A MODEL OF A FIELD THEORY WITH NONVANISHING RENORMALIZED CHARGE

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Two fermion fields are considered in a space of one dimension (and in time), with interaction of each field with itself and of the two with each other. The first term in the expansion of the vertex part in an asymptotic series of well known form is obtained. It is shown that within certain limits the renormalized charge can have an arbitrary (nonvanishing) value.

In a number of papers[1-3] attempts have been made to deal with a relativistically invariant model of a field theory — the one-dimensional four-fermion interaction.

It was expected that the relation between the renormalized and bare charges in this theory would be such that the renormalized charge would not vanish when one goes to the limit of a point interaction.[2] A more detailed examination showed, however, that on account of a certain special cancellation of terms in the perturbation-theory series the theory contains no divergences at all, so that there is no charge renormalization and the question of the vanishing of the charge simply does not arise. In the present paper we consider two spinor fields in a one-dimensional space, with interaction of the four-fermion type between the two fields and of each field with itself. There is then a logarithmic divergence in the vertex part, and consequently an infinite charge renormalization. Unlike the various three-dimensional types of field theory, however, in which up to now it has not been possible to avoid the vanishing of the renormalized charge in the point-interaction theory, in the present model it is possible to construct a renormalized solution (or, more precisely, its asymptotic form for large momenta) with an arbitrary value of the renormalized charge.

1. THE EQUATIONS FOR THE VERTEX PART

Let us consider two fermion fields ψ and χ that depend on a single space coordinate x and the time x_0.

In our case the Hamiltonian can be chosen in the form:

$$H = \sum_\mu [(g_1/4)(\bar{\psi}\sigma_\mu\psi)(\bar{\psi}\sigma_\mu\psi) + (g_2/4)(\bar{\chi}\sigma_\mu\chi)(\bar{\chi}\sigma_\mu\chi) + (g_3/2)(\bar{\psi}\sigma_\mu\psi)(\chi\sigma_\mu\chi)], \quad \sum_\mu \sigma_\mu \times \sigma_\mu = \sigma \times \sigma - 1 \times 1. \quad (1)$$

The product $\sum_\mu \sigma_\mu \times \sigma_\mu$ is a mixture of interaction types $-S + P + V$, which in the one-dimensional case is the only completely antisymmetric type of interaction.[2] For the first two terms in Eq. (1) this spinor form is necessary; in the third term $\sigma_\mu \times \sigma_\mu$ can in principle be replaced by an arbitrary invariant operator $O_j \times O_j$, so that $\sigma_\mu \times \sigma_\mu$ has been kept here just for simplicity. Thus we are dealing with the one-dimensional analogue of the "universal interaction" of the type proposed by Feynman and Gell-Mann.

The introduction of terms of the type $(\bar{\psi}\sigma_\mu\chi) \times (\bar{\psi}\sigma_\mu\chi)$ into the Hamiltonian makes the problem much more complicated, and this case is not considered here. We now introduce a field $\Phi_\alpha = \begin{pmatrix}\psi \\ \chi\end{pmatrix}$ with the components ψ and χ. Then Eq. (1) can be rewritten in the form

$$H = \sum_\mu [(g_1/4)(\bar{\Phi}\sigma_\mu\tau_1\Phi)(\bar{\Phi}\sigma_\mu\tau_1\Phi) + (g_2/4)(\bar{\Phi}\sigma_\mu\tau_2\Phi)(\bar{\Phi}\sigma_\mu\tau_2\Phi) + (g_3/2)(\bar{\Phi}\sigma_\mu\tau_1\Phi)(\bar{\Phi}\sigma_\mu\tau_2\Phi),$$

$$\tau_1 = \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}, \quad \tau_2 = \begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}. \quad (2)$$

The matrices τ_i act on the "isotopic" index of the function Φ_α.

The most general form of the vertex part is

$$\Gamma = \sum_{i,k} \alpha_{ik}(\tau_i \times \tau_k) \sum_\mu (\sigma_\mu \times \sigma_\mu)$$

where α_{ik} is a set of scalar functions, and τ_i and τ_k are all possible two-rowed matrices. It can be shown, however, that the equations for Γ are satisfied by the following spinor form:*

*This is a consequence of the law of conservation of the number of particles ψ and χ, which follows from the Hamiltonian (1).

$$\Gamma = [\alpha_1(\tau_1 \times \tau_2) + \alpha_2(\tau_2 \times \tau_2) + 2\alpha_3(\tau_1 \times \tau_2)] \sum_\mu \sigma_\mu \times \sigma_\mu. \quad (3)$$

