REACTIONS INVOLVING POLARIZED PARTICLES OF ZERO REST MASS

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In this paper the group-theoretical point of view is used for the description of the spin states of particles of zero rest mass. Complete sets of operators and of their eigenfunctions for a system of two particles are found in the representation of the momenta and spins. The statistical tensors for the particles produced in a reaction of the type a + b → c + d or a → c + d are obtained in the case in which one of these particles has no rest mass. The most general selection rules are derived for the reaction a + b → c + d in the form of relations between the statistical tensors, under the condition that the space and time parities are conserved. The wave functions are calculated for a system of two identical particles of zero rest mass.

1. INTRODUCTION

IN reference 1 a study was made of the relativistic theory of reactions involving polarized particles with non-zero rest masses. This theory is not applicable to reactions involving photons and neutrinos. In what follows we shall call a particle (a photon, neutrino, or graviton) with zero rest mass a γ-particle. The purpose of the present paper is to study the theory of reactions involving γ-particles.

The general theory of reactions involving photons has already been extensively developed in papers of Simon[2] and of Morita and others;[3] these authors describe the spin states of photons by a vector potential, using the Lorentz supplementary condition to exclude the longitudinal and scalar components. Such a procedure is complicated, and it is difficult to extend it to cases of higher spins.

The group-theoretical description of the spin states of γ-particles has been given by Wigner and by Yu. M. Shirokov.[4] Starting from the theory of the irreducible representations of the inhomogeneous Lorentz group, they showed that the spin states of free γ-particles are completely determined by the operator

$$\hat{\Sigma} = (\mathbf{j} \cdot \mathbf{n}), \qquad (1)$$

where $\mathbf{j}$ is the total angular momentum of the γ-particle; $\mathbf{n}$ is the unit vector parallel to the momentum of the γ-particle. $\hat{\Sigma}$ is a Lorentz-invariant operator with integral and half-integral values of i. Since the total angular momentum $\mathbf{j}$ is an axial vector and $\mathbf{n}$ is a polar vector, the operator $\hat{\Sigma}$ transforms as a pseudoscalar under space reflections. Thus if a γ-particle has a definite parity, then for each momentum there exist two spin states with eigenvalues $\pm i$ of the operator $\hat{\Sigma}$. If, on the other hand, it does not have a definite parity, then it can have only one spin state (of longitudinal polarization).

In the present paper we use the operator $\hat{\Sigma}$ for the description of the spin states of γ-particles. As can be seen from Eq. (1), the operator $\hat{\Sigma}$ is closely connected with the direction of the momentum. Therefore we shall work directly in the momentum representation. We emphasize that a spin vector does not exist for a γ-particle, and the total angular momentum $\mathbf{j}$ cannot be represented in the form of the sum of an orbital angular momentum and a spin. Because of this the complete sets of operators for the description of the states of a system containing γ-particles, given in Sec. 2 of the present paper, differ from the sets generally used for a system without γ-particles. In Sec. 2 we shall find the complete sets of operators and of their eigenfunctions for systems of two particles. To obtain the formulas for the statistical tensors (s-tensors for short) of reaction products, and to obtain the selection rules under the condition that space and time parities are conserved, we use a method developed by M. I. Shirokov.[5,6]

Landau and Yang have established the existence of specific selection rules for the decay of a particle into two photons.[7] Generalized selection rules of this type have been obtained by Shapiro.[8] Because of the peculiarities of a system of two identical γ-particles it is of interest to calculate the explicit form of the wave functions of this sys-

tem for a definite total angular momentum, angular momentum component, and parity; these functions can be used to obtain angular distributions and polarizations for the decay of particles into pairs of identical γ-particles. It is shown that certain selection rules remain valid when parity is not conserved in a decay.

2. COMPLETE SETS OF OPERATORS AND OF THEIR EIGENFUNCTIONS

We denote state vectors of γ-particles characterized by the momentum $\mathbf{p}$ and the spin operator $\hat{\Sigma}$ by symbols $|\mathbf{p}, \mu\rangle$, where $\mu = \pm i$ is the eigenvalue of the operator $\hat{\Sigma}$. Since $\hat{\Sigma}$ is invariant under the proper Lorentz group, μ remains unchanged by a Lorentz transformation. Therefore state vectors in different coordinate systems differ only by a phase factor

$$|\mathbf{p}, \mu\rangle \overset{L}{\to} |\mathbf{p}, \mu\rangle' = \eta_\mu(L, \mathbf{p}) |L^{-1}\mathbf{p}, \mu\rangle, \qquad (2)$$

where L is a Lorentz transformation; $\eta_\mu(L, \mathbf{p})$ satisfies the group relation

$$\eta_\mu(L_1, \mathbf{p}) \eta_\mu(L_2, L_1^{-1}\mathbf{p}) = \eta_\mu(L_2 L_1, \mathbf{p}).$$

The explicit form of the factor η_μ depends on the choice of the coordinate system in which the total angular momentum is quantized. If we choose the axis of quantization of the total angular momentum in such a way that the z axis is parallel to $\mathbf{p}$ and the x axis to $[\mathbf{v} \times \mathbf{p}]$, where $\mathbf{v}$ is the velocity of the new coordinate system relative to the old for the Lorentz transformation L, then by means of the formalism developed in reference 9 it can easily be shown that in this case $\eta_\mu = 1$.*

