A NEW CLASS OF REPRESENTATIONS OF THE FULL LORENTZ GROUP

G. A. SOKOLIK

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All representations of the full Lorentz group are found. It is shown that these representations reduce to direct products of spinors belonging to three classes. An attempt is made to interpret isotopic spin in terms of the generalized parity operator without introducing new degrees of freedom.

It is shown that the connection between spin and statistics may not hold for spinors which transform according to commuting representations.

In the paper by Gel'fand and Tsetlin[1] the possibility was noted of representing the group of space-time reflections by matrices which commute in the case of spinor representations and which anti-commute in the case of representations corresponding to particles of integral spin.

The author has shown[2] that the Dirac equation is invariant with respect to the commuting spinor representation of the Lorentz group only in the extended eight-dimensional form. In this connection it was noted that the doubling of dimensionality obtained in this way may be interpreted as the introduction of isotopic spin. This gives rise to the possibility of obtaining the isotopic doublet without introducing new degrees of freedom only by means of generalizing the representation of the full Lorentz group, while the majority of the attempts to interpret isotopic spin undertaken recently were based on the generalization of the Lorentz group itself (cf. for example, reference 3).

Thus, it appears to be possible to classify elementary particles according to the representations of the group of space-time reflections. In this case the parity operator will play the role of isotopic spin. In connection with this the problem arises of finding all the irreducible representations of the full Lorentz group. In this way we shall obtain all the invariants and pseudoinvariants expressed in terms of polylinear combinations of three kinds of spinors. In other words, the solution of this problem will make it possible to carry out the classification of interactions of elementary particles (conserving or nonconserving parity) in terms of the representations of the full Lorentz group. This problem is of particular interest in connection with the fusion theory, according to which all the particles reduce to one or several spinor particles,[4] and also in connection with attempts of creating a single nonlinear field theory taking into account various types of fundamental spinors.[5] The normal and the anomalous (commuting and anticommuting)[1] representations of the full Lorentz group are effectively specified by means of the contracted direct product of the following three spinor representations:

$$T_{01} = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}; \qquad T_{10} = \begin{pmatrix} 0 & iI \\ iI & 0 \end{pmatrix}; \qquad T_{11} = \begin{pmatrix} iI & 0 \\ 0 & iI \end{pmatrix}, \tag{1}$$

$$T_{01} = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}; \qquad T_{10} = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}; \qquad T_{11} = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \tag{2}$$

$$T_{01} = \begin{pmatrix} 0 & -iI \\ iI & 0 \end{pmatrix}; \qquad T_{10} = \begin{pmatrix} 0 & iI \\ iI & 0 \end{pmatrix}; \qquad T_{11} = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}. \tag{3}$$

The representation (1) is normal, while the representations (2) and (3) are anomalous. The operators T_{01}, T_{10}, T_{11}, which satisfy the relations $T_{10}T_{11} = T_{01}$; $T_{10}T_{01} = T_{11}$, operate on the bispinors $(x_1 x_2 \bar{x}_1 \bar{x}_2)$, with T_{01}, T_{10}, T_{11} corresponding to reflections of space, time and space-time respectively. The representations are determined up to a factor of modulus 1. As was shown in reference 2, the spinors transforming according to formulas (3) may be interpreted in terms of the five-dimensional rotation group.

We note that the spinors (1) may be regarded as spinors of the first kind with respect to spatial reflections, and of the second kind with respect to time reflections.[6,7] The representations of the full Lorentz group are given[6] by multiplying the spinor representations $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$.

By expanding the generating polynomials D_{jk}^{+} and D_{jk}^{-}:

$$D_{jk}^{+} = (u_1 x_1 + u_2 x_2)^{2j} (v_1 \bar{x}_1 + v_2 \bar{x}_2)^{2k},$$

$$D_{jk}^{-} = (u_1 x_1 + u_2 x_2)^{2j} (v_1 \bar{x}_1 + v_2 \bar{x}_2)^{2k} \begin{Bmatrix} x_1 x_2' - x_2 x_1' \\ \bar{x}_1 \bar{x}_2' - \bar{x}_2 \bar{x}_1' \end{Bmatrix},$$

we obtain the quantities Y^{jk}_{mp} and Z^{jk}_{mp} which transform according to the representations D^{+}_{jk} and D^{-}_{jk}:

$$Y^{jk}_{mp} = x_1^{j+m} x_2^{j-m} \bar{x}_1^{k+p} \bar{x}_2^{k-p},$$

$$Y^{kj}_{pm} = x_1^{k+p} x_2^{k-p} \bar{x}_1^{j+m} \bar{x}_2^{j-m},$$

$$Z^{jk}_{mp} = x_1^{j+m} x_2^{j-m} \bar{x}_1^{k+p} \bar{x}_2^{k-p} (x_1 x_2' - x_2 x_1'),$$

$$Z^{kj}_{pm} = x_1^{k+p} x_2^{k-p} \bar{x}_1^{j+m} \bar{x}_2^{j-m} (\bar{x}_1 \bar{x}_2' - \bar{x}_2 \bar{x}_1').$$

