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A GROUP-THEORETICAL CONSIDERATION OF THE BASIS OF RELATIVISTIC QUANTUM MECHANICS V.

The Irreducible Representations of the Inhomogeneous Lorentz Group, Including Space Inversion and Time Reversal*

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A complete classification is obtained of all the irreducible representations of the inhomogeneous Lorentz group, including space inversion and time reversal. It is shown that the concept of time parity cannot be introduced for particles with nonvanishing rest mass; for particles with zero rest mass and given spin there are two nonequivalent representations that differ in their space-time parity properties. On the basis of the concept of a universal covering group it is shown that for particles with half-integral spins the number of possible representations with different reflection laws is larger than has previously been supposed.

1. STATEMENT OF THE PROBLEM

The problem of the invariance of a quantum theory with respect to time reversal is a comparatively complicated one, and despite the large number of papers on the subject (cf., e.g., the review articles, references 5 and 6) it cannot be regarded as having been exhaustively studied up to the present.

Papers in recent years have mainly dealt with the law of time reversal for particles with spin $\frac{1}{2}$, with the Dirac equation taken as fundamental.

In order to eliminate the negative energies, however; one has to subtract out in the quantized Dirac equation the infinite energy of the negative-energy sea, i.e., to carry out a certain nonidentical operation which, as it turns out, changes the transformation properties with respect to time reversal. Moreover, in studies based on concrete equations of motion there is always the danger of getting a result relating only to particles that obey the equation in question, and not to arbitrary particles of the given spin. These limitations are not present in group-theoretical methods based only on the geometrical properties of space and on the linearity of the equations of quantum mechanics. Furthermore the work must be based on the inhomogeneous Lorentz group containing four-dimensional rotations and displacements, since the irreducible representations of precisely this group describe the free motions of relativistic particles.

Usually the group-theoretical formulation of invariance under certain transformations consists of the assumption that the wave function transforms according to one of the representations of the group in question. In treatments of time reversal, however, it is more customary to use a different formulation, due to Wigner,[7] in which a change to the complex conjugate of the wave function is associated with time reversal.

The purpose of the present paper is a consistent study of the law of time reversal in the framework of the ordinary theory of the representations of the inhomogeneous Lorentz group, without the use of a concrete form of the equations of motion. A separate paper will be devoted to the Wigner formulation of the law of time reversal. The present paper is also not concerned with problems connected with the CPT theorem, which requires the introduction of a charge operator.

The inhomogeneous Lorentz group, including all possible space-time reflections (we denote this group by G_{st}) is obtained from the group G_s, which contains only the space reflections, by adjoining the operation I_{st} of the reflection of all four coordinates to the operations contained in G_s. According to Eqs. (I.36) and (IV.43) the operator I_{st} satisfies the commutation relations

*Notations introduced without explanation are the same as in references 1–4. Equation numbers such as (I.36), (IV.1) refer to formulas in references 1–4.

$$[I_{st}, P_\mu]_+ = 0, \quad [I_{st}, M_{\mu\nu}]_- = 0, \quad [I_{st}, \Gamma_\sigma]_+ = 0. \quad (1)$$

2. THE IMPROPER GROUPS $G_1 - G_8$ OF TRANSFORMATIONS OF THE COORDINATES

In order completely to define the group G_{st} we must prescribe the commutation relations (I.31) between the components of the operators $M_{\mu\nu}$, P_λ, the relations (IV.1) between the inversion I_s and the operators $M_{\mu\nu}$, P_λ, the relations (1) for the operator I_{st}, and also all possible products of the operators I_s and I_{st} by themselves and each other. In this connection, as was already pointed out in reference 4, in order not to have to deal with two-valued representations, one must consider instead of the proper inhomogeneous Lorentz group its universal covering group.

