DISPERSION RELATIONS AND CHEW-LOW TYPE EQUATIONS FOR INELASTIC MESON PROCESSES IN THE FIXED SOURCE CASE

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The static dispersion relations and Chew-Low equations are established for the process $\pi + N \to n\pi + N$. It turns out that for this process one can obtain physically different dispersion relations and Chew-Low equations, depending on how the variables are chosen.

THE progress in theoretical π-meson physics of the last few years has been achieved mainly by means of the theory of dispersion relations and by means of the so-called Chew-Low equations. There exists a connection between these two theories. This has already been investigated by several authors.[1,2] In the present paper we will establish the dispersion relations and the Chew-Low equations for the process $\pi + N \to n\pi + N$ in the fixed-source approximation.

As in the case of elastic scattering one has to utilize the causality conditions in order to establish the dispersion relations. Utilizing the Bogolyubov formalism[3] it is possible to introduce the causality conditions in an explicit manner in the static case[4] to which we shall restrict ourselves. This is as is well known not possible in the formalism of Wick, Chew, and Low.[5]

In setting up the dispersion relations we employ the retarded and advanced transition amplitudes of the considered process. Since there does not hold an "optical theorem" for the case $n > 1$ and because of the appearance of the unphysical region it seems that it is not possible to apply the dispersion relations in an exact fashion. Therefore we shall go over from the static dispersion equations to the appropriate equation of the Chew-Low type.

Depending on the way how one fixes the variables in the case $n > 1$, one can obtain different dispersion relations. Presumably the exact solutions to these different dispersion relations will coincide.

On the other hand it turns out that the results obtained by using approximations definitely depend on the choice of variables. This is also true for the different Chew-Low type equations which correspond to the different dispersion relations. It should be mentioned that these differences are due to physical reasons and are not connected in any way with the frequently discussed nonuniqueness of the solutions to the Chew-Low equations (see references 6 and 7). We shall not consider this point in the present paper.

1. STRUCTURE OF THE S MATRIX

An element of the S matrix of the process $\pi + N \to n\pi + N$ can be written in the form

$$S_{n,1} = \langle s', \mathbf{q}_n, \ldots, \mathbf{q}_1 | S | s, \mathbf{q}_0 \rangle = \frac{v_n \ldots v_0 \mathbf{q}_n \ldots \mathbf{q}_0}{(2\pi)^{3(n+1)/2}(2^{n+1}E_n \ldots E_0)^{1/2}}$$
$$\times (-i)^n i \int dt_n \ldots dt_0 e^{i\{E_n t_n + \ldots + E_1 t_1 - E_0 t_0\}}$$
$$\times \langle s' \left| \frac{\delta^n [(\delta S / \delta\varphi_0(t_0)) S^+]}{\delta\varphi_n(t_n) \ldots \delta\varphi_1(t_1)} \right| s \rangle, \quad (1)$$

where $\mathbf{q}_0$ is the momentum of the incoming π mesons and $\mathbf{q}_n \ldots \mathbf{q}_1 \ldots$ are the momenta of the outgoing mesons; each of the momenta $\mathbf{q}_n$ is understood to form a scalar product with the corresponding $\delta S/\delta\varphi$; $v_i = v(|\mathbf{q}_i|)$ is the Fourier transform of the smeared out fixed nucleon, and $\varphi_i(t_i)$ can be written in the form

$$\varphi_i(t_i) \equiv \varphi_{\gamma_i \rho_i}(t_i)$$
$$= \frac{-i}{(2\pi)^{8/2}} \int \frac{d\mathbf{q}_i}{\sqrt{2E_i}} q_{\gamma_i} v_i \left\{ a_{\rho_i}^{(+)}(\mathbf{q}_i) e^{iE_i t_i} - a_{\rho_i}^{(-)}(\mathbf{q}_i) e^{-iE_i t_i} \right\} \quad (2)$$

and is essentially the field operator of the i-th π meson.