Let us examine the case D^{+}_{jk} and D^{+}_{kj}. For spinors which transform according to formulas (1), (2), and (3), we have

$$T_{01} = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}; \quad T_{10} = \begin{pmatrix} 0 & i^{2(k+j)} \\ i^{2(k+j)} & 0 \end{pmatrix};$$

$$T_{11} = \begin{pmatrix} i^{2(k+j)} & 0 \\ 0 & i^{2(k+j)} \end{pmatrix}, \qquad (4)$$

$$T_{01} = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}; \quad T_{10} = \begin{pmatrix} 0 & (-I)^{2k} \\ (-I)^{2j} & 0 \end{pmatrix};$$

$$T_{11} = \begin{pmatrix} (-I)^{2j} & 0 \\ 0 & (-I)^{2k} \end{pmatrix}, \qquad (5)$$

$$T_{01} = \begin{pmatrix} 0 & (-i)^{2j} i^{2k} \\ (-i)^{2k} i^{2j} & 0 \end{pmatrix}; \quad T_{10} = \begin{pmatrix} 0 & i^{2(j+k)} \\ i^{2(j+k)} & 0 \end{pmatrix};$$

$$T_{11} = \begin{pmatrix} (-I)^{2j} & 0 \\ 0 & (-I)^{2k} \end{pmatrix} \qquad (6)$$

respectively. The representations (5) and (6) are anomalous in the case when one of the jk is integral, while the other one is half-integral; in all other cases these representations are normal. The representation (4) is always normal.

Let us examine the improper representation D^{-}_{jk} and D^{-}_{kj}. It may be easily shown that Z^{jk}_{mp} transforms according to formulas (4), (5), and (6), if $x\bar{x}$ and $x'\bar{x}'$ transform according to (1), (2), and (3). In the case when $x\bar{x}$ transforms according to (1), while $x'\bar{x}'$ transforms according to (2), the representation D^{-}_{jk}, D^{-}_{kj} is specified by the anticommuting operators

$$T_{10} = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}; \quad T_{01} = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}; \quad T_{11} = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}. \qquad (7)$$

We see that in this case the representation of the full Lorentz group specified by the direct product of spinors of different type decomposes into two representations of the proper group related by reflection even in the case when j = k. When $x\bar{x}$ and $x'\bar{x}'$ transform according to (1) and (3) or (2) and (3), respectively, the following representations hold

$$T_{01} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}; \quad T_{10} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}; \quad T_{11} = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \qquad (8)$$

$$T_{01} = \begin{pmatrix} 0 & i^{2k} \\ (-i)^{2k} & 0 \end{pmatrix}; \quad T_{10} = \begin{pmatrix} 0 & (-I)^{2j} i^{2k} \\ i^{2k} & 0 \end{pmatrix};$$

$$T_{11} = \begin{pmatrix} (-I)^{2k} & 0 \\ 0 & (-I)^{2j} \end{pmatrix}. \qquad (9)$$

In the last case the operators commute when kj are integral or half-integral, and anticommute if 2j and 2k have opposite parity.

The representation of the full Lorentz group may also be specified by the generating polynomials

$$B^{+}_{jk} = (u_1 x_1 + u_2 x_2)^{2j} (v_1 \bar{x}_1' + v_2 \bar{x}_2')^{2k},$$

$$B^{-}_{jk} = (u_1 x_1 + u_2 x_2)^{2j} (v_1 \bar{x}_1' + v_2 \bar{x}_2')^{2k} \begin{Bmatrix} x_1 x_2' - x_2 x_1' \\ \bar{x}_1 \bar{x}_2' - \bar{x}_2 \bar{x}_1' \end{Bmatrix}. \qquad (10)$$

In this case the basis of the representation is specified by the products of the monomials

$$Z^{jk}_{mp} = x_1^{j+m} x_2^{j-m} \bar{x}_1'^{k+p} \bar{x}_2'^{k-p},$$
$$\bar{Z}^{jk}_{mp} = \bar{x}_1^{j+m} \bar{x}_2^{j-m} x_1'^{k+p} x_2'^{k-p},$$
$$Z^{kj}_{pm} = x_1^{k+p} x_2^{k-p} \bar{x}_1'^{j+m} \bar{x}_2'^{j-m},$$
$$\bar{Z}^{kj}_{pm} = \bar{x}_1^{k+p} \bar{x}_2^{k-p} x_1'^{j+m} x_2'^{j-m},$$
$$Z^{00} = x_1 x_2' - x_2 x_1', \quad \bar{Z}^{00} = \bar{x}_1 \bar{x}_2' - \bar{x}_2 \bar{x}_1',$$

where x and x′ behave in the same way under rotations. If x and x′ transform according to (1) and (2), $Z^{jk}_{mp} \bar{Z}^{jk}_{mp}$ and $Z^{kj}_{pm} \bar{Z}^{kj}_{pm}$ transform according to the following representations:

$$T_{01} = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}; \quad T_{10} = \begin{pmatrix} 0 & (-I)^{2j} i^{2k} \\ i^{2k} & 0 \end{pmatrix};$$

$$T_{11} = \begin{pmatrix} i^{2k} & 0 \\ 0 & (-I)^{2j} i^{2k} \end{pmatrix} \qquad (11)$$

and

$$T_{01} = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}; \quad T_{10} = \begin{pmatrix} 0 & (-I)^{2k} i^{2j} \\ i^{2j} & 0 \end{pmatrix}; \quad T_{11} = \begin{pmatrix} i^{2j} & 0 \\ 0 & (-I)^{2k} i^{2j} \end{pmatrix}.$$

