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OSCILLATIONS OF A COMPLETELY IONIZED PLASMA IN A CYLINDRICAL CAVITY

L. M. KOVRIZHNYKH

P. N. Lebedev Physics Institute, Academy of Sciences, U.S.S.R.

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Within the framework of magnetohydrodynamics, under the assumption of ideal conductivity, a study has been made of oscillations of a cylindral cavity in a completely ionized plasma located in a magnetic field. It is shown that such a system is stable and that under certain conditions no waves can propagate along the cavity.

THE stability of an ideal conducting gas in a cylinder with respect to small perturbations has been studied by Kruskal and Schwarzschild[1] and by Shafranov.[2]

In the present paper an analogous method will be applied to solve the problem of plasma oscillations in an infinite cylindrical cavity of radius a, containing the conducting medium in a coaxial cylinder of radius a_0, in which a current I_0 is flowing. In the equilibrium state the pressure of the plasma on the cavity boundary is balanced by the magnetic pressure resulting from the surface currents flowing at the plasma-vacuum interface.

The starting point is the system of equations consisting of the magnetohydrodynamic equations for an ideal fluid conductor

$$\rho\, d\mathbf{v}/dt = [\mathbf{j}\times\mathbf{H}]/c - \nabla p;$$

$$\partial\rho/\partial t + \nabla\rho\mathbf{v} = 0, \quad p = \mathrm{const}\cdot\rho^{\gamma}; \tag{1}$$

$$\partial\mathbf{H}/\partial t = \mathbf{curl}\,[\mathbf{v}\times\mathbf{H}]; \quad \mathbf{curl}\,\mathbf{H} = = 4\pi\,\mathbf{j}/c \tag{2}$$

for the region within the plasma; the equations

$$\mathbf{H} = \nabla\varphi, \quad \Delta\varphi = 0 \tag{3}$$

for the vacuum; and the boundary conditions at the interface,

$$\bar{\mathbf{H}}\{\mathbf{H}\} = -4\pi\{p\}; \quad \mathbf{n}\{\mathbf{H}\} = 0. \tag{4}$$

Here $\bar{\mathbf{H}}$ denotes the mean value of the field at the boundary, and the letters in brackets denote the magnitudes of the discontinuities in the corresponding quantities at the interface, and **n** is the unit vector normal to the plasma surface, directed into the plasma. Since this surface moves with the plasma, the relationship

$$d\mathbf{n}/dt = [\mathbf{n}\times[\mathbf{n}\times\nabla u]], \quad u = \mathbf{n}\ \mathbf{v}. \tag{5}$$

must be satisfied.

In the equilibrium state $\mathbf{v} = 0$, and the plasma is uniform in the z and φ directions, and the non-zero components of the magnetic field are H_z^0 and H_φ^0. It will be assumed that within the plasma $H_\varphi^0 = 0$ and H_z^0 is uniform, i.e.,

$$H_{\varphi 1}^0 = 2I/cr, \quad H_{\varphi 2}^0 = 0, \quad H_{z1}^0 = \mathrm{const}, \quad H_{z2}^0 = \mathrm{const}, \tag{6}$$

where the indices 1 and 2 denote that the corresponding value of the quantity refers to the cavity and the plasma, respectively. Then according to Eq. (4) we must have

$$H_{z1}^{02} - H_{z2}^{02} + H_{\varphi 1}^{02} = 8\pi p_0. \tag{7}$$

Solving the system (1) and (2) by the method of small oscillations, subject to the boundary conditions (4) and (5), we obtain the desired dispersion relation*

$$\Omega^2 = f_m(\Omega^2, k) = \frac{2}{\gamma'}(k^2 - \Omega^2)\left[h_2{}^2 - \frac{1}{\zeta}\frac{K_m'(\zeta)}{K_m(\zeta)}\alpha_m(k)\right] \tag{8}$$

under the conditions $-\pi/2 < \arg\zeta \le \pi/2$, where

$$\Omega^2 = \frac{\omega^2 a^2}{v_T^2}, \quad v_T^2 = \gamma\frac{p_0}{\rho_0}, \quad \zeta^2 = \frac{(\Omega^2 - k^2)(\Omega^2 - q^2 k^2)}{q^2 k^2 - (1 + q^2)\,\Omega^2},$$

$$q^2 = 2h_2{}^2/\gamma', \quad \gamma' = \gamma[h_1{}^2 - h_2{}^2 + 1],$$

$$h_i = H_{zi}^0/H_\varphi^0, \quad x_0 = a_0/a,$$

$$\alpha_m(k) = \left\{1 + \frac{I_m(k)K_m'(kx_0) - I_m'(kx_0)K_m(k)}{I_m'(k)K_m'(kx_0) - I_m'(kx_0)K_m'(k)}\,\frac{(m + kh_1)^2}{k}\right\}, \tag{9}$$

I_m and K_m being the modified Bessel functions of order m.