It is important to note that the universal covering group is uniquely defined by the topological properties of the corresponding continuous group and is locally isomorphic to the latter. The transition to the universal covering group $\widetilde{G}$ is accomplished by adjoining to the generators of the group G the element $I_{2\pi}$ of a rotation through the angle 2π, which commutes with all the elements of the group and satisfies the relations

$$I_{2\pi}^2 = I, \qquad \lim_{\varphi \to 2\pi} I_\varphi = I_{2\pi} \tag{2}$$

for one of the space rotations.

As in reference 4, when we go to the improper group $\widetilde{G}_{st}$ that includes in itself the space and time reflections we must prescribe the values of the squares of the operators I_s, I_{st}, and of the operator I_t defined by the relation

$$I_{st} = I_s I_t. \tag{3}$$

The application of any reflection operator twice over returns the system to its original position. But there are two operations that leave the directions of all the axes unchanged: the identity I and the rotation $I_{2\pi}$ through the angle 2π. Since the reflections I_s, I_t, I_{st} are not reducible to each other, the square of any one of them, independent of the others, can be equal to either I or $I_{2\pi}$. Accordingly we find that there are eight different improper groups of space-time transformations $\widetilde{G}_1 - \widetilde{G}_8$, for which the squares of the reflections have the values

Group	$\widetilde{G}_1$	$\widetilde{G}_2$	$\widetilde{G}_3$	$\widetilde{G}_4$	$\widetilde{G}_5$	$\widetilde{G}_6$	$\widetilde{G}_7$	$\widetilde{G}_8$
I_s^2	I	I	$I_{2\pi}$	$I_{2\pi}$	$I_{2\pi}$	$I_{2\pi}$	I	I
I_t^2	I	$I_{2\pi}$	I	$I_{2\pi}$	$I_{2\pi}$	I	$I_{2\pi}$	I
I_{st}^2	I	$I_{2\pi}$	$I_{2\pi}$	I	$I_{2\pi}$	I	I	$I_{2\pi}$

(4)

Besides the groups $\widetilde{G}_1 - \widetilde{G}_8$ there also exist eight groups $\widetilde{G}'_1 - \widetilde{G}'_8$ for which instead of Eq. (3) we have the relation

$$I_{st} = I_{2\pi} I_s I_t. \tag{3a}$$

But substituting in Eq. (3a)

$$I_s I_{2\pi} = I'_s, \qquad I'_s I_{2\pi} = I_s,$$

we find that because of the lack of real distinction between the operators I_s and I'_s the primed groups are indistinguishable from the corresponding unprimed groups. The relations (3), (4) completely define the improper elements of each of the groups $\widetilde{G}_1 - \widetilde{G}_8$, in the sense that they suffice for the calculation of the product of any two of the improper elements (with use of Eq. (2) and of the fact that all the operators commute with $I_{2\pi}$). Multiplying Eq. (3) on the left by I_s, and also on the right by I_t, we get for all eight groups the respective relations

$$I_s^2 I_t = I_s I_{st}, \tag{5}$$

$$I_t^2 I_s = I_{st} I_t. \tag{6}$$

Multiplying Eq. (6) by I_{st}, we get

$$I_t^2 I_{st} I_s = I_{st}^2 I_t. \tag{7}$$

From Eqs. (5) and (7) it follows that

$$I_{st} I_s = (I_s^2 I_t^2 I_{st}^2) I_s I_{st}. \tag{8}$$

In the derivation of Eq. (8) we have made use of the fact that for all the groups the squares of the reflections commute with all the elements of the group, and their fourth powers are equal to the identity. In analogous fashion one can obtain the relations

$$I_{st} I_t = (I_s^2 I_t^2 I_{st}^2) I_t I_{st}, \qquad I_s I_t = (I_s^2 I_t^2 I_{st}^2) I_t I_s. \tag{9}$$

For the groups $\widetilde{G}_1 - \widetilde{G}_4$

$$I_s^2 I_t^2 I_{st}^2 = I, \tag{10}$$

and the operators I_s, I_t, I_{st} commute with each other in all representations. For the groups $\widetilde{G}_5 - \widetilde{G}_8$

$$I_s^2 I_t^2 I_{st}^2 = I_{2\pi}, \tag{11}$$

so that the operators I_s, I_t, I_{st} commute in the single-valued representations and anticommute in the double-valued representations. (We shall continue to use the usual terms "single-valued" and "double-valued" representations although, strictly speaking, with respect to the groups $\tilde{G}_1 - \tilde{G}_8$ all the representations are single-valued, and the representations with integral and half-integral spins differ in the eigenvalues of the invariant $I_{2\pi}$, which are one and minus one, respectively.)