The representations obtained above are anomalous for j and k having half-integral values respectively and are normal in all other cases. By considering x and x′ as transforming according to (2), (3) and (1), (3), it may be shown in general that in the case when $x\bar{x}$ transform according to the normal representations while $x'\bar{x}'$ transform according to anomalous representations the resulting representation of B^{+}_{jk}, B^{+}_{kj}, is anomalous if j or k is half-integral and is normal in all other cases. If both spinor representations are anomalous then B^{+}_{jk}, B^{+}_{kj} is anomalous only when one of the numbers j, k is integral and the other one is half-integral.

In conclusion we note that both new variants of the representation of the full Lorentz group (the commuting spinor and the anticommuting boson representations) lead to a doubling of the dimensionality of the ψ-function.

As an example we shall consider two pseudoscalar irreducible representations

$$Z^{00} = x_1x_2' - x_2x_1', \quad \bar{Z}^{00} = \bar{x}_1\bar{x}_2' - \bar{x}_2\bar{x}_1', \qquad (12)$$

which transform according to the Pauli matrices

$$T_{10} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}; \quad T_{10} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \quad T_{11} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \qquad (13)$$

The representation obtained above may be used to describe a scalar particle which may exist in states with different parities. In order to do this we shall go over to another representation

$$T = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}; \quad T'_{ik} = T^{-1}T_{ik}T,$$

$$T_{10} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}; \quad T_{01} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}; \quad T_{11} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \qquad (14)$$

In this case we have the doublet

$$\phi = \psi'^{+}L\psi, \quad \Phi = \psi'^{+}C\psi, \qquad L = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

$$C = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \qquad \psi = \begin{bmatrix} x_1 \\ x_2 \\ \bar{x}_1 \\ \bar{x}_2 \end{bmatrix},$$

which may be interpreted as one of the K-meson isotopic doublets $[K^0K^+]$ or $[-K^{0*}K^-]$.

With reference to the spinor particles we see that in accordance with (2) the Dirac equation $\gamma_i\partial\psi/\partial x_i + m\psi = 0$ is invariant with respect to (3), while the equation $\gamma_i\partial\psi/\partial x_i + m\gamma_5\psi = 0$ is invariant with respect to (2). The normal representation gives the eight-dimensional Dirac equation

$$\Gamma_i\partial\Psi/\partial x_i + m\Psi = 0, \qquad \Gamma_i = \begin{pmatrix} \gamma_i & 0 \\ 0 & -\gamma_i \end{pmatrix}.$$

The spinor Ψ may be expressed in terms of x_1x_2 and $\bar{x}_1\bar{x}_2$ in the following manner. $\Psi = \Pi \times Z$ where

$$\Pi = \begin{pmatrix} x_1 \\ x_2 \\ \bar{x}_1 \\ \bar{x}_2 \end{pmatrix}, \qquad Z = \begin{pmatrix} Z^{00} \\ \bar{Z}^{00} \end{pmatrix}.$$

It may be easily shown that in the case of spinors which transform according to the normal representation, time reversal is equivalent to charge conjugation. Indeed, if $\psi = C\psi^*$, where $CD^*C = -D$

$$\gamma_i p_i = D = \begin{pmatrix} 0 & 0 & \partial_4 + \partial_3 & \partial_1 - i\partial_2 \\ 0 & 0 & \partial_1 + i\partial_2 & \partial_4 - \partial_3 \\ \partial_4 - \partial_3 & -\partial_1 + i\partial_2 & 0 & 0 \\ -\partial_1 - i\partial_2 & \partial_4 + \partial_3 & 0 & 0 \end{pmatrix};$$

$$p_i \equiv \partial_i \equiv \partial/\partial x_i.$$

On the other hand $\gamma_4 D_{-t}\gamma_4 = -D_t$ with $\psi' = \gamma_4\psi$. Consequently the operation TC in the case of the normal spinor representation preserves the Dirac equation and, thereby, also the Lagrangian $\mathcal{L} = \tilde{\psi}^*(D\psi + m\psi)$; here $\tilde{\psi}^*$ and ψ are not subjected to the anticommutation condition as in the case of the anomalous spinor representations.[8]

Thus the normal spinors are quantized according to Bose statistics.[2] The argument presented above is applicable to the Dirac equation interacting with the electromagnetic field

$$\gamma_i(\partial/\partial x_i - ieA_i)\psi + m\psi = 0; \qquad \Box A_i = j_i/c,$$

since $j = \psi^+\gamma_i\psi$ and A_i transform under reflections like $\partial/\partial x_i$. I wish to thank O. A. Germogenova and A. M. Brodskiĭ for valuable suggestions and fruitful discussions.

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