From Eq. (2) we easily find the transformation function between the state vectors of a γ-particle in two different coordinate systems 1 and 2:

$$\langle \mathbf{p}_1, \mu_1 | \mathbf{p}_2, \mu_2 \rangle$$

$$= \sqrt{p_1/p_2}\, \delta^3 \left(p_{1i} - \sum_{j=1}^{4} a_{ij}\, p_{2j} \right) \delta_{\mu_1 \mu_2}, \quad i = 1, 2, 3, \qquad (3)$$

where a_{ij} is the matrix of the Lorentz transformation, and the factor $(p_1/p_2)^{1/2}$ is the square root of the Jacobian of the transformation,

$$[\partial(p_{1x}, p_{1y}, p_{1z})/\partial(p_{2x}, p_{2y}, p_{2z})]^{1/2}.$$

We note that if the rest mass of the particle is not equal to zero but its speed approaches that of light, then the angle of rotation of the spin in a Lorentz transformation (cf. reference 1) goes to

zero,[10] so that the corresponding transformation function approaches the form (3).

Let us consider the state of a system of two free particles 1 and 2. The rest mass of the system of two free particles is in general not zero.* As is shown in reference 1, for such a system of particles there exists a reference system in which the total momentum is zero and the total angular momentum equals the angular momentum relative to the center of mass of the system of particles. In what follows all results are given in the center-of-mass system. They can be expressed in another coordinate system, if need be, by means of the transformation function (3).

Let us denote the total angular momentum of the system by $\mathbf{J}$, and the total angular momentum of particle α by $\mathbf{j}_\alpha$. We suppose at first that the rest mass of particle 2 is not zero. In this case $\mathbf{j}_2$ can be represented as a sum of an orbital angular momentum $\mathbf{l}_2$ and a spin $\mathbf{s}_2$.[1] We introduce a new vector $\mathbf{j}$ in the following way:

$$\mathbf{J} = \mathbf{j}_1 + \mathbf{j}_2 = \mathbf{j} + \mathbf{s}_2, \quad \mathbf{j} = \mathbf{j}_1 + \mathbf{l}_2. \qquad (4)$$

The spin operator of particle 2 commutes with the operator $\mathbf{j}$, by definition. Since $\mathbf{J}$ and $\mathbf{s}_2$ satisfy the usual commutation relations for angular momenta, $\mathbf{j}$ also satisfies these relations.

In the center-of-mass system a complete set of operators for the two-particle system consists of the following operators:

$$\mathbf{J}^2, \ J_z, \ \mathbf{j}^2, \ \hat{\Sigma}_1, \ \mathbf{p}. \qquad (5)$$

If particles 1 and 2 have definite parities and parity is conserved in the reaction under consideration, then it is better to replace the operator $\hat{\Sigma}_1$ in the set (5) by the space-inversion operator I.

To express observable quantities in terms of diagonal elements of the R matrix ($R = S - 1$, where S is the scattering matrix), we must find the eigenfunctions of the complete set of operators (5) in the representation of the momenta and spins of the free particles. We write these eigenfunctions in the center-of-mass system in the form

$$\langle \mathbf{n}, \mu_1, \mu_2, p | J, M, j, \mu_1', p' \rangle, \qquad (6)$$

where μ_α is the eigenvalue of the operator $(\mathbf{j}_\alpha \mathbf{n})$, M is the eigenvalue of the operator J_z, and $\mathbf{n}$ is the unit vector parallel to the direction of the relative momentum $\mathbf{p}$.

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*If the coordinate system is chosen as in reference 9, then $\eta_\mu(L, \mathbf{p}) = \exp[i\mu\varphi(L, \mathbf{p})]$, where $\varphi(L, \mathbf{p})$ is given by Eq. (1.91) of reference 9.

*The rest mass M of the system is given by

$$M^2 = (\sqrt{p_1^2 + M_1^2} + \sqrt{p_2^2 + M_2^2})^2 - (p_1 + p_2)^2,$$

and is zero only when the rest masses M_1 and M_2 of the two particles are zero and their momenta are parallel ($p_1 \parallel p_2$).

The functions (6) are determined by the following conditions.

1. If we choose the z axis in the direction **n**, then

$$\langle \mathbf{k}, \mu_1, \mu_2, p \,|\, J, M, j, \mu_1', p' \rangle = C\delta_{\mu_1\mu_1'}\, \delta_{\mu_1+\mu_2, M}\, \delta_{pp'}, \quad (7)$$

where **k** is the unit vector in the direction of the z axis, and C is a constant which depends on J, j, and μ_α. The relation (7) follows from the equation $J_z = \hat{\Sigma}_1 + \hat{\Sigma}_2$, if the z axis is parallel to **n**.

2. Under a three-dimensional rotation the functions (6) transform in the following way (like spherical functions):

$$\langle \mathbf{n}, \mu_1, \mu_2, p \,|\, J, M, j, \mu_1', p' \rangle$$
$$= \sum_{M'} \langle \mathbf{n}', \mu_1, \mu_2, p \,|\, J, M', j, \mu_1', p' \rangle D_{M'M}^J(g), \quad (8)$$

if the transformation of the unit vector **n** under the rotation is written $\mathbf{n}' = g\mathbf{n}$ (cf. Note 4 in reference 5). In what follows we shall use the Euler angles $\{\varphi_1, \vartheta, \varphi_2\}$ and the functions $D_{M'M}^J$ introduced in reference 5. We emphasize that the angle ϑ is defined here as a rotation around the y axis.

3. The total angular momentum **J** is the vector sum of the operators **j** and $\mathbf{s}_2$ (Eq. (4)).

4. The functions (6) form a complete set of orthogonal and normalized eigenfunctions. The normalization is based on a volume V with the radius R.