It follows from the self-adjoint nature of the operator which occurs in the linearized equation of motion that the expression (8) has no roots in the complex region. Hence in studying this expression it is sufficient to limit ourselves to real values of Ω^2.

*Here we assume, as usual, that all quantities are proportional to exp $i(kz + m\phi + \omega t)$.

First of all, we observe that, although as Ω^2 varies from zero to $-\infty$ the function $f_m(\Omega^2, k)$ increases monotonically and

$$f_m(0, k) = \frac{2}{\gamma'} k^2 \left[h_2^2 - \frac{1}{k} \frac{K'_m(k)}{K_m(k)} \alpha_m(k) \right] > 0,$$

yet the dispersion equation has no solution for $\Omega^2 < 0$. Thus Ω is always real, and the system is stable with respect to small perturbations for which $k \neq 0$. However, if $k = 0$, the dispersion equation has the solution $\Omega^2 = 0$ for any arbitrary value of m when $H_\varphi^0 = 0$, but only for $m = 0$ when $H_\varphi^0 \neq 0$; i.e., in this case we have a state of neutral equilibrium.

Let us now consider equation (8) in more detail for the case $\Omega^2 > 0$.

1. $q^2 < 1$, i.e., $v_T^2 > v_H^2 = H_{Z2}^{02}/4\pi\rho_0$.

(a) $m = 0$. As the frequency varies from zero to $q^2k^2/(1 + q^2)$, the function $f_m(\Omega^2, k)$ decreases, and for $\Omega^2 \to q^2k^2/(1 + q^2)$ it tends to the value $q^2k^2/(1 + q^2)$; at the same time ζ tends toward infinity. With further increase in the frequency, f_m becomes complex, and then when $\Omega^2 = q^2k^2$ it returns to the real domain again, varying from $+\infty$ for $\Omega^2 = q^2k^2$ to zero when $\Omega^2 = k^2$. In the region $\Omega^2 > k^2$ the function f_m is always complex, i.e., there is no solution. There is therefore only a single branch, coinciding with the acoustic branch at small values of k and q^2.

(b) $m \neq 0$. Just as in the case $m = 0$, the dispersion equation has no roots $\Omega^2 < q^2k^2$. For $\Omega^2 > q^2k^2$ the function f_m is positive, and varies from $+\infty$ at $\Omega^2 = q^2k^2$ to the value

$$f_m(k^2, k) = \frac{2m}{\gamma'} \frac{\alpha_m(k)}{1 - q^2} \tag{10}$$

at $\Omega^2 = k^2$, and then becomes complex. Consequently, solutions exist only for those cases where

$$k^2 > f_m(k^2, k), \tag{11}$$

i.e., the only waves which can propagate in the system are those with a wavelength greater than some critical wavelength $\lambda_{cr}^{(m)} = 2\pi/k_{cr}^{(m)}$, where $k_{cr}^{(m)}$ is the smallest root of the equation $k^2 = f_m(k^2, k)$.

In both of the above cases the frequency Ω^2 lies in the interval from q^2k^2 to k^2.

2. $q^2 > 1$. In this case f_m is positive only when $\Omega^2 < q^2k^2/(1 + q^2)$. For values of Ω^2 greater than $q^2k^2/(1 + q^2)$ the function f_m is either complex or negative; that is, the dispersion equation has no solutions at all.

Thus we have seen that the dispersion equation (8) has no solutions corresponding to acoustic or Alfven waves in a gas. There is only a single mode that vanishes as $q \to 1$. For $q^2 > 1$ no solutions are possible at all; i.e., no wave-like motion is possible.

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[1] M. Kruskal and M. Schwarzschild, Proc. Roy. Soc. **A223**, 348 (1954).

[2] V. D. Shafranov, Атомная энергия (Atomic Energy) **5**, 38 (1956).

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