3. THE IRREDUCIBLE REPRESENTATIONS OF THE GROUPS $\tilde{G}_1 - \tilde{G}_8$

The construction of the irreducible representations of one of the groups $\tilde{G}_1 - \tilde{G}_8$ reduces to the finding of an irreducible manifold of operators $M_{\mu\nu}$, p_λ, I_s, I_t, I_{st} that satisfy the commutation relations (I.31) that define the proper Lorentz group and also the relations (IV.1), (1), and the corresponding column of the table (4).

For the construction of the complete system of irreducible representations one can proceed as in Sec. 3 of reference 4, using the properties of the invariants of the representations of the proper group G that contains no reflections.

Repeating for the operators I_s, I_{st} (by Eq. (3) the operator I_t is not independent) the arguments given in Sec. 1 of reference 4 for the operator I_s, we get the result that irreducible representations of the group that includes the space and time reflections can either coincide with representations of the proper group or be direct sums of two or four representations. Therefore the operators I_s, I_t, I_{st} can be written in the form

$$I_s = I_{s0}\lambda_s g_s, \quad (12)$$

$$I_t = I_{t0}\lambda_t g_t, \quad (13)$$

$$I_{st} = I_{st0}\lambda_{st} g_{st}, \quad (14)$$

where I_{s0}, I_{t0}, I_{st0} are operators that act only on the variables of the representations of the proper group. λ_s, λ_t, λ_{st} are factors that do not act on the variables of the representations of the proper group (but that may, for example, cause an interchange of representations of the proper group). g_s, g_t, g_{st} are numerical coefficients equal to unity for all single-valued representations and for the double-valued representations of the groups $\tilde{G}_1$ and $\tilde{G}_5$.

For the double-valued representations of the other groups the table (4) requires that the factors g_s, g_t, g_{st} have the values

Coefficient	Group			
	$\tilde{G}_1, \tilde{G}_5$	$\tilde{G}_2, \tilde{G}_6$	$\tilde{G}_3, \tilde{G}_7$	$\tilde{G}_4, \tilde{G}_8$
g_s	1	1	i	i
g_t	1	i	1	$-i$
g_{st}	1	i	i	1

(15)

It is obvious that the operators I_{s0}, I_{t0}, I_{st0} commute with λ_s, λ_t, and λ_{st}, and that the numerical factors they contain can be chosen so that

$$I_{s0} I_{t0} = I_{st0}, \quad (16)$$

$$\lambda_s\lambda_t = \lambda_{st}. \quad (17)$$

Let us consider first irreducible representations that do not contain noncommuting reflection operators, i.e., the single-valued representations of all the groups and the double-valued representations of groups $\tilde{G}_1 - \tilde{G}_4$. According to Sec. 3 of reference 4, in the construction of the representations of groups including reflections an important part is played by the transformation properties under the reflections of the invariants of the corresponding representations of the proper group.