5. The time-reversal operator T is represented by the product of a unitary operator K and the operation of complex conjugation. The phase factor of the function (6) is determined by the condition[11]

$$K\langle \mathbf{n}, \mu_1, \mu_2, p \,|\, J, M, j, \mu_1', p' \rangle^*$$
$$= \eta_T \langle -\mathbf{n}, \mu_1, \mu_2, p \,|\, J, M, j, \mu_1', p' \rangle^*$$
$$= (-1)^{J+M} \langle \mathbf{n}, \mu_1, \mu_2, p \,|\, J, -M, j, \mu_1', p' \rangle. \quad (9)$$

Taking $\mathbf{n}' = \mathbf{k}$ in Eq. (8) and substituting Eq. (7) in the right member of Eq. (8), we get

$$\langle \mathbf{n}, \mu_1, \mu_2, p \,|\, J, M, j, \mu_1', p' \rangle = C\delta_{\mu_1\mu_1'}\, \delta_{pp'}\, D_{\mu_1+\mu_2, M}^J(g_\mathbf{n}), \quad (10)$$

where $g_\mathbf{n}$ is the rotation of the coordinate system zyx into the system $z_1 y_1 x_1$ (z_1 axis parallel to **n**). If we choose the y_1 axis in the direction of $[\mathbf{k}\times\mathbf{n}]$, the Euler angles for the rotation $g_\mathbf{n}$ are $\{-\pi, \vartheta, \pi - \varphi\}$ where ϑ and φ are the spherical angles of the unit vector **n** in the zyx coordinate system.

The constant C is determined by conditions 3, 4, and 5. Choosing the phase factor of the state vector $|\mathbf{n}, \mu_1, \mu_2, \mathbf{p}\rangle$ so that $\eta_T = (-1)^{\mu_1+\mu_2}$, and using the relation

$$D_{\mu_1+\mu_2, M}^{J*}(g_{-\mathbf{n}}) = (-1)^{J+M-\mu_1-\mu_2} D_{\mu_1+\mu_2, -M}^J(g_\mathbf{n}),$$

we find the phase factor C from Eq. (9). We thus get

$$C = \frac{2\pi\hbar}{p}\frac{\sqrt{R}}{V\sqrt{V}}\left(\frac{2j+1}{4\pi}\right)^{1/2} C_{j\mu_1 s_2\mu_2}^{J\mu_1+\mu_2}. \quad (11)$$

Under the reflection I the function (10) with the constant C from Eq. (11) transforms in the following way:

$$I\langle \mathbf{n}, \mu_1, \mu_2, p \,|\, J, M, j, \mu_1', p' \rangle$$
$$= I_0' \eta_I \langle -\mathbf{n}, -\mu_1, -\mu_2, p \,|\, J, M, j, \mu_1', p' \rangle$$
$$= I_0' \eta_I (-1)^{-j-s_2} \langle \mathbf{n}, \mu_1, \mu_2, p \,|\, J, M, j, -\mu_1', p' \rangle, \quad (12)$$

where I_0' is the product of the intrinsic parities of the two particles.

We assume that $I^2 = 1$ in Eq. (12). This condition determines the factor η_I apart from its sign: $\eta_I = \pm(-1)^{s_2+i}$, or

$$I\langle \mathbf{n}, \mu_1, \mu_2, p \,|\, J, M, j, \mu_1', p' \rangle$$
$$= \pm(-1)^{j-i_1} I_0' \langle \mathbf{n}, \mu_1, \mu_2, p \,|\, J, M, j, -\mu_1', p' \rangle, \quad (13)$$

where $i_1 = |\mu_1|$. The eigenfunctions with definite parity I' have the form

$$\langle \mathbf{n}, \mu_1, \mu_2, p \,|\, J, M, j, I', p' \rangle$$
$$= 2^{-1/2}(1 + I'I)\langle \mathbf{n}, \mu_1, \mu_2, p \,|\, J, M, j, i_1, p' \rangle, \quad (14)$$

where I' is an eigenvalue of the operator I.

We introduce new variables t:

$$I' = \pm(-1)^{j-i_1+t}.$$

Here t = 0 or t = 1. In reactions with photons one value corresponds to electric radiation, and the other to magnetic radiation (cf. reference 3). The lack of uniqueness in the sign affects only the value of t, but cannot affect the final observable results for the density matrix. Therefore in what follows we can consider the expression (13) with the plus sign only.

Substituting Eqs. (10), (11), and (13) in Eq. (14), we get the eigenfunctions of the complete set of operators $\mathbf{J}^2$, J_z, $\mathbf{j}^2$, I, and **p** in the representation of the momenta and spins:

$$\langle \mathbf{n}, \mu_1, \mu_2, p \,|\, J, M, j, I', p' \rangle$$
$$= \frac{2\pi\hbar}{p}\frac{\sqrt{R}}{V\sqrt{V}}\left(\frac{2j+1}{4\pi}\right)^{1/2} C_{j\mu_1 s_2\mu_2}^{J\mu_1+\mu_2} D_{\mu_1+\mu_2, M}^J(g_\mathbf{n}) \delta_{pp'} \sigma^t, \quad (16)$$
where

$$\mu_1 = i_1\sigma, \quad \sigma = \pm 1.$$

If the rest masses of both particles are zero, we cannot introduce the operator **j**. In this case we use the following complete set of operators:

$$\mathbf{J}^2, J_z, \hat{\Sigma}_1, \hat{\Sigma}_2, p. \quad (17)$$

The eigenfunctions of the operators (17) are found in a similar way, and have the form

$$\langle \mathbf{n}, \mu_1, \mu_2, p \mid J, M, \mu_1', \mu_2', p' \rangle$$

$$= \frac{2\pi\hbar}{p} \frac{V R}{V V} \left(\frac{2J+1}{4\pi} \right)^{1/2} D^J_{\mu_1+\mu_2, M} (g_\mathbf{n}) \, \delta_{\mu_1\mu_1'} \delta_{\mu_2\mu_2'} \delta_{pp'}. \quad (18)$$