Owing to the assumption of a fixed and smeared out nucleon, (2) depends only on the time variable.

Between the π meson creation and annihilation operators and the operator S there hold the following commutation relations:

$$[a_\rho^{(-)}(\mathbf{q}), S] = -\frac{i}{(2\pi)^{3/2}} \frac{vq_\gamma}{\sqrt{2E}} \int e^{-iEt} \frac{\delta S}{\delta\varphi_{\gamma\rho}(t)} dt,$$

$$[S, a_\rho^{(+)}(\mathbf{q})] = +\frac{i}{(2\pi)^{3/2}} \frac{vq_\gamma}{\sqrt{2E}} \int e^{+iEt} \frac{\delta S}{\delta\varphi_{\gamma\rho}(t)} dt. \quad (3)$$

The pseudoscalar nature of the π mesons leads to the well known factors in (1) and (3). They are of no particular importance for our purposes.

In order to bring the matrix element of the considered process to the form (1) one has to utilize the stationary character of

$$S|\alpha\rangle = |\alpha\rangle, \qquad (4)$$

where $|\alpha\rangle$ is either the vacuum or a one-particle state. This way one can immediately express the S matrix element in terms of the retarded transition amplitude. For this purpose we utilize the translational invariance of the expression and write $S_{n,1}$ in the following form:

$$S_{n,1} = -2\pi i\delta(E_n + \ldots + E_1 - E_0) \times \frac{v_n \ldots v_0}{(2^{n+1}E_n \ldots E_0)^{1/2}} T^{ret}_{n,1}(E_n, \ldots, E_1; E_0),$$

$$T^{ret}_{n,1}(E_n, \ldots E_1; E_0) = = \frac{q_n \ldots q_0}{(2\pi)^{3(n+1)/2}}(-i)^n i \int dt_n \ldots dt_1 e^{i\{E_n t_n + \ldots + E_1 t_1\}}$$

$$\left\langle s' \left| \frac{i\delta^n[(\delta S/\delta\varphi_0(0))S^+]}{\delta\varphi_n(t_n)\ldots\delta\varphi_1(t_1)} \right| s \right\rangle. \qquad (5)$$

Starting with the Hermitian conjugate of the matrix element $S_{n,1}$ we find for the advanced transition amplitude

$$T^{av}_{n,1}(E_n, \ldots, E_1; E_0) = \frac{q_n \ldots q_0}{(2\pi)^{3(n+1)/2}}(-i)\, i \int dt_n \ldots dt_1 e^{i\{E_n t_n + \ldots + E_1 t_1\}}$$

$$\times \left\langle s' \left| \frac{-i\delta^n[(\delta S^+/\delta\varphi_0(0))S]}{\delta\varphi_n(t_n)\ldots\delta\varphi_1(t_1)} \right| s \right\rangle. \qquad (6)$$

The causality conditions can be expressed in our static case in the following two ways:

$$\delta\left(\frac{\delta S}{\delta\varphi(0)} S^+\right) \Big/ \delta\varphi'(t') = 0 \qquad \text{for } t' < 0,$$

$$\delta\left(\frac{\delta S^+}{\delta\varphi(0)} S\right) \Big/ \delta\varphi'(t') = 0 \qquad \text{for } t' > 0. \qquad (7)$$

From this it follows that

$$T^{ret}_{n,1} = 0 \quad \text{for } t_l < 0, \quad l = 1, \ldots n;$$

$$T^{av}_{n,1} = 0 \quad \text{for } t_l > 0, \quad l = 1, \ldots, n, \qquad (8)$$

which is the justification for the terms "advanced" and "retarded."

The conditions (8) will allow to establish dispersion relations.