The basic invariants p_μ^2 and Γ_σ^2 of the proper group commute with all the reflection operators. Therefore in the classes P_π^s, P_π^a, P_π^b, P_π^α (cf. reference 2), in which there are no additional invariants, each representation of the proper group G will at the same time be also a representation of the improper group $\tilde{G}_i$ (we must remember that we are not as yet considering the double-valued representations of the groups $\tilde{G}_5 - \tilde{G}_8$). For these representations the operators I_{s0}, I_{t0}, I_{st0} are given by

$$I_{s0} = A(\mathbf{p}),\ I_{t0} = A(p_0),\ I_{st0} = A(\mathbf{p})\,A(p_0), \quad (18)$$

where A denotes an operator that changes the sign of the indicated quantity in the wave function. For example

$$A(\mathbf{p})\,\psi(\mathbf{p}) = \psi(-\mathbf{p}). \quad (19)$$

If, in accordance with Eq. (II.62), we write a wave function of a representation of the class P_π in four-dimensional polar coordinates

$$p_1 = \Pi\,\mathrm{ch}\,\chi \sin\vartheta \cos\varphi, \quad p_2 = \Pi\,\mathrm{ch}\,\chi \sin\vartheta \sin\varphi, \quad (20)$$
$$p_3 = \Pi\,\mathrm{ch}\,\chi \cos\vartheta, \quad p_0 = \Pi\,\mathrm{sh}\,\chi,$$

the action of the operator is just to replace χ by $-\chi$.

A similar situation exists for representations of class O_0 with the invariant W equal to zero (cf. reference 3)

$$W = \mathbf{M}\cdot\mathbf{N} = 0.$$

In this case also, since the remaining additional invariant F,

$$F = \mathbf{M}^2 - \mathbf{N}^2,$$

is a scalar with respect to space and time reflections, a representation of the proper group is at the same time a representation of the group $\tilde{G}_i$. In this case the operator I_{s0} is identical with that defined in Eq. (IV.14), the operator I_{t0}, is equal to I_{s0}, and the operator I_{st0} is equal to the identity. The quantities belonging to representations of this type include, in particular, four-dimensional vectors and tensors.

In this present case of representations of the classes $P_{\pi}^{s,a,b,\alpha}$, and O_0 with $W = 0$, the operators λ_s, λ_t, λ_{st} are numbers. The square of each of them is unity. Moreover, they satisfy Eq. (17). Therefore the factors λ_s, λ_t can independently take the values

$$\lambda_s = \pm 1, \tag{21}$$

$$\lambda_t = \pm 1. \tag{22}$$

Thus to each irreducible representation of the group G of any of the classes enumerated above there correspond four nonequivalent representations of the group $\tilde{G}_i$, having the same number of dimensions and differing in space and time parities. The existence of time parity for the vector and tensor representations of the homogeneous Lorentz group was first pointed out by Watanabe[8] and Shapiro.[9]

In all the other classes of irreducible representations of the proper group there are additional invariants that change sign on one or more of the reflections. Therefore in these cases an irreducible representation of a group $\tilde{G}_i$ cannot be an irreducible representation with respect to the proper group G, and will be the direct sum of representations corresponding to different signs of the additional invariant in question.

In the representations of greatest importance for physics, of the class $P_{\pm m}^{s}$ (cf. reference 2), and also in representations of the classes $P_{\pm 0}^{c}$, $P_{\pm 0}^{-c}$, $P_{\pm 0}^{\alpha}$ (cf. reference 3), the only additional invariant is the sign of the energy, S_H (cf. reference 2, Sec. 3). The operator S_H commutes with I_s and anticommutes with I_t, I_{st}

$$[S_H, I_s]_- = 0, \quad [S_H, I_t]_+ = 0, \quad [S_H, I_{st}]_+ = 0. \tag{23}$$

Since by definition the operators I_{s0}, I_{t0}, I_{st0} act only on the variables of the proper group, they must commute with the invariant S_H of the proper group. Therefore it follows from Eqs. (12) — (14) and (23) that

$$[\lambda_s, S_H]_- = 0, \ [\lambda_t, S_H]_+ = 0, \ [\lambda_{st}, S_H]_+ = 0. \tag{24}$$