The complete sets (5) and (18) can also be used for a system of two particles with nonvanishing rest masses, which is usually described by the set $\mathbf{J}^2$, $\mathbf{J}_Z$, $\mathbf{l}^2$, $\mathbf{S}^2$, $\mathbf{p}$, where l is the orbital angular momentum of the relative motion, and $\mathbf{S} = \mathbf{l}_1 + \mathbf{l}_2$ is the sum of the spins of the two particles. Between the eigenfunctions of these sets of operators there exist unitary transformations (transformation functions)

$$\langle J, M, j, \mu, p \mid J', M', l, S, p' \rangle$$

$$= \delta_{JJ'} \delta_{MM'} \delta_{pp'} (2l+1)^{1/2} (2S+1)^{1/2} C^{J\mu}_{l0S\mu} W (lSji_2, Ji_1)$$

and

$$\langle J, M, \mu_1, \mu_2, p \mid J', M', l, S, p' \rangle$$

$$= \delta_{JJ'} \delta_{MM'} \delta_{pp'} \{ (2l+1)/(2J+1) \}^{1/2} C^{S\mu_1+\mu_2}_{i_1\mu_1 i_2\mu_2} C^{J\mu_1+\mu_2}_{l0S\mu_1+\mu_2},$$

where W (abcd, ef) is the Racah coefficient.

3. GENERAL FORMULAS FOR THE ANGULAR DISTRIBUTIONS AND THE POLARIZATION VECTORS AND TENSORS FOR THE REACTIONS a + b → c + d AND a → c + d

Let us first consider the density matrix of the γ-particles, $\langle \mu_\gamma \mid \rho \mid \mu_\gamma' \rangle$. Since μ_γ takes only the two values $\pm i$, we cannot construct the s-tensors from $\langle \mu_\gamma \mid \rho \mid \mu_\gamma' \rangle$ in the ordinary way. Fano[12] has shown that for photons it is convenient to use the Stokes parmeters. This idea is easily generalized and is adopted in the present work. The Stokes parameters are related to the density matrix of the γ-particles in the following way:

$$\rho (q_\gamma \chi_\gamma) = 2^{1/2} \sum_{\sigma_\gamma, \sigma_\gamma'} (-1)^{1/2-1/2\sigma_\gamma'} C^{q_\gamma \chi_\gamma}_{1/2, 1/2\sigma_\gamma, 1/2, -1/2\sigma_\gamma'} \langle \mu_\gamma \mid \rho \mid \mu_\gamma' \rangle, \quad (20)$$

where $\rho (q_\gamma, \chi_\gamma)$ are the Stokes parameters, $\mu_\gamma = i_\gamma \sigma_\gamma$, and $\mu_\gamma' = i_\gamma \sigma_\gamma'$.

We emphasize that $\rho (1, \chi_\gamma)$ defined by Eq. (20), do not transform like a vector under rotations. The physical interpretation of the Stokes parameters given in Fano's paper for photons is also correct for any γ-particles. We shall not repeat it. In this paper the Stokes parameters will for convenience also be called s-tensors.

The calculations of the s-tensors of the reaction-product particles is made by the method of M. I. Shirokov.[5] We use the notations introduced in reference 5. If the masses of particles 1 and 2 in the initial and final states of the reaction are not zero, we use the complete set of operators

$\mathbf{J}^2$, $\mathbf{J}_Z$, $\mathbf{l}^2$, $\mathbf{S}^2$, $\mathbf{p}$, where l is the orbital angular momentum and $\mathbf{S}$ is the total spin of the two particles. We get the following results:

1. $\gamma + b \rightarrow c + d$.

$$\rho' (q_c \chi_c q_d \chi_d; \mathbf{n}_c)$$

$$= [N_1/(4\pi)^2] [(2i_c + 1) (2i_d + 1) 2^{-1} (2i_b + 1)^{-1}]^{1/4}$$

$$\times \sum Y^*_{q_c \chi_c q_d \chi_d} (J_1 l_1' S_1', \ Jq'\chi', \ J_2 l_2' S_2') \langle S_1' l_1' \alpha \mid R^{J_1} \mid j_1 t_1 \alpha_1 \rangle$$

$$\times \langle S_2' l_2' \alpha' \mid R^{J_2} \mid j_2 t_2 \alpha_2 \rangle^* \ Y_{q_\gamma \chi_\gamma q_b \chi_b} (J_1 j_1 t_1, \ Jq\chi, \ J_2 j_2 t_2)$$

$$\times D^J_{\chi'\chi} (g_c g_\gamma^{-1}) \rho (q_\gamma \chi_\gamma q_b \chi_b, \ \mathbf{n}_\gamma), \quad (21)$$