2. DISPERSION RELATIONS

We consider the expressions (5) and (6) to be functions of n complex variables $E_l = a_l + ib_l$ ($l = 1 \ldots n$). One sees from the causality conditions (7) that $T^{ret}_{n,1}(E_n, \ldots E_1; E)$ is an analytic function of E_l for $\mathrm{Im}\, E_l > 0$ while $T^{av}_{n,1}(E_n, \ldots \ldots E_1; E)$ is analytic for $\mathrm{Im}\, E_l < 0$.

In order to establish the behavior of the functions $T^{ret}_{n,1}$ and $T^{av}_{n,1}$ for real values of the arguments we consider the difference of these functions which is proportional to the antihermitian part of the transition amplitude, $A_{n,1}$. For this we exchange $(\delta S^+/\delta\varphi_0)S$ in $T^{av}_{n,1}$ by $-S^+(\delta S/\delta\varphi_0)$ and perform explicitly the functional derivative. Then we obtain the following expression (here E_l are real)

$$T^{ret}_{n,1} - T^{av}_{n,1} = 2iA_{n,1}$$

$$= \frac{q_n \ldots q_0}{(2\pi)^{3(n+1)/2}}(-i)^n i \int dt_n \ldots dt_1 e^{i\{E_n t_n + \ldots + E_1 t_1\}}$$

$$\times \mathfrak{S} i \Big\{ -\left\langle s' \left| \frac{-i\delta S^+}{\delta\varphi_n} \frac{i\delta^n S}{\delta\varphi_{n-1}\ldots\delta\varphi_0} \right| s\right\rangle$$

$$+ \left\langle s' \left| \frac{i\delta^n S}{\delta\varphi_0 \ldots \delta\varphi_{n-1}} \frac{-i\delta S^+}{\delta\varphi_n} \right| s\right\rangle - \left\langle s' \left| \frac{-i\delta^2 S^+}{\delta\varphi_n\delta\varphi_{n-1}} \frac{i\delta^{n-1}S}{\delta\varphi_{n-2}\ldots\delta\varphi_0} \right| s\right\rangle$$

$$+ \left\langle s' \left| \frac{i\delta^{n-1}S}{\delta\varphi_0\ldots\delta\varphi_{n-2}} \frac{-i\delta^2 S^+}{\delta\varphi_{n-1}\delta\varphi_n} \right| s\right\rangle$$

$$-\ldots- \left\langle s' \left| \frac{-i\delta^n S^+}{\delta\varphi_n\ldots\delta\varphi_1} \frac{i\delta S}{\delta\varphi_0} \right| s\right\rangle$$

$$+ \left\langle s' \left| \frac{i\delta S}{\delta\varphi_0} \frac{-i\delta^n S^+}{\delta\varphi_1\ldots\delta\varphi_n} \right| s\right\rangle \Big\}. \qquad (9)$$

Here $\mathfrak{S}$ is a symmetrization operator defined by the following relations:

$$\mathfrak{S}f(n; n-1, \ldots, 1) = f(n; n-1, \ldots, 1) + f(n-1; n, n-2, \ldots, 1) + \ldots;$$

$$\mathfrak{S}f(n, n-1; n-2, \ldots, 1) = f(n, n-1; n-2, \ldots, 1)$$
$$+ f(n, n-2; n-1, n-3, \ldots, 1) + \ldots$$
$$\ldots + f(n, 1; n-2, \ldots, 2, n-1)$$
$$+ f(n-1, n-2; n, n-3, \ldots, 1) + \ldots$$
$$\ldots + f(n-1, 1; n, n-3, \ldots, 2, n-2)$$
$$+ \ldots + f(2, 1; n, \ldots, 3).$$

In an analogous fashion one can write down at once expressions for $\mathfrak{S}f(n, \ldots, k; k-1, \ldots, 1)$.