Since according to Eq. (23) the operators I_t, I_{st} acting on a wave function change the sign of the energy, in a representation irreducible with respect to $\tilde{G}_i$ there must appear representations of G with both signs of the energy. Therefore the representations irreducible with respect to $\tilde{G}_i$ will be direct sums

$$P^s_{+m} \dotplus P^s_{-m} \tag{25}$$

and similarly for the classes $P_{\pm 0}^{c,-c}$. A wave function of the representation (25) will depend on the variables of the proper group, i.e., on the three-dimensional momentum $\mathbf{p}$ and the spin component, and also will be a two-rowed matrix, with indices referring to the representations P_{+m}, P_{-m} (P_{+0}, P_{-0}). In accordance with Eqs. (I.11), (I.12), (IV.1), (IV.11), (1), (12) — (14), (19), and (24), the operators $\mathbf{M}$, $\mathbf{N}$, $\mathbf{p}$, p_0, I_{s0}, I_{t0}, I_{st0} have for the representation (25) the forms, to within a unitary transformation,

$$\begin{gathered}\mathbf{M} = -i\,[\mathbf{p}\partial/\partial\mathbf{p}] + \mathbf{S}, \quad \mathbf{N} = iE_p\partial/\partial\mathbf{p} - [\mathbf{S}\mathbf{p}]/(E_p + m), \\ \hat{\mathbf{p}} = \mathbf{p}, \ p_0 = \rho_3 E_p, \ I_{s0} = A(\mathbf{p}), \ I_{t0} = \mathrm{I}, \ I_{st0} = A(\mathbf{p}), \\ \lambda_s = \pm \mathrm{I}, \ \lambda_t = \rho_1, \ \lambda_{st} = \lambda_s\lambda_t,\end{gathered} \tag{26}$$

Here and in our other formulas the matrices ρ_1, ρ_2, ρ_3 have the forms of the Pauli matrices

$$\rho_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \rho_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \rho_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{27}$$

and act on the wave function of the representation (25), namely

$$\Omega = \begin{pmatrix} \Omega_{+m} \\ \Omega_{-m} \end{pmatrix}. \tag{28}$$

The double sign on the λ_s in Eq. (26) means that for given spin and absolute value of the mass there exist two nonequivalent representations that differ only in the spatial parity. It might seem that, independently of λ_s, one could also assign both signs to the operator λ_t and thus introduce two more representations, differing in the time parity. But the representations with $\lambda_t = -\rho_1$ reduce to those in Eq. (26) by the transformation

$$\Omega = \rho_3\Omega' \tag{29}$$

Since representations of the class P_m^s describe free particles (with non-zero rest mass) we can say, contrary to the statement in references 5 and 6, that from the point of view of the theory of the representations of the inhomogeneous Lorentz group the concept of time parity cannot be introduced for physical particles. We note further that here, as in Sec. 5 of reference 4, for the double-valued representations there exists only the relative parity of a pair of particles, but not absolute parity of each individual particle. The irreducible representations of the classes $P_{\pm 0}^{c,-c,\alpha}$ have the same structure

$$P_{+0}^{c,-c,\alpha} \dot{+} P_{-0}^{c,-c,\alpha} \tag{30}$$

and are constructed in analogous ways. With the exception of the operators **M** and **N**, which must be chosen in accordance with Sec. 1 of reference 3, all of the operators have the forms shown in Eq. (26).

Let us now examine the representations of class O_0 with the invariant W not equal to zero,

$$W = \mathbf{M}\cdot\mathbf{N} \neq 0. \tag{31}$$

According to Eqs. (IV.1) and (1) the operator W commutes with I_{st} and anticommutes with I_s and I_t

$$[I_s,\ W]_+ = 0,\ [I_t,\ W]_+ = 0,\ \ [I_{st},\ W]_- = 0. \tag{32}$$

Therefore the direct sum

$$P_W \dot{+} P_{-W}, \tag{33}$$

is a representation that is irreducible with respect to $\widetilde{G}_i$, and the operators λ_s, λ_t, λ_{st} will have the forms

$$\lambda_s = \pm\rho_1,\ \lambda_t = \rho_1,\ \lambda_{st} = \lambda_s\lambda_t. \tag{34}$$