A method for obtaining an integral equation for Γ in the asymptotic region $p^2 \gg m^2$ has been developed by Dyatlov, Sudakov, and Ter-Matirosyan.[4] In the present case we have to do with a situation entirely analogous to that considered in reference 2. Carrying out some simple algebraic manipulations, we get the following system for the "lying" and "standing bricks" (the definition of these functions and some explations about the equations are given in reference 2):

$$f_1(\xi) = -\frac{1}{2\pi}\int_\xi^L \alpha_1^2(z)\,dz,$$

$$f_2(\xi) = -\frac{1}{2\pi}\int_\xi^L \alpha_2^2(z)\,dz, \quad f_3(\xi) = -\frac{1}{2\pi}\int_\xi^L \alpha_3^2(z)\,dz,$$

$$\varphi_1(\xi) = \frac{1}{2\pi}\int_\xi^L [\alpha_1^2(z) + \alpha_3^2(z)]\,dz,$$

$$\varphi_2(\xi) = \frac{1}{2\pi}\int_\xi^L [\alpha_2^2(z) + \alpha_3^2(z)]\,dz,$$

$$\varphi_3(\xi) = \frac{1}{2\pi}\int_\xi^L [\alpha_1(z) + \alpha_2(z)]\,\alpha_3(z)\,dz,$$

$$\alpha_i = g_i + f_i + \varphi_i, \quad \xi = \ln(p^2/m^2), \quad L = \ln(\Lambda^2/m^2), \quad (4)$$

Λ is the maximum momentum, and all the momenta entering and leaving the vertex part are of the order of p. In Eq. (4) the Green's function is taken equal to its zeroth approximation, since it is in general free from divergences.[2] From Eq. (4) it at once follows that

$$\alpha_1(\xi) = g_1 + \frac{1}{2\pi}\int_\xi^L \alpha_3^2(z)\,dz, \quad \alpha_2(\xi) = g_2 + \frac{1}{2\pi}\int_\xi^L \alpha_3^2(z)\,dz,$$

$$\alpha_3(\xi) = g_3 + \frac{1}{2\pi}\int_\xi^L [\alpha_1(z) + \alpha_2(z) - \alpha_3(z)]\,\alpha_3(z)\,dz. \quad (5)$$

2. SOLUTION OF THE INTEGRAL EQUATIONS

Combining the first and second equations of the system (5), we get the following system, which includes the third equation:

$$f(\xi) = \lambda + \frac{1}{\pi}\int_\xi^L \varphi^2(z)\,dz,$$

$$\varphi(\xi) = \nu + \frac{1}{2\pi}\int_\xi^L [f(z) - \varphi(z)]\,\varphi(z)\,dz. \quad (6)$$

Here we have introduced the notations

$$f = \alpha_1 + \alpha_2, \quad \varphi = \alpha_3, \quad \lambda = g_1 + g_2, \quad \nu = g_3. \quad (7)$$

Since the independent variable does not appear explicitly in Eq. (6), it is convenient first to try to find f as a function of φ. The first equation of the system (6) then has the form:

$$f = \lambda + \frac{1}{\pi}\int_\varphi^\nu \varphi^2\left(\frac{d\xi}{d\varphi}\right)d\varphi. \quad (8)$$

From the second equation it follows that

$$d\varphi/d\xi = -(f - \varphi)\,\varphi/2\pi, \quad (9)$$

that is,

$$f(\varphi) = \lambda - 2\int_\varphi^\nu \frac{\varphi'\,d\varphi'}{f(\varphi') - \varphi'}, \quad (10)$$

or

$$df/d\varphi = 2\varphi/(f - \varphi). \quad (11)$$

Equation (11) is a homogeneous equation; integrating it in the usual way, we easily get

$$(f - 2\varphi)(f + \varphi)^2 = (\lambda - 2\nu)(\lambda + \nu)^2 \equiv G^3. \quad (12)$$

Let us consider the important case $G = 0$. Here we have either: (a) $\lambda = 2\nu$, $g_1 + g_2 = 2g_3$, $f = 2\varphi$, or else (b) $\lambda = -\nu$, $g_1 + g_2 = -g_3$, $f = -\varphi$. In each case we can substitute φ, expressed in terms of f, in the first equation of (6); we thus easily get the following expressions for the vertex parts:

(a) $$f(\xi) = \frac{2\nu}{1 - (\nu/2\pi)(L - \xi)}, \quad \varphi(\xi) = \frac{\nu}{1 - (\nu/2\pi)(L - \xi)},$$

$$\alpha_1(\xi) = \frac{g_1 - g_2}{2} + \frac{\nu}{1 - (\nu/2\pi)(L - \xi)},$$

$$\alpha_2(\xi) = -\frac{g_1 - g_2}{2} + \frac{\nu}{1 - (\nu/2\pi)(L - \xi)},$$

(b) $$f(\xi) = \frac{-\nu}{1 + (\nu/\pi)(L - \xi)}, \quad \varphi(\xi) = \frac{\nu}{1 + (\nu/\pi)(L - \xi)},$$

$$\alpha_1(\xi) = \frac{g_1 - g_2}{2} - \frac{\nu/2}{1 + (\nu/\pi)(L - \xi)},$$

$$\alpha_2(\xi) = -\frac{g_1 - g_2}{2} - \frac{\nu/2}{1 + (\nu/\pi)(L - \xi)}. \quad (13)$$