Utilizing the translational invariance of the matrix elements one can write (9) as a sum of terms of the following form (we take as an example the fourth term of (9))

$$2\pi i \sum_i \mathfrak{S}\delta(E_n + E_{n-1} + E_i)$$
$$\times T_{n-1,i}(-E_0, E_1, \ldots, E_{n-2}; E_i)\, T^+_{i,2}(E_i; -E_{n-1}, -E_n),$$

where

$$T_{n-1,i}(-E_0, E_1, \ldots, E_{n-2}; E_i)$$
$$= \frac{q_0 \cdots q_{n-2}}{(2\pi)^{3(n-1)/2}} (-i)^{n-2}\, i \int dt_1 \ldots dt_{n-2} e^{i\{E_1 t_1 + \ldots E_{n-2} t_{n-2}\}}$$
$$\times \langle s' \left| \frac{i\delta^{n-1} S}{\delta\varphi_0 \ldots \delta\varphi_{n-2}} \right| s, i\rangle.$$

We now introduce new variables for the E_l

$$E_l = \nu_l E, \quad l = 1, \ldots, n, \quad E_0 = E;$$
$$\sum_{l=1}^{n} \nu_l = 1, \quad \nu_l = \text{const} \quad (\text{real}) \tag{10}$$

(on the nonuniqueness of this choice of variables and its consequences see Sec. 4). Then $T^{\text{ret}}_{n,1}$ and $T^{\text{av}}_{n,1}$ are analytic functions in the upper and lower half of the complex E-plane respectively.

If we now consider only strong interactions in our process then the energy E_i can assume only the values $E_i = 0$ and $E_i \geq \mu$ where μ is the mass of the π meson. As a consequence of this the difference $T^{\text{ret}}_{n,1} - T^{\text{av}}_{n,1}$ has at $E = 0$ a δ-function singularity and equals zero for $0 < |E| < \mu$. One can thus make the following statement: $T^{\text{ret}}_{n,1}(E, \nu)$ and $T^{\text{av}}_{n,1}(E, \nu)$ define in the complex E plane a single analytic function which has branch cuts only on the real axis and has there one δ-function singularity (see Fig. 1).

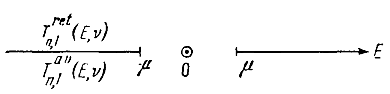

FIG. 1

One can apply the Cauchy theorem to this function thus obtaining dispersion relations. Assuming that $A_{n,1}(E, \nu)$ decreases for $E \to \infty$ like $1/E$ or faster these have the form

$$D_{n,1}(E_n, \ldots, E_1; E) = \frac{1}{\pi} P \int_{-\infty}^{+\infty} \frac{A_{n,1}(\varepsilon, \nu)}{\varepsilon - E} d\varepsilon. \tag{11}$$

For negative energies the integral can be obtained from the relations

$$T_{n,1}(-E_n, \ldots, -E_1; -E) = P_{s's} T^{+}_{1,n}(E; E_1, \ldots, E_n)$$
$$= P_{s's} T^{*}_{n,1}(E_n, \ldots, E_1; E),$$

where $P_{s's}$ — operator exchanging initial and final spin and isospin of the nucleon.

Finally, writing explicitly the one-nucleon term we obtain for the process $\pi + N \to n\pi + N$ the dispersion relation

$$D_{n,1}(E_n, \ldots, E_1; E) = \frac{1}{\pi} P \int_{\mu}^{\infty} \left\{ \frac{A_{n,1}(\varepsilon, \nu)}{\varepsilon - E} + \frac{P_{s's} A^{+}_{n,1}(\varepsilon, \nu)}{\varepsilon + E} \right\} d\varepsilon$$
$$+ \sum_{s_i} \mathfrak{S} \left\{ \frac{1}{E_n} (T^{+}_{1,0} T_{0,n} - T_{n,0} T^{+}_{0,1})_{\varepsilon=0} \right.$$
$$+ \ldots + \frac{1}{E_n + E_{n-1}} (T^{+}_{2,0} T_{0,n-1} - T_{n-1,0} T^{+}_{0,2})_{\varepsilon=0}$$
$$\left. + \frac{1}{E} (T^{+}_{n,0} T_{0,1} - T_{1,0} T^{+}_{0,n})_{\varepsilon=0} \right\}. \tag{12}$$

The summation is over the spin and isospin indices of the nucleon in the intermediate state.