The operator I_{s0}, which we denote by B for this case, is determined in accordance with Eq. (IV.14), and the operator I_{st0} is equal to the identity, so that we can write

$$I_{s0} = I_{t0} = B,\quad I_{st0} = \mathrm{I}. \tag{35}$$

Thus for representations of the type (33) (in particular for the Dirac bispinor) it would also seem that there is a distinction between representations for the space parity and that the concept of time parity does not exist. Such a conclusion, however, is not entirely correct, since on applying to the representation

$$\lambda_s = -\rho_1,\ \lambda_t = \rho_1,\ \lambda_{st} = -1$$

the equivalence transformation

$$\Omega = \rho_3\Omega',$$

we find that for representations of this type the time parity exists and space parity does not. Obviously both of these conclusions lack invariance under equivalence transformations. The difference from the preceding case is that here λ_{st} is a multiple of the unit matrix, instead of λ_s. Strictly speaking, therefore, for this type of representation only space-time parity exists.

The representations $P_{\pi}^{\pm l}$ have as an additional invariant the sign of the fourth component of the pseudovector Γ_σ, denoted in reference 4 by the symbol S_Γ. According to Eqs. (IV.1) and (1) the operator S_Γ commutes with I_t and anticommutes with I_s, I_{st}

$$[I_s,\ S_\Gamma]_+ = 0,\ \ [I_t,\ \ S_\Gamma]_- = 0,\ [I_{st},\ S_\Gamma]_+ = 0. \tag{36}$$

The direct sum

$$P_\Pi^{+l} \dot{+} P_\Pi^{-l}, \tag{37}$$

is a representation irreducible with respect to $\widetilde{G}_i$, and the operators λ_s, λ_t, λ_{st} have the forms

$$\lambda_s = \rho_1,\ \ \lambda_t = \pm 1,\ \lambda_{st} = \lambda_s\lambda_t. \tag{38}$$

The operators I_{s0}, I_{t0}, I_{st0} have the values (18). In this case the time parity exists, but there is no space parity.

Finally, in the representations of the class $P_{\pm 0}^{\pm\Sigma}$, which is important for physics (the wave functions of particles of zero rest mass transform according to representations of this type), there exist the two additional invariants S_H and S_Γ, which do not commute with the reflections [cf. Eqs. (23) and (26)]. Therefore here a representation irreducible with respect to $\widetilde{G}_i$ will be the direct sum of four representations corresponding to the four possible choices of the signs of the invariants S_H, S_Γ

$$P_{+0}^{+\Sigma} \dot{+} P_{+0}^{-\Sigma} \dot{+} P_{-0}^{+\Sigma} \dot{+} P_{-0}^{-\Sigma}.$$

Here the operators I_{s0}, I_{t0}, I_{st0} are the same as in Eq. (26). For the operators λ_s, λ_t, λ_{st} there are two possible inequivalent systems of four-rowed matrices

$$\lambda_s = \rho_1,\ \ \lambda_t = \rho_1',\ \lambda_{st} = \rho_1\rho_1', \tag{39}$$

and

$$\lambda_s = \rho_1\rho_3',\ \lambda_t = \rho_3\rho_1',\ \lambda_{st} = \rho_2\rho_2'. \tag{40}$$

A wave function has the form

$$\Omega = \begin{pmatrix}\Omega_{+0}\\ \Omega_{-0}\end{pmatrix},\quad \Omega_{+0} = \begin{pmatrix}\Omega_{+0}^{+\Sigma}\\ \Omega_{+0}^{-\Sigma}\end{pmatrix},\quad \Omega_{-0} = \begin{pmatrix}\Omega_{-0}^{+\Sigma}\\ \Omega_{-0}^{-\Sigma}\end{pmatrix}. \tag{41}$$

The primed matrices ρ' act on the first of the functions (41), i.e., on the variable sign of the energy; the unprimed matrices act on the variable sign S_Γ.