where

$$Y_{q_c \chi_c q_d \chi_d} (J_1 l_1' S_1', \ Jq'\chi', \ J_2 l_2' S_2')$$

$$= [(2q_c + 1) (2q_d + 1)]^{1/2} C^{q'\chi'}_{q_c \chi_c q_d \chi_d}$$

$$\times (-1)^{q'+\chi'} i^{\,l_1'-l_2'} X(i_c q_c i_c, S_1' q' S_2', \ i_d q_d i_d)$$

$$\times G_{\chi'} (J_1 l_1' S_1', \ Jq, \ J_2 l_2' S_2'); \quad (22)$$

$$Y_{q_\gamma \chi_\gamma q_b \chi_b} (J_1 j_1 t_1, \ Jq\chi, \ J_2 j_2 t_2) = \sum [(2J_1 + 1) (2J_2 + 1)$$

$$\times (2j_1 + 1) (2j_2 + 1) (2J + 1) (2q + 1)]^{1/2}$$

$$\times C^{h 2i_\gamma \chi_\gamma}_{J\chi q_b - \chi_b} (-1)^{l_1 + i_\gamma \chi_\gamma + q_b + \chi_b}$$

$$\times X (J_1 J J_2, j_1 h j_2, i_b q_b i_b) F_h (j_1 t_1, j_2 t_2 q\chi\chi_\gamma); \quad (23)$$

$$F_h (j_1 t_1, j_2 t_2, q_\gamma 0)$$

$$= (-1)^{-i_\gamma} [1 + I_{t_1'} I_{t_2} (-1)^{h+q_\gamma}] C^{q_\gamma 0}_{1/2 1/2, 1/2 -1/2} C^{h 0}_{j_1 i_\gamma j_1 - i_\gamma}, \quad (24)$$

$$F_h (j_1 t_1, j_2 t_2, 11) = (-1)^{j_1 + j_2 + h + t_2 + 1} C^{h - 2i_\gamma}_{j_1 - i_\gamma j_2 - i_\gamma},$$

$$F_h (j_1 t_1, j_2 t_2, 1 - 1) = (-1)^{j_1 + j_2 + h + t_1} C^{h 2i_\gamma}_{j_1 i_\gamma j_2 i_\gamma};$$

$$I_{t_\alpha} = (-1)^{j_\alpha - i_\gamma + t_\alpha}, \quad N_1 = (2\pi\hbar)^4 R^2 [V^2 p_b^2 p_d^2]^{-1} (p_b; p_d)^2.$$

The sum is taken over q', χ', J', l_1', S_1', J_2, l_2', S_2', j_1, t_1, j_2, t_2, J, q, χ, q_γ, χ_γ, q_b, χ_b.

2. $a + b \rightarrow \gamma + d$.

$$\rho' (q_\gamma \chi_\gamma q_d \chi_d; \mathbf{n}_\gamma)$$

$$= [N_1/(4\pi)^2] [2 (2i_d + 1) (2i_a + 1)^{-1} (2i_b + 1)^{-1}]^{1/2}$$

$$\times \sum Y^*_{q_\gamma \chi_\gamma q_d \chi_d} (J_1 j_1 t_1, \ Jq'\chi', \ J_2 j_2 t_2)$$

$$\times \langle j_1' t_1' \alpha \mid R^{J_1} \mid S_1 l_1 \alpha_1 \rangle \langle j_2' t_2' \alpha' \mid R^{J_2} \mid S_2 l_2 \alpha_2 \rangle$$

$$\times Y_{q_a \chi_a q_b \chi_b} (J_1 l_1 S_1, \ Jq\chi, \ J_2 l_2 S_2)$$

$$\times D^J_{\chi'\chi} (g_\gamma g_a^{-1}) \rho (q_a \chi_a q_b \chi_b; \mathbf{n}_a). \quad (25)$$

3. $\gamma + b \rightarrow \gamma' + d$.

$$\rho'(q_{\gamma'}\chi_{\gamma'}, q_d\chi_d; \mathbf{n}_{\gamma'}) = [N_1/(4\pi)^2][(2i_d+1)(2i_b+1)^{-1}]^{1/2}$$

$$\times \sum Y^*_{q_{\gamma'}\chi_{\gamma'}q_d\chi_d}(J_1 j' t'_1, Jq'\chi', J_2 j'_2 t'_2)$$

$$\times \langle j'_1 t'_1 \alpha \,|\, R^{J_1} \,|\, j_1 t_1 \alpha_1 \rangle \langle j'_2 t'_2 \alpha' \,|\, R^{J_1} \,|\, j_2 t_2 \alpha_2 \rangle$$

$$\times Y_{q_{\gamma}\chi_{\gamma}q_b\chi_b}(J_1 j_1 t_1, Jq\chi, J_2 j_2 t_2)$$

$$\times D^J_{\chi'\chi}(g_{\gamma'}g_{\gamma}^{-1})\,\rho(q_{\gamma}\chi_{\gamma}q_b\chi_b; \mathbf{n}_{\gamma}). \tag{26}$$

4. a → γ + d.

$$\rho'(q_{\gamma}\chi_{\gamma}q_d\chi_d, \mathbf{n}_{\gamma}) = (N_2/4\pi)[2(2i_d+1)(2i_a+1)^{-1}]^{1/2}$$

$$\times \sum Y^*_{q_{\gamma}\chi_{\gamma}q_d\chi_d}(i_a j_1 t'_1, q_a q'\chi', i_a j'_2 t'_2)\langle j'_1 t'_1 \alpha \,|\, R^{i_a} \,|\, \alpha_1 \rangle$$

$$\times \langle j'_2 t'_2 \alpha' \,|\, R^{i_a} \,|\, \alpha_2 \rangle^* D^{q_a}_{\chi'\chi_a}(g_{\gamma})\,\rho(q_a, \chi_a), \tag{27}$$

where $N_2 = (2\pi\hbar)^3 R(Vp^2)^{-1}(E_{\gamma}|E_a)$. The sum is taken over j'_1, t'_1, j'_2, t'_2, q_a, χ_a, q', χ'.