When $\nu = 0$ we have for both sets of relations between the constants the results $\alpha_1 = g_1$, $\alpha_2 = g_2$, $\alpha_3 = 0$, in agreement with the results obtained in reference 2. If $g_1 + g_2 < 0$, then for the usual reasons the renormalized charge goes to zero, but if $g_1 + g_2 > 0$ the zero charge does not appear. Furthermore, the connection between the renormalized and bare charges is given by the following relation (for example, in case a) or:

$$\nu_c = \frac{\nu}{1 - \nu L/2\pi}, \quad g_{1c} = \frac{g_1 - g_2}{2} + \frac{\nu}{1 - \nu L/2\pi},$$

$$g_{2c} = -\frac{g_1 - g_2}{2} + \frac{\nu}{1 - \nu L/2\pi}, \quad (14)$$

and the expressions for the renormalized vertex parts have no nonphysical pole:

$$\varphi_c(\xi) = \frac{\nu_c}{1+\nu_c\xi/2\pi}, \quad \alpha_{1c}(\xi) = \frac{g_{1c}-g_{2c}}{2} + \frac{\nu_c}{1+\nu_c\xi/2\pi},$$

$$\alpha_{2c}(\xi) = -\frac{g_{1c}-g_{2c}}{2} + \frac{\nu_c}{1+\nu_c\xi/2\pi}. \qquad (15)$$

Let us now go back to the determination of the functions f and φ in the general case $G = (\lambda-2\nu)^{1/3}(\lambda+\nu)^{2/3} \neq 0$. Equation (12) in principle makes it possible to express the function φ in terms of f, after which the first of the equations (6) reduces to a quadrature. We shall, however, proceed in a different way, which allows us to get the solution in a more compact form. We define $F(\xi)$ by the relation

$$f + \varphi = (\lambda+\nu)/F, \qquad (16a)$$

and then from Eq. (12) we get

$$f - 2\varphi = (\lambda-2\nu)F^2, \qquad (16b)$$

from which we have

$$f = \frac{2(\lambda+\nu)}{3}\frac{1}{F} + \frac{\lambda-2\nu}{3}F^2,$$

$$\varphi = \frac{\lambda+\nu}{3}\frac{1}{F} - \frac{\lambda-2\nu}{3}F^2. \qquad (17)$$

Substituting these expressions in the first equation of (6) and differentiating it with respect to ξ, we find a differential equation for F. Separating the variables and integrating, we find

$$3\alpha^2\int_1^F \frac{dx}{x^3-\alpha^3} = \frac{G}{2\pi}(L-\xi),$$

$$\alpha = \left(\frac{\lambda+\nu}{\lambda-2\nu}\right)^{1/3}, \quad G = (\lambda-2\nu)^{1/3}(\lambda+\nu)^{2/3},$$

$$3\alpha^2\int_1^F \frac{dx}{x^3-\alpha^3} = \frac{1}{2}\ln\frac{(F-\alpha)^2(1+\alpha+\alpha^2)}{(F^2+\alpha F+\alpha^2)(1-\alpha)^2}$$

$$-\sqrt{3}\arctan\frac{2F+\alpha}{\alpha\sqrt{3}} + \sqrt{3}\arctan\frac{2+\alpha}{\alpha\sqrt{3}}. \qquad (18)$$

Together with the (17), Eq. (18), which implicitly determines F, gives the solution of our stated problem. We shall now show that the solution we have obtained has the property of normalizability, and shall carry out the renormalization program explicitly. We introduce the renormalized charges λ_c and ν_c as the values of the vertex parts f and φ when the incoming and outgoing momenta are real ($p^2 = -m^2$, $\xi = 0$):

$$\lambda_c = \frac{2(\lambda+\nu)}{3}\frac{1}{F_0} + \frac{\lambda-2\nu}{3}F_0^2,$$

$$\nu_c = \frac{\lambda+\nu}{3}\frac{1}{F_0} - \frac{\lambda-2\nu}{3}F_0^2,$$

$$3\alpha^2\int_1^{F_0}\frac{dx}{x^3-\alpha^3} = \frac{G}{2\pi}L. \qquad (19)$$

From this we have

$$\lambda+\nu = (\lambda_c+\nu_c)F_0, \quad \lambda-2\nu = (\lambda_c-2\nu_c)F_0^{-2} \qquad (20)$$