Thus to each value of $|\Sigma|$ there correspond two nonequivalent irreducible representations of the group $\widetilde{G}_i$.

Let us now consider the double-valued representations of the groups $\widetilde{G}_5 - \widetilde{G}_8$, in which the operators I_s, I_t, I_{st} anticommute with each other. The operators I_{s0}, I_{t0}, I_{st0} are not changed in this case. The operators λ_s, λ_t, λ_{st} are changed in two respects. Firstly, they must now anticommute with each other; secondly, in accordance with the table (15) the square of each of them must be equal to minus one. In the foregoing discussion, in representations of the classes P_π^s, P_π^a, P_π^b, P_π^α, and O_0 with $W = 0$, the operators λ were all numbers. Therefore, in order to make these operators anticommuting quantities, one has to double the dimensionality of the representation. Thus for the classes listed above the double-valued representations of the groups $\tilde{G}_5 - \tilde{G}_8$ have doubled dimensionalities as compared with the analogous representations of the groups $\tilde{G}_1 - \tilde{G}_4$. In all the other classes the operators λ are matrices that can be taken to be anticommuting without changing the dimensionality of the representation. On going over to the double-valued representations of the groups $\tilde{G}_5 - \tilde{G}_8$ the number of nonequivalent representations, differing as to parity, remains unchanged in all classes. The final results as to the factors λ can be summarized as follows:

		$P_\Pi^{s,a,b,\alpha}$ O_0 for $W = 0$	$P^s_{\pm m}$ $P^{c,-c,\alpha}_{\pm 0}$	O_0 for $W \neq 0$	$P_\Pi^{\pm l}$	$P^{\pm\Sigma}_{\pm 0}$	
$\tilde{G}_1 - \tilde{G}_4$	λ_s	$\pm I$	$\pm I$	$\pm\rho_1$	ρ_1	ρ_1	$\rho_1\rho_3'$
	λ_t	$\pm I$	ρ_1	ρ_1	$\pm I$	ρ_1'	$\rho_3\rho_1'$
$\tilde{G}_5 - \tilde{G}_8$	λ_s	$\pm i\rho_3$	$\pm i\rho_3$	$\pm i\rho_1$	$i\rho_2$	$i\rho_2$	$i\rho_1\rho_3'$
	λ_t	$\pm i\rho_2$	$i\rho_2$	$i\rho_2$	$\pm i\rho_3$	$i\rho_3\rho_1'$	$i\rho_2'$

(42)

If we compare the representations of the group $\tilde{G}_s$ (G_s') containing only the space reflections with the representations of the group $\tilde{G}_i$ (other than the double-valued representations of groups $\tilde{G}_5 - \tilde{G}_8$), it turns out that when the time reflection is included the dimensionalities of the irreducible representations are doubled in the classes $P_{\pm m}$, $P_{\pm 0}$, and are unchanged in the other cases. We note also for completeness that besides the representations enumerated above the groups $\tilde{G}_1 - \tilde{G}_8$ also have three single-valued one-dimensional irreducible representations J_s, J_t, J_{st}, for which

$$M_{\mu\nu} \equiv 0, \quad p_\lambda \equiv 0, \tag{43}$$

and the operators I_s, I_t, I_{st} are shown in the table:

Representation	Coefficient I_s	I_t	I_{st}
J_s	-1	1	-1
J_t	1	-1	-1
J_{st}	-1	-1	1

(44)

4. DISCUSSION OF RESULTS

In the preceding section we have found all the irreducible representations of the inhomogeneous Lorentz group that contains the space and time reflections, and now we can discuss the problem of what physical results can be obtained from a discussion of time reversal carried out in the framework of the usual representation theory (without the Wigner change to the complex conjugate of the wave function.

The discussion here is more general than those in previous papers on this problem in the following three ways. First, no use has been made of concrete equations of motion, so that the results depend only on the geometrical properties of space-time. Second, the treatment has been based on the inhomogeneous Lorentz group, including not only four-dimensional rotations, but also four-dimensional displacements. Third, use has been made of the concept of the universal covering group, which has made possible a more precise examination of the problem of the factors ± 1, $\pm i$ in the reflection operators. These differences in the statement of the problem have led to a number of differences in the final results.

Ordinarily the study of the transformation properties under reflections has been made not with free-particle wave functions, transforming according to the infinite-dimensional (in the momentum coordinates) representations of the inhomogeneous Lorentz group, P_m^s, $P_{\pm 0}^\Sigma$, but with the spin parts of the wave functions as defined for concrete equations of motion. These spin parts of the wave functions transform according to representations of the class O_0; for integral spins $W = 0$, and for half-integral spins $W \neq 0$. This artificial and actually incorrect separation of the coordinate (momentum) part of the wave function has led to a number of results that do not follow from the rigorous statement of the problem.

The differences that thus arise are easily traced out in the table (42). We see that the representations of the class O_0 with $W = 0$ (scalars, vectors, and so on) have independent space and time parities.[8,9] For real particles with nonvanishing rest mass, however, (class P_m) the time parity is absent. On the other hand, for particles with zero rest mass (class P_0), according to reference 4, the intrinsic space parity is absent, but still, according to the table (42), there exist two systems of representations that differ in their parity properties with respect to space and time reflections. Another difference between our treatment and previous ones lies in the establishing of the eight groups $\tilde{G}_1 - \tilde{G}_8$, with different double-valued representations. Inclusion of the fact that the squares of the reflections can be either positive or negative for the double-valued representations, together with the stipulation that there cannot exist at the same time particles with different signs for the squares of the reflections,[6] is equivalent to the introduction of the universal covering group. In reference 6, however, only reflections that anticommute with each other were considered, i.e., only the groups $\tilde{G}_5 - \tilde{G}_8$. The representations of the groups $\tilde{G}_1 - \tilde{G}_4$ were rejected in reference 6 on the grounds that in this case the Dirac equation leads to an incorrect relation between energy and momentum. The present treatment shows that wave functions can also transform according to representations of the groups $\tilde{G}_1 - \tilde{G}_4$ with commuting reflection operators in the double-valued representations.

We note also that according to our treatment the existence of antiparticles does not follow from the transformation properties under reflections but, according to reference 4, is a direct consequence of the pseudoscalar nature of the charge for charged particles.

All of the results obtained here can be applied to arbitrary kinds of particles, including those that obey the Dirac equation, both in the quantized and also in the unquantized form; a complete analysis of the relations will be given in a separate paper.

5. ON THE INVARIANCE OF THE EQUATIONS OF QUANTUM THEORY

The equations of a quantum theory must be covariant in the sense that the physical results must not depend on the sense of the time axis.[6] From the covariance of an equation, however, it by no means follows that it must be invariant. Moreover, a correct equation for the wave function (state vector), containing no infinities of any kind, cannot be invariant with respect to time reversal. In fact, the condition for the invariance of the equation

$$L\Omega = 0, \tag{45}$$

where L is a linear operator, has the form

$$[L, I_{st}]_- = 0. \tag{46}$$

Therefore, if Ω is a solution of Eq. (45), in virtue of Eq. (46) the function $I_{st}\Omega$ will also be a solution. According to the first of the relations (1) the functions Ω and $I_{st}\Omega$ describe physical systems with different signs of the energy and mass. But masses of different signs not only are never observed in nature, but also are not permissible, since a system of two particles with masses of different signs can have a space-like energy-momentum vector, which leads to signals faster than light and the loss of causality. Thus when we use representations of the usual type the equations of motion must be noninvariant with respect to time reversal and have solutions corresponding to only one sign of the mass. On time reversal the signs of all masses and energies are changed. The formal invariance of a number of equations in general use exists either because they have solutions with both signs of the mass (the unquantized Dirac equation) or because of an infinite vacuum background (the quantized scalar and Dirac equations).

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159