The physical region of the integration (11) begins at the point $E_\pi = \mu/\bar{\nu}$ where $\bar{\nu} = \min(\nu_1, \ldots, \nu_n)$. Therefore the inequality $E_\pi \geq n\mu$ always holds, the equal sign applying when the outgoing mesons all have equal energy. One sees from this that except in the case of elastic scattering ($n=1$) there always exists a large unobservable region in the dispersion integral. It therefore is appropriate to bo over from the dispersion relation to the corresponding equation of the Chew-Low type.

3. EQUATIONS OF THE CHEW-LOW TYPE

If one knows the dispersion relation for any process in the fixed nucleon approximation one can immediately obtain the Chew Low type equation for this process.

To this end we insert into the dispersion relation the explicit expression for the antihermitian part of the transition amplitude. The expression thus obtained can be easily integrated because of the δ-function. In our case we shall this way obtain an explicit expression for $D_{n,1}(E\ , \ldots, E_1; E)$. Utilizing

$$T_{n,1} = D_{n,1} + iA_{n,1} \quad \text{и} \quad \frac{1}{x \pm i\delta} = P\frac{1}{x} \mp i\pi\delta(x)$$

one can immediately write down the following relation:

$$T_{n,1}(E_n, \ldots, E_1; E)$$
$$= -\sum_i \mathfrak{S} \left\{ \frac{\left(T^{+}_{1,i}(\varepsilon_n; E_i)\, T_{i,n}(E_i; -\varepsilon_{n-1}, \ldots, -\varepsilon_1, \varepsilon)\right)_{\varepsilon_n = E_i}}{E_i - E_n - i\delta} \right.$$
$$+ \frac{\left(T_{n,i}(-\varepsilon, \varepsilon_1, \ldots, \varepsilon_{n-1}; E_i)\, T^{+}_{i,1}(E_i; -\varepsilon_n)\right)_{\varepsilon_n = -E_i}}{E_i + E_n + i\delta}$$
$$+ \frac{\left(T^{+}_{2,i}(\varepsilon_n, \varepsilon_{n-1}; E_i)\, T_{i,n-1}(E_i; -\varepsilon_{n-2}, \ldots, -\varepsilon_1, \varepsilon)\right)_{\varepsilon_n + \varepsilon_{n-1} = E_i}}{E_i - E_n - E_{n-1} - i\delta}$$
$$+ \frac{\left(T_{n-1,i}(-\varepsilon, \varepsilon_1, \ldots, \varepsilon_{n-2}; E_i)\, T^{+}_{i,2}(E_i; -\varepsilon_{n-1}, -\varepsilon_n)\right)_{\varepsilon_n + \varepsilon_{n-1} = -E_i}}{E_i + E_n + E_{n-1} + i\delta}$$
$$+ \ldots + \frac{\left(T^{+}_{n,i}(\varepsilon_n, \ldots, \varepsilon_1; E_i)\, T_{i,1}(E_i; \varepsilon)\right)_{\varepsilon = E_i}}{E_i - E - i\delta}$$
$$\left. + \frac{\left(T_{1,i}(-\varepsilon; E_i)\, T^{+}_{i,n}(E_i; -\varepsilon_1, \ldots, -\varepsilon_n)\right)_{\varepsilon = -E_i}}{E_i + E + i\delta} \right\}. \tag{13}$$

In view of the equality

$$\frac{\delta S}{\delta\varphi}S^{+}-S^{+}\frac{\delta S}{\delta\varphi}=\frac{\delta S^{+}}{\delta\varphi}S-S\frac{\delta S^{+}}{\delta\varphi}$$

one can replace on the right hand side $T^{+}_{m,k}(E_m; E_k) \rightleftarrows T^{+}_{m,k}(E_m; E_k)$ (this not identical with hermitian conjugation).