In particular for $q_{\gamma} = q_b = q_c = q_d = 0$ the formula (21) gives the angular distribution of the products of the reaction $\gamma + b \to c + d$ for unpolarized incident beam and unpolarized target. In this case our formula agrees with the result obtained by Morita et al., if we eliminate certain phase factors by means of a redefinition of the elements of the R matrix. Our results confirm the conclusion that Simon's formulas are erroneous.

4. SELECTION RULES

If parity is conserved selection rules exist in the form of relations between the s-tensors. These selection rules have been obtained in references 2, 5, 6, 13 for the case in which all the particles have nonvanishing rest masses. The results are easily extended to reactions involving γ-particles. As an example we shall consider a reaction of the type $\gamma + b \to c + d$.

The conservation of spatial parity gives the following equation:

$$I_{t_1}I_{t_2} = (-1)^{l'_1+l'_2}. \tag{28}$$

From the properties of the D-function and Eqs. (22) — (24) it follows that:

$$D^J_{\chi'\chi}(g_c g_{\gamma}^{-1}) = D^J_{\chi'\chi}(-\pi, \vartheta_c, \pi-\varphi_c)$$

$$= (-1)^{\chi'-\chi}D^J_{-\chi'-\chi}(-\pi, \vartheta_c, \pi+\varphi_c), \tag{29}$$

$$Y_{q_c\chi_c q_d\chi_d}(J_1 l'_1 S'_1, Jq'\chi', J_2 l'_2 S_2) = (-1)^{l'_1+l'_2+J+q_c+q_d}$$

$$\times Y_{q_c,-\chi_c,q_d,-\chi_d}(J_1 l'_1 S'_1, Jq'-\chi',J_2 l'_2 S'_2), \tag{30}$$

$$Y_{q_{\gamma}\chi_{\gamma}q_b\chi_b}(J_1 j_1 t_1, Jq\chi, J_2 j_2 t_2) = (-1)^{J+q_{\gamma}+q_b}$$

$$\times I_{t_1}I_{t_2}Y_{q_{\gamma}-\chi_{\gamma}q_b-\chi_b}(J_1 j_1 t_1, Jq-\chi, J_2 j_2 t_2). \tag{31}$$

Equation (31) is obtained by the use of the equations $(-1)^{q}q = (-1)^{\chi_{\gamma}}$ for $\chi_{\gamma} = \pm 1$ and $(-1)^h = I_{t_1}I_{t_2}(-1)^{q_{\gamma}}$ for $\chi_{\gamma} = 0$. Substituting Eqs. (29),

(30), and (31) in Eq. (21), we get the most general selection rules holding when parity is conserved; these rules are expressed as follows: if we replace the s-tensors $\rho(q_{\gamma}\chi_{\gamma}q_b\chi_b, \mathbf{n}_{\gamma})$ of the initial state by

$$\rho_I = (-1)^{q_{\gamma}+2i_{\gamma}\chi_{\gamma}+q_b+\chi_b}\,\rho(q_{\gamma}, -\chi_{\gamma}, q_b, -\chi_b; \mathbf{n}_{\gamma}),$$

then we get in the final state the s-tensors

$$\rho'_I = (-1)^{q_c+\chi_c+q_d+\chi_d}\,\rho'(q_c-\chi_c q_d-\chi_d; \vartheta, -\varphi).$$

The results obtained here for the reaction involving γ-particles and the writer's results[13] for the reaction without γ-particles are consistent with each other. We emphasize that the spin states of the γ-particles are described not by s-tensors, but by Stokes parameters. We shall not consider here the special cases dealt with in reference 13, for which the results have just the same form.

The conservation of time parity gives a relation between the s-tensors of the direct $(\gamma + b \to c + d)$ and inverse $(c + d \to \gamma + b)$ reactions. Such relations have been obtained by M. I. Shirokov[6] for the reaction without γ-particles. His formulas require only a slight change for the reactions involving γ-particles. In what follows we suppose that spatial parity is also conserved.

Repeating the calculations of reference 6 step by step and using the same notations, we get the relation for the reaction with γ-particles that corresponds to Eq. (8) of reference 6:

$$\langle q_{\gamma}, \chi_{\gamma}, q_b, \chi_b \,|\, W_I(\varphi_1, \vartheta, -\varphi_2)\,|\, q_c, \chi_c, q_d, \chi_d\rangle$$

$$= (-1)^{q_{\gamma}+2i_{\gamma}\chi_{\gamma}+q_b+\chi_b+q_c+\chi_c+q_d+\chi_d}\langle q_c,-\chi_c, q_d,-\chi_d,$$

$$\times | W_D(-\pi+\varphi_2, \vartheta, \pi-\varphi_1)\,|\,q_{\gamma}, -\chi_{\gamma}, q_b,-\chi_b\rangle, \tag{32}$$

