ω'_θ and ω''_θ, is always equal to $1/v\cos\theta$.

We note that Eqs. (2.15) and (2.16) do not contain ω_0 and, consequently, are valid both for the Doppler effect and for the Vavilov-Cerenkov radiation.

If one of the radiated frequencies is known, for example ω'_θ, then, considering ω''_θ as the variable in (2.16) or (2.17) it is possible to determine in principle all the remaining frequencies of a given type.

3. CONNECTION BETWEEN THE VELOCITY OF THE RADIATOR AND THE GROUP VELOCITY OF LIGHT

Based on the foregoing, it still remains quite unclear why the resultant relations refer specifically to the projection of the velocity of the radiator and the direction of the beam. This can be explained as follows. It is obvious that the spectrum of the frequencies radiated in the direction θ is given by the quantity ω_0 and by the product $v \cos\theta$. We obtain the same frequency spectrum at a velocity $v' = v\cos\theta$ and the same value of ω_0. (The natural frequency of the system should be changed in this case, since $\omega_0 = \omega_0' \sqrt{1-\beta^2}$). It corresponds to forward radiation if $v' > 0$, i.e., $\cos\theta > 0$, or to backward radiation if $v' < 0$, i.e., $\cos\theta$ is negative. If we disregard the case when the angle θ has the threshold value for the production of radiation, we find, from the fact that a component satisfying (2.10) always exists, that the radiation spectrum should contain a frequency for which

$$v'/w(\omega') < 1. \qquad (3.1)$$

Let v' be positive. This means that the light emitted forward should have a frequency such that the group velocity is greater than the velocity of motion. Thus, in the case of forward radiation the radiated energy must outpace the motion of the source.

Let us assume that v' is negative. Condition (3.1) it is satisfied in this case for all $w > 0$. Thus, no limitations are imposed on the group velocity of the light radiated backwards.

The consequences of inequality (3.1) can be extended to include also the case of negative group velocity. This case deserves a special analysis and we shall treat it here only briefly.* At negative group velocities the directions of the phase and group velocities are opposite. If radiation does take place, i.e., energy is carried away, the group velocity should be directed along the ray drawn from the radiator and consequently the phase velocity should be directed towards the radiator. Fig. 3a shows the picture obtained when $w < 0$ for an acute angle θ (as before, θ is the angle between the direction of the phase velocity u and the velocity of the radiator v). It is seen from Fig. 3a that the radiated energy makes in

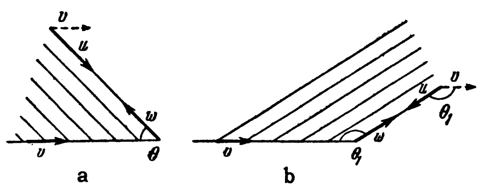

FIG. 3

*Certain features connected with negative group velocity for the Vavilov-Cerenkov effect in crystals are discussed by V. E. Pafomov.[4]

this case an acute angle $(\pi-\theta)$ to the direction of the velocity. Figure 3b shows the case $\theta = \theta_1 > \pi/2$. In this case the energy is obviously carried away at an acute angle $(\pi-\theta_1)$ to the direction of motion.

Let us turn now to the consequences of inequality (3.1) at negative w. The angle $\theta < \pi/2$ corresponds to positive v'. Inequality (3.1) will be fulfilled in this case for any negative w. In this case the energy is carried away, as we have seen, backward and no conditions whatever, as before, are imposed on its removal.

The quantity w can be negative also when $v' < 0$, corresponding to $\theta_1 > \pi/2$. In this case condition (3.1) is satisfied only if the absolute value of w is greater than that of v'. Since the energy is carried away forward in this case, this means that here too we have a case when the velocity of energy propagation outpaces the velocity of the radiator.

In our analysis we went from radiation at an angle θ to the direction of the velocity to forward radiation at the same frequency, i.e., to $\theta = 0$. Obviously, we can proceed in an opposite sequence Were the radiation in a forward direction to contain no components that outpace the radiator, then for the same ω_0 and $v\cos\theta = v'$, we would find that there exists no component satisfying (2.10), which, as we have seen, is impossible.

Complex radiation effects are accompanied by the appearance of components or a component satisfying (2.11). Such an anomalous component must be accompanied by a normal one, satisfying (2.10) and having a higher frequency at that. Thus, the normal component that outpaces the radiator carries more energy per unit frequency interval than the anomalous one.

The relations between the quantity $v\cos\theta$ and the group velocity are a direct consequence of the fact that the radiator cannot overtake fully the light signal that it radiates in the direction of its own motion.

Condition (2.12), corresponding to a threshold of appearance or vanishing of the frequencies, means for $\theta' = 0$ that the group of waves moves at the same velocity as the radiator. Obviously, in this case there is no radiation, merely the motion of the field with the radiator. That this is a threshold case at $\theta' = 0$ is almost obvious.

No conditions whatever were imposed in this analysis on the value of ω_0 and on the corresponding initial phase of the oscillations. Thus the spectrum of ω_0 is arbitrary and, consequently, everything said here concerning the role of the group velocity applies also for any source of light uniformly moving in a refractive medium.

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Translated by J. G. Adashko
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