and

$$G = G_c, \quad \alpha = \alpha_c F_0. \qquad (21)$$

These formulas enable us to eliminate from Eqs. (17) and (18) the renormalized charges and the logarithm of the cut-off L:

$$3\alpha_c^2F_0^2\int_{F_0}^F\frac{dx}{x^3-\alpha_c^3F_0^3} = -\frac{G_c}{2\pi}\xi,$$

$$f = \frac{2(\lambda_c+\nu_c)}{3}\frac{F_0}{F} + \frac{\lambda_c-2\nu_c}{3}\frac{F^2}{F_0^2},$$

$$\varphi = \frac{\lambda_c+\nu_c}{3}\frac{F_0}{F} - \frac{\lambda_c-2\nu_c}{3}\frac{F^2}{F_0^2}. \qquad (22)$$

Introducing the quantity $F_c(\xi) = F(\xi)/F_0$, we can verify that the renormalization invariants $f_c = f$, $\varphi_c = \varphi$ can be expressed in terms of the renormalized quantities only:

$$f_c(\xi) = \frac{2(\lambda_c+\nu_c)}{3}\frac{1}{F_c} + \frac{\lambda_c-2\nu_c}{3}F_c^2,$$

$$\varphi_c(\xi) = \frac{\lambda_c+\nu_c}{3}\frac{1}{F_c} - \frac{\lambda_c-2\nu_c}{3}F_c^2,$$

$$3\alpha_c^2\int_1^{F_c}\frac{dx}{x^3-\alpha_c^3} = -\frac{G_c}{2\pi}\xi. \qquad (23)$$

We have still to examine the problem of the zero charge in the general case. For this purpose we rewrite Eq. (19), which defines the function F_0, in the following form:

$$\frac{1}{2}\ln\frac{(1-\alpha_c)^2(1+\alpha_cF_0+\alpha_c^2F_0^2)}{(1+\alpha_c+\alpha_c^2)(1-\alpha_cF_0)^2}$$

$$-\sqrt{3}\arctan\frac{2+\alpha_c}{\alpha_c\sqrt{3}} + \sqrt{3}\arctan\frac{2+\alpha_cF_0}{\alpha_cF_0\sqrt{3}} = \frac{G_c}{2\pi}L. \qquad (24)$$

For $L \to \infty$ the right member of this equation goes to $+\infty$ for $G_c > 0$ and to $-\infty$ for $G_c < 0$. The left member can go to $+\infty$ if $1-\alpha_cF_0 \to 0$.* Thus for $G > 0$, $\lambda > 2\nu$ the zero-charge situation does not arise, and the bare charges approach the following limiting values [cf. Ea. (20)]:

$$\lambda+\nu \to (\lambda_c+\nu_c)\alpha_c^{-1} = G_c, \quad \lambda-2\nu \to (\lambda_c-2\nu_c)\alpha_c^2 = G_c,$$

that is,

$$\lambda \to G_c, \quad \nu \to 0. \qquad (25)$$

This limiting behavior can be understood in a qualitative way. All the divergence in the theory is due to the presence of the third term in the

*If $\alpha_c^2F_0^2 + \alpha_cF_0 + 1 \to 0$, then F_0 is complex and the theory is non-Hermitian.

Hamiltonian (1) (cf. reference 2). When this term is absent ($\nu = 0$, $\nu_c = 0$), λ requires no renormalization, $\lambda = \lambda_c$. In the general case the renormalization of λ still turns out to be a finite one.

3. CONCLUSION

Various attempts to construct a consistent physically useful field theory with nonvanishing renormalized charge have so far been unsuccessful.[5] Therefore the idea has been put forward recently that it is impossible in principle for there to be a relativistically invariant Hermitian theory with a point interaction. We have analyzed here an example of such a theory, though of course it is of no use for the description of physical phenomena. It seems to us that the very fact of the existence of such a scheme shows how complicated the problem is of proving rigorously the zero-charge result

[1] W. Thirring, Annals of Physics **3**, 91 (1958).
[2] A. A. Ansel'm, J. Exptl. Theoret. Phys. (U.S.S.R.) **35**, 1522 (1958), Soviet Phys. JETP **8**, 1065 (1959).
[3] M. E. Maĭer and D. V. Shirkov, Joint Institute of Nuclear Studies Preprint, 1958.
[4] Dyatlov, Sudakov, and Ter-Martirosyan, J. Exptl. Theoret. Phys. (U.S.S.R.) **32**, 767 (1957), Soviet Phys. JETP **5**, 631 (1957).
[5] Pomeranchuk, Sudakov, and Ter-Martirosyan, Phys. Rev. **103**, 784 (1956); Abrikosov, Galanin, Gorkov, Landau, Pomeranchuk, and Ter-Martirosyan, Phys. Rev. **111**, 321 (1958).

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