where the Euler angles $\{-\pi+\varphi_2, \vartheta, \pi-\varphi_1\}$ are those for the rotation $g_c g_{\gamma}^{-1}$ in the direct reaction, and $\{\varphi_1, \vartheta, -\varphi_2\}$ are those for the rotation $g_{\gamma}g_c^{-1}$ in the inverse reaction. We emphasize that in the direct reaction the y_c axis is not parallel to the vector $[\mathbf{n}_{\gamma}, \mathbf{n}_c]$, but is obtained by clockwise rotation of this vector through the angle φ_2 around the z_c axis. The angles ϑ and φ_1 are equal to the spherical angles of the vector $\mathbf{n}_c$ in the coordinate system $z_{\gamma}y_{\gamma}x_{\gamma}$. In the inverse reaction ϑ and $\pi+\varphi_2$ are equal to the spherical angles of the vector $\mathbf{n}_{\gamma}$ in the system $z_c y_c x_c$, and the y_{γ} axis makes the angle $\pi+\varphi_1$ with the direction of $[\mathbf{n}_c \times \mathbf{n}_{\gamma}]$.*

Replacing the factor F in Eq. (9) of reference 6 by

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*We note that Eq. (8) of reference 6 is valid only in the specially chosen systems $z_c y_c x_c$ and $z_a y_a x_a$ in which $\varphi_1 = \varphi_2$.

$$F = [(2i_c + 1)(2i_d + 1)]^{1/4} [2(2i_b + 1)]^{-1/4} \qquad (33)$$

and $(-1)^{q_a + \chi_a}$ by $(-1)^{q\gamma + 2i\gamma\chi\gamma}$, we get relations corresponding to Eqs. (9) and (10) in reference 6.

Let us go on to the consideration of the special cases studied in reference 6. The relation of detailed balancing between the differential cross-sections of the direct and inverse reactions for unpolarized initial state has the well known form

$$p_\gamma^2 2 (2i_b + 1)\, \sigma_D(\vartheta) = p_c^2 (2i_c + 1)(2i_d + 1)\, \sigma_I(\vartheta). \qquad (34)$$

The difference $\sigma_D'(\vartheta, \varphi_1) - \sigma_D(\vartheta, \varphi_1)$ between the cross-sections with a polarized beam of γ-particles and with an unpolarized beam in the direct reaction can be expressed in the form

$$\sigma_D'(\vartheta, \varphi_1) - \sigma_D(\vartheta, \varphi_1) = -F^2 p_c^2 p_\gamma^{-2} [(-1)^{2l_\gamma} \rho_1 \rho_{-1} + \rho_0' \rho_0$$
$$+ \rho_{-1}' \rho_1 (-1)^{2l_\gamma}] = -F^2 p_c^2 p_\gamma^{-2} (\rho'(\varphi_1, \vartheta, -\varphi_2)\rho), \qquad (35)$$

where ρ_{-1}, ρ_0, and ρ_1 are the Stokes parameters characterizing the spin state of the incident γ-particles in the direct reactions; ρ_{-1}', ρ_0', and ρ_1' are the Stokes parameters for the spin state of the γ-particles produced in the inverse reaction with unpolarized initial state.

5. SYSTEM WITH TWO IDENTICAL γ-PARTICLES

Let us construct symmetric or antisymmetric wave functions with definite parity, total angular momentum, z-component of angular momentum, and energy, for a system of two identical γ-particles. The state vector $|J, M, \mu_1', \mu_2', p\rangle$ transforms under reflection in the following way:

$$I | J, M, \mu_1', \mu_2', p\rangle = I_0' \eta_I' | J, M, -\mu_1', -\mu_2', p\rangle,$$

where I_0' is the product of the intrinsic parities of the two identical particles, which is always equal to unity; η_I' is a phase factor.

Since the state vector $|J, M, \mu_1', \mu_2', p\rangle$ transforms under rotations according to an irreducible representation of the rotation group with the weight J, a rotation through the angle φ around the direction $\mathbf{n}$ of the relative momentum of the particles gives a factor $\exp[i(\mu_1' + \mu_2')\varphi]$. Accordingly, in the rotated coordinate system the factor η_I' is replaced by $\eta_I'' = \eta_I' \exp[2i(\mu_1' + \mu_2')\varphi]$. Therefore the value of the phase factor η_I' depends on the direction of the x (or y) axis in the coordinate system with the z axis parallel to $\mathbf{n}$. In view of the fact that the system of two γ-particles occurs only in the final state of some reaction, we choose the coordinate system with axes $z \parallel \mathbf{n}$, $y \parallel \zeta - (\zeta \cdot \mathbf{n})\mathbf{n}$, $x \parallel [\zeta \times \mathbf{n}]$, where ζ is the polarization vector of the particles in the initial state

of the reaction. After reflection the z and x axes change their directions, but the y axis remains unchanged. The "reflected" coordinate system differs from the original system only by a rotation through the angle π around the y axis. Therefore from the relation $D_{\mu'\mu}^J(0, \pi, 0) = (-1)^{J-\mu} \delta_{\mu', -\mu}$ we get

$$I | J, M, \mu_1', \mu_2', p\rangle$$
$$= I_1' \sum_{\mu_1''\mu_2''} | J, M, \mu_1'', \mu_2''; p\rangle D_{\mu_1''+\mu_2'', \mu_1'+\mu_2'}^J (0, \pi, 0)$$
$$= I_1'(-1)^{J-\mu_1'-\mu_2'} | J, M, -\mu_1, -\mu_2, p\rangle. \qquad (36)$$