Equations (12) and (13) are symmetrical with respect to the outgoing mesons and, furthermore, have the required crossing symmetry. The terms of (13) correspond to the diagrams shown in Fig. 2 (for the case $n = 3$).

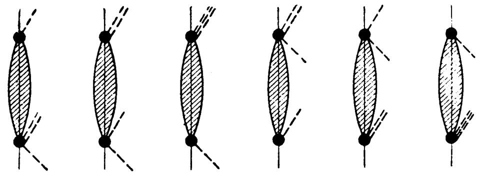

FIG. 2

4. THE PROCESS $\pi + N \rightarrow 2\pi + N$

We now shall show by means of the example $\pi + N \rightarrow 2\pi + N$ what are the consequences of the choice (10) of the variables.

For this purpose we compare the results of the present paper with those of Zoellner et al.[4] where the choice had been made

$$E_1=(E+\Delta)/2, \quad E_2=(E-\Delta)/2. \qquad (14)$$

In both cases we have two variables of which one has to be kept constant in order to establish dispersion relations. This can be done in several ways (particularly in the case of $n\pi$ mesons). In reference 4 the differnce $E_1 - E_2 = \Delta$ was kept constant while in the present paper this was done with their ratio $E_1/E_2 = \nu_1/\nu_2$.

This possibility of choosing different variables has both a mathematical and a physical meaning. It was shown in reference 4 that the choice (14) for the variables leads to limiting conditions for the existence of the dispersion relation or of the Chew-Low equation since the region on the real axis where $T^{ret} = T^{av}$ exists only for $|\Delta| < 2\mu$.

However, as can be seen from the present work, there do not exist such limiting conditions for the ν_l if the choice (10) has been made (even in the case of arbitrary n).

In fixing the variables one has to make sure that the analyticity of the amplitude is guaranteed and that the symmetry of the system is not disturbed. For example, if one chooses the variables in the form $E_2 = \text{const} = c$ and $E_1 = E - c$, then one finds that part of the spectrum of the amplitude A does not depend on E due to the presence of terms of the type $\sim \sum_i \delta(E_i \pm E_2)(E_i \geq \mu)$. It is thus impossible, within the framework of the present model, to establish exact relations for D (dispersion relations) or for T (Chew-Low type equations).

This way one can obtain for the process $\pi + N \rightarrow n\pi + N$ (with $n \geq 2$) in the fixed nucleon case different dispersion relations and Chew-Low type equations which differ from each other by the different ways of choosing the $(n-1)$ appearing parameters.*

One can suppose that the final results of an exact evaluation of the different dispersion relations (which, obviously, for $n > 1$ is practically impossible to achieve) will be identical. However, this cannot be assumed for the approximate expressions following from the respective dispersion relations or Chew-Low type equations. So, for example, the one-nucleon terms in reference 4 have the form† $T_{1,1}(\Delta)$ while here they are $T_{1,1}(0)$.

The common characteristic of all these variants of the dispersion relations and Chew-Low equations is that in all expressions the energy is conserved. This is due to the circumstance that always the hermitian part of the transition amplitude is expressed as a dispersion integral over the antihermitian part.

In references 9—11, Chew-Low equations have been obtained for the process $\pi + N \rightarrow 2\pi + N$ where the utilized quantities did not lie on the energy shell. The authors, for example, assumed that the argument of the amplitude of the elastic process (the one-nucleon term) lies in the observable energy region. In the here considered case where energy conservation is always required the argument of the one-nucleon term lies in the unobservable region of the process and has to be calculated by means of the dispersion relations for the case of elastic scattering. The results of the corresponding computations will be published in a subsequent paper.

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*In the relativistic case the number of variables is still considerably larger (see, e.g., reference 8).

†In several expressions of reference 4, the symbols used differ from those in the present paper.

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