Let us now consider the operator S that replaces the first particle by the second and vice versa. In the chosen coordinate system the operator S can be expressed as the product of the operator I, which changes the direction of the relative momentum, and the operator that interchanges the spin indices μ_1' and μ_2'. Thus

$$S | J, M, \mu_1', \mu_2', p'\rangle$$
$$= (-1)^{J-\mu_1'-\mu_2'} | J, M, -\mu_2', -\mu_1', p'\rangle. \qquad (37)$$

From Eqs. (36) and (37) we get the symmetric or antisymmetric wave vectors with definite parity I', total angular momentum, z-component of angular momentum, and energy

$$| J, M, \mu, I', p\rangle = A[1 + I'I + S'S$$
$$+ I'S'IS] | J, M, \mu_1', \mu_2', p\rangle \, \delta_{|\mu_1'+\mu_2'|, \mu}, \qquad (38)$$

where $\mu = 2i$ or 0 is the eigenvalue of the operator $|\hat{\Sigma}_1 + \hat{\Sigma}_2|$, A is a normalization constant, and $S' = +1$ for particles obeying Bose statistics and $S' = -1$ for Fermi statistics. Multiplying Eq. (38) from the left by $\langle \mathbf{n}, \mu_1, \mu_2, p |$ and using Eqs. (19), (36), and (37), we get the wave functions for this system in the momentum-spins representation.

1. $\mu = 2i$, $\mu_1 = \mu_2 = \pm i$.

$$\langle \mathbf{n}, \mu_1, \mu_2, p | J, M, 2i, I', p'\rangle$$
$$= (1/2\sqrt{2})(1 + S'I')(2\pi\hbar \sqrt{R}/p \sqrt{V})\left(\frac{2J+1}{4\pi}\right)^{1/2} \delta_{pp'}$$
$$\times [D_{2i, M}^J (g_\mathbf{n}) \delta_{\mu_1+\mu_2, 2i} + I'(-1)^{J+2i} D_{-2i, M}^J (g_\mathbf{n}) \delta_{\mu_1+\mu_2, -2i}]. \qquad (39)$$

2. $\mu = 0$, $\mu_1 = -\mu_2 = \pm i$.

$$\langle \mathbf{n}, \mu_1, \mu_2, p | J, M, 0, I', p\rangle$$
$$= (1/2\sqrt{2})(1 + S'(-1)^J)(2\pi\hbar \sqrt{R}/p \sqrt{V})[(2J+1)/4\pi]^{1/2} \delta_{pp'}$$
$$\times D_{0M}^J (g_\mathbf{n})(\delta_{\mu_1 i}\delta_{\mu_2 -i} + I'(-1)^J \delta_{\mu_1 -i}\delta_{\mu_2 i}). \qquad (40)$$

As can be seen from Eqs. (39) and (40), the wave functions are nonvanishing only when $S'I' = 1$ for $\mu = 2i$, and $S'(-1)^J = 1$ for $\mu = 0$. These are just the specific selection rules obtained by Landau

and by Yang for photons and by Shapiro in the general case. We note that when parity is not conserved the selection rule $S'(-1)^J = 1$ for $\mu = 0$ remains valid, but the factor $\delta_{\mu_1 i}\delta_{\mu_2-i} + I'(-1)^J \times \delta_{\mu_1-i}\delta_{\mu_2 i}$, which characterizes the polarization of the system, is changed. Therefore in the decay of a particle with spin $J < 2i$ into two identical γ-particles a measurement of the correlation of the polarizations of the γ-particles not only will give the parity of this particle when parity is conserved (cf. Yang, reference 7), but also will give information about the nonconservation of parity in the decay.

The wave functions (39) and (40) can be written together in a single formula:

$$\langle \mathbf{n}, \mu_1, \mu_2, p \mid J, M, \mu, I', p' \rangle = \frac{1}{2\sqrt{2}} \, F_\mu \, \frac{2\pi\hbar\sqrt{R}}{p\sqrt{V}}$$

$$\times \left(\frac{2J+1}{4\pi}\right)^{1/2} D^J_{\mu_1+\mu_2,M}(g_\mathbf{n}) \, \sigma^t \delta_{|\mu_1+\mu_2|,\mu}, \qquad (41)$$

where $\sigma i = \mu_1$, $I' = (-1)^{J-2i+t}$,

$$F_\mu = (1 + S'I') \text{ for } \mu = 2i,$$

$$F_\mu = (1 + S'(-1)^J) \text{ for } \mu = 0.$$

By means of Eq. (41) one can easily calculate the angular distributions and the polarizations for the decay of a particle into two identical γ-particles. We shall not do this here.

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ERRATA TO VOLUME 9

Page	Reads	Should read
115, Col. 2, line 18 from top	R. Gatto and M. A. Ruderman, [Nuovo cimento 8, 775, (1958)]	T. Goto, Nuovo cimento 8, 625 (1958)
294, Col. 2, line 4 from bottom	$N = N_{exp}(p, \theta)\, F(p, \theta)$	$N = N_{exp}(p, \theta)\, 1 + F(p, \theta)$
462, Col. 1, line 8 from top	which are approximately 13 Z ev	and approximately equal to 13 Z ev
646, Col. 1, line 3 from top	$\langle j'_1 t'_1 \alpha \| R^{J_2} \| j_1 t_1 \alpha_1 \rangle$	$\langle j'_1 t'_1 \alpha \| R^{J_1} \| j_1 t_1 \alpha_1 \rangle$
661, Col 1, line 6 from top	$\lambda = 2.14 \times 10^{-13}$	$\lambda = 1.04 \times 10